HINTS & SOLUTIONS

EXERCISE - 1
Single Choice

1. Given $\frac{x}{3} = \cos t + \sin t$ & $\frac{y}{4} = \cos t - \sin t$

Squaring these two,

$\Rightarrow \frac{x^2}{9} = 1 + 2\cos t \sin t$ .....(i)

$\frac{y^2}{16} = 1 - 2\sin t \cos t$ .....(ii)

Adding (i) & (ii)

$$\frac{x^2}{9} + \frac{y^2}{16} = 2 \quad \Rightarrow \quad \frac{x^2}{18} + \frac{y^2}{32} = 1$$

3. Consider P(θ), $Q\left(\theta + \frac{2\pi}{3}\right)$, and $R\left(\theta + \frac{4\pi}{3}\right)$. Then,

$P' \equiv (a\cos\theta, b\sin\theta)$

$$Q' \equiv \left(a\cos\left(\theta + \frac{2\pi}{3}\right), b\sin\left(\theta + \frac{2\pi}{3}\right)\right)$$

and $$R' \equiv \left(a\cos\left(\theta + \frac{4\pi}{3}\right), b\sin\left(\theta + \frac{4\pi}{3}\right)\right)$$

Let the centroid of ΔP'Q'R' be (x', y').

$$x' = a\left[\frac{\cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{4\pi}{3}\right)}{3}\right]$$

$$= \frac{a}{3}\left[\cos\theta + 2\cos(\theta + \pi)\cos\frac{\pi}{3}\right] = 0$$

$$y' = \frac{a}{3}\left[\sin\theta + \sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta + \frac{4\pi}{3}\right)\right]$$

$$= \frac{a}{3}\left[\sin\theta + 2\sin(\theta + \pi)\sin\frac{\pi}{3}\right] = 0$$

$= 0$

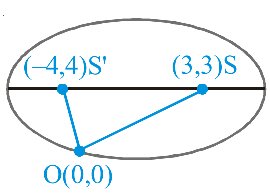
4. Here S is (3, 3) & S' is (–4, 4).

$\Rightarrow SS' = \sqrt{50} = 2ae$ .....(i)

Now $OS + OS' = 2a$

$3\sqrt{2} + 4\sqrt{2} = 2a$

$7\sqrt{2} = 2a$ .....(ii)



From (i) & (ii)

$$e = \frac{5}{7}$$

5. $(5x - 10)^2 + (5y + 15)^2 = \frac{(3x + 4y + 7)^2}{4}$

or $(x - 2)^2 + (y + 3)^2 = \left(\frac{1}{2}\frac{3x - 4y - 7}{5}\right)^2$

or $\sqrt{(x-2)^2 + (y+3)^2} = \frac{1}{2}\frac{|3x - 4y - 7|}{5}$

It is an ellipse, whose focus is (2, –3), directrix is $3x - 4y + 7 = 0$, and eccentricity is 1/2.

length of perpendicular from the focus to the directrix is

$$\frac{|3 \times 2 - 4(-3) + 7|}{5} = 5$$

or $\frac{a}{e} - ae = 5$

or $2a - \frac{a}{2} = 5$

or $a = \frac{10}{3}$

So, the length of the major axis is 20/3.

7. Since major axis is along y-axis.

∴ Equation of tangent is $x = my + \sqrt{b^2m^2 + a^2}$

slope of tangent $= \frac{1}{m} = \frac{-4}{3} \Rightarrow m = \frac{-3}{4}$

Hence equation of tangent is $4x + 3y = 24$

or $\frac{x}{6} + \frac{y}{8} = 1$

Its intercepts on the axes are 6 and 8.

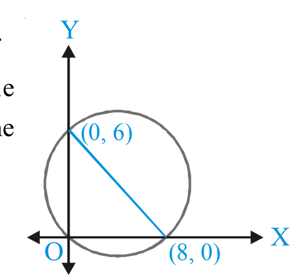
Area $(\Delta AOB) = \frac{1}{2} \times 6 \times 8 = 24$ sq.unit.

8.

The center of the family of ellipse is (4, 3) and the distance of focus from the center is ae = 5/2.

Hence, the locus is

$$(x - 4)^2 + (y - 3)^2 = \frac{25}{4}$$



For the chord to subtend a right angle at the origin,

$(1-m)+(2+\sin^2\alpha-1)=0$

(As sum of the coefficient of $x^2+y^2=10$)

or $\sin^2\alpha = m-2$

or $0 \le m-2 \le 1$

or $2 \le m \le 3$

EXERCISE - 2

Part # I : Multiple Choice

1. $r^2 - r - 6 > 0$ and $r^2 - 6r + 5 > 0$

or $(r-3)(r+2) > 0$ and $(r-1)(r-5) > 0$

or $(r < -2$ or $r > 3)$ and $(r < 1$ or $r > 5)$

i.e., $r < -2$ or $r > 5$

Also, $r^2 - r - 6 \ne r^2 - 6r + 5$

or $r \ne \frac{11}{5}$

2. If both the foci are fixed, then the ellipse is fixed, that is, both the directrices can be decided (eccentricity is given). Similar is the case for option (c). Thus, (a) and (c) are the correct choices. In the remaining cases, the size of the ellipse is fixed, but its position is not fixed.

4. The ellipse is $16x^2 + 11y^2 = 256$

The equation of tangent is $\left(4\cos\theta, 16\sqrt{11}\cos\theta\right)$ is

$$16x(4\cos\theta) + 11y\left(\frac{16}{\sqrt{11}}\sin\theta\right) = 256$$

It touches $(x-1)^2 + y^2 = 4^2$ if

$$\left|\frac{4\cos\theta - 16}{\sqrt{16\cos^2\theta + 11\sin\theta}}\right| = 4$$

or $(\cos\theta - 4)^2 = 16\cos^2\theta + 11\sin^2\theta$

or $4\cos^2\theta + 8\cos\theta - 5 = 0$

or $\cos\theta = \frac{1}{2}$

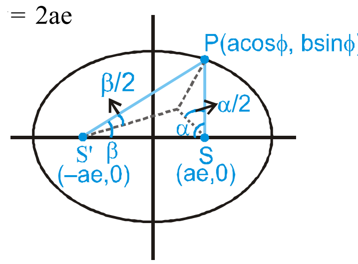
$\therefore$ $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

5. By definition of ellipse

PS + PS' = 2a if a > b

PS + PS' = 2b if a < b

and SS' = 2ae



$(\sqrt{5}\sec\theta)x - (2\ \text{cosec}\theta)y = 5 - 4$

or $\quad(\sqrt{5}\sec\theta)x - (2\ \text{cosec}\theta)y = 1 \qquad$ ....**(ii)**

Given that **(i)** and **(ii)** represent the same line. Then

$$\frac{\sqrt{5}\sec\theta}{1} = \frac{-2\,\text{cosec}\theta}{-t} = \frac{-1}{t^2}$$

or $\quad t = \frac{2}{\sqrt{5}}\cot\theta$ and $t = -\frac{1}{2}\sin\theta$

or $\quad \frac{2}{\sqrt{5}}\cot\theta = -\frac{1}{2}\sin\theta$

or $\quad 4\cos\theta = -\sqrt{5}\sin^2\theta$

or $\quad 4\cos\theta = -\sqrt{5}(1 - \cos^2\theta)$

or $\quad \sqrt{5}\cos^2\theta - 4\cos\theta - \sqrt{5} = 0$

or $\quad (\cos\theta - \sqrt{5})(\sqrt{5}\cos\theta + 1) = 0$

or $\quad \cos\theta = -\frac{1}{\sqrt{5}} \qquad [\because \cos\theta \neq -\sqrt{5}]$

or $\quad \theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$

Putting $\cos\theta = -1/\sqrt{5}$ in $t = -(1/2)\sin\theta$, we get

$$t = -\frac{1}{2}\sqrt{1-\frac{1}{5}} = -\frac{1}{\sqrt{5}}$$

Hence, $\theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$ and $t = -\frac{1}{\sqrt{5}}$

Part # II : Assertion & Reason

2. The chord of contact of the ellipse

$$\frac{x^2}{4} + \frac{y^2}{2} = 1$$

w.r.t. the point (8, 6) is

$$\frac{8x}{4} + \frac{6x}{2} = 1$$

or $\quad 2x + 3y = 1$

Hence, statement 1 is correct. Also, statement 2 is correct and explains statement 1.

3. $x^2 + y^2 + xy = 1$

Replacing x by –x & y by –y we get the same equation.

$\therefore$ Centre of conic is (0, 0) and every chord passing through the centre is bisected by the point. Hence st. I & st. II both are true & st I explains st. II.

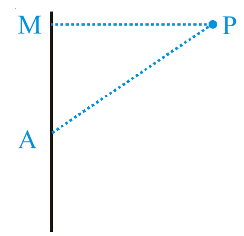
4. $|ay - bx| = c\sqrt{(x-a)^2 + (y-b)^2}$

or $\quad \frac{|ay - bx|}{\sqrt{a^2+b^2}} = \frac{c}{\sqrt{a^2+b^2}}\sqrt{(x-a)^2+(y-b)^2}$

or $\quad PM = kPA$

where m is the length of perpendicular from P on the line $ay - bx = 0$, PA is the length of line segment joining P to the point A(a, b), and A lies on the line. So, the locus of P is a straight line through A inclined at an angle $\sin^{-1}(c/\sqrt{a^2+b^2})$ to the given line

(provided $c < \sqrt{a^2+b^2}$).



Homogenizing the equation of ellipse

$$\frac{x^2}{9}+\frac{y^2}{5}=\left(\frac{hx}{9}+\frac{ky}{5}\right)^2$$

$$x^2\left(\frac{h^2}{81}-\frac{1}{9}\right)+y^2\left(\frac{k^2}{25}-\frac{1}{5}\right)+\frac{2hk}{45}xy=0$$

coefficient of x^2 + coefficient of $y^2 = 0$

$$\frac{h^2}{81}-\frac{1}{9}+\frac{k^2}{25}-\frac{1}{5}=0 \Rightarrow 25x^2+81y^2=630$$

3. Chord of contact of A(h,k) is

$$\frac{hx}{9}+\frac{ky}{5}=1 \qquad \text{......(1)}$$

$$\frac{x}{3}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{\sqrt{5}}.\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right) \qquad \text{.....(2)}$$

Comparing (1) & (2)

$$\frac{h}{3\cos\left(\frac{\alpha+\beta}{2}\right)}=\frac{k}{\sqrt{5}\sin\left(\frac{\alpha+\beta}{2}\right)}=\frac{1}{\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$\frac{h}{3\cos\left(\frac{\alpha+\beta}{2}\right)}=\frac{k}{\sqrt{5}\sin\left(\frac{\alpha+\beta}{2}\right)}=\frac{2}{\sqrt{3}}$$

$$\cos\left(\frac{\alpha+\beta}{2}\right)=\frac{h}{2\sqrt{3}}\;;\;\sin\left(\frac{\alpha+\beta}{2}\right)=\frac{\sqrt{3}k}{2\sqrt{5}}$$

$$\Rightarrow \frac{x^2}{12}+\frac{3y^2}{20}=1$$

$$\Rightarrow 5x^2+9y^2=60$$

EXERCISE - 4
Subjective Type

2. $\frac{x^2}{25}+\frac{y^2}{16}=1$

Since point $(7-\frac{5}{4}\alpha, \alpha)$ lies inside the ellipse,

$\therefore\; S_1<0$

$$\Rightarrow 16\left(7-\frac{5}{4}\alpha\right)^2+25.\alpha^2<400$$

$$\Rightarrow (28-5\alpha)^2+25\alpha^2<400$$

$$\Rightarrow 50\alpha^2-280\alpha+384<0$$

$$\Rightarrow 25\alpha^2-140\alpha+192<0$$

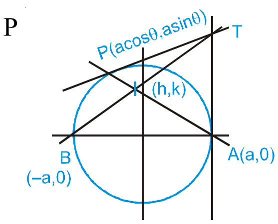
$$\Rightarrow \alpha\in\left(\frac{12}{5},\frac{16}{5}\right)$$

3. Equation of tangent at P

$x\cos\theta + y\sin\theta = a$

which meets $x = a$ at T

$\therefore$ T (a, a tan θ/2)



Equation of AP $\rightarrow$ $y=-\cot(\theta/2)(x-a)$ ....(1)

Equation of BT $\rightarrow$ $y=\frac{\tan(\theta/2)}{2}(x+a)$ ....(2)

From (1) & (2)

$$y^2=-\frac{1}{2}(x^2-a^2)$$

$$x^2+2y^2=a^2$$

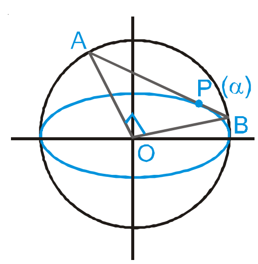
4. Equation of auxiliary circle is $x^2+y^2=a^2$ ....(1)

Equation of tangent at point P (acosα, b sinα)

is $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$ ....(2)

Equation of pair of lines OA, OB is obtained by homogenous equation of (1) with the help of (2)

$$\therefore\; x^2+y^2=a^2\left(\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha\right)^2$$



$$e^2 = 1 - \frac{a^2}{b^2}$$

foci lie in the inner circle then

$$\frac{a^2}{b^2} = 1 - \frac{a^2}{b^2} \qquad [a = be]$$

$$\frac{a^2}{b^2} = \frac{1}{2} \qquad \Rightarrow \quad \frac{a}{b} = \frac{1}{\sqrt{2}} = e$$

10. The equation of any tangent PQ to the ellipse

$$\frac{x^2}{y^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \qquad \text{....(i)}$$

This tangent cuts the ellipse $\frac{x^2}{c^2} + \frac{y^2}{d^2} = a$ at the point P and Q.

Let the tangents at P and Q intersect the point R(h, k).

Then PQ becomes the chord of contact with respect to the point R for the ellipse $\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$, i.e., the equation of PQ is $\frac{hx}{c^2} + \frac{ky}{d^2} = 1$ ....**(ii)**

Equation **(i)** and **(ii)** represent same straight lines.

Therefore, $\frac{(\cos\theta)/a}{h/c^2} = \frac{(\sin\theta)/a}{k/d^2} = 1$

or $\quad \cos\theta = \frac{ah}{c^2} \sin\theta = \frac{bk}{d^2}$

Squaring and adding, we get

$$\frac{a^2h^2}{c^4} + \frac{b^2k^2}{d^4} = 1$$

or $\quad \frac{a^2x^2}{c^4} + \frac{b^2y^2}{d^4} = 1$ ....**(iii)**

which is the locus of the point R(h, k)

If R(h, k) is the point of intersection of two perpendicular tangents, then the locus of R should be the director circle of the ellipse.

$$\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$$

i.e., $x^2 + y^2 = c^2 + d^2$

i.e. $\quad \frac{x^2}{c^2+d^2} + \frac{y^2}{c^2+d^2} = 1$ ....**(iv)**

Equations **(iii)** and **(iv)** represent the same locus. Therefore,

$$\frac{a^2}{c^4} = \frac{1}{c^2+d^2}$$

and $\quad \frac{b^2}{a^4} = \frac{1}{c^2+d^2}$

or $\quad \frac{a^2}{c^2} + \frac{b^2}{d^2} = 1$

12. Tangent at point $(t^2, 2t)$ on parabola $y^2 = 4x$ is

$ty = x + t^2$ .....**(i)**

Normal at $(\sqrt{5}\cos\phi, 2\sin\phi)$ on ellipse $4x^2 + 5y^2 = 20$ is

$\sqrt{5}\, x\sec\phi - 2y\operatorname{cosec}\phi = 1$ .....**(ii)**

(i) & **(ii)** are same lines, hence by comparing

$$\frac{-\sqrt{5}}{\cos\phi} = \frac{-2}{t\sin\phi} = \frac{1}{t^2}$$

$\Rightarrow \quad \cos\phi = -\sqrt{5}\, t^2$ ..... **(iii)**

$\sin\phi = \frac{-2t^2}{t}$ ......**(iv)**

Square & add **(iii)** & **(iv)** we get

$$t = \pm\frac{1}{\sqrt{5}}, t = 0$$

when $t = \frac{-1}{\sqrt{5}}$, $\tan\phi = -2 \quad \Rightarrow \quad \phi = \pi - \tan^{-1}2$

$t = \frac{1}{\sqrt{5}}$, $\tan\phi = 2 \quad \Rightarrow \quad \phi = \pi + \tan^{-1}2$

$t = 0$, $\phi = \frac{\pi}{2}, \frac{3\pi}{2}$

13. Equation of tangent at point P $(a\cos\theta, b\sin\theta)$

on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

foci $F_1 \equiv (ae, 0)$, $F_2 = (-ae, 0)$

and $d = \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}} = \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$

Now $4a^2\left(1 - \frac{b^2}{d^2}\right)$

$$= 4a^2\left(1 - \frac{b^2(a^2\sin^2\theta + b^2\cos^2\theta)}{a^2b^2}\right)$$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

1. Foci are $(\pm ae, 0)$. Therefore accoording to the condition,

$2ae = 2b$ or $ae = b$

Also, $b^2 = a^2(1 - e^2) \Rightarrow e^2 = (1 - e^2) \Rightarrow e = \dfrac{1}{\sqrt{2}}$

2. Since directrix is parallel to y-axis, hence axes of the ellipse are parallel to x-axis.

Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ (a > b)$$

$$e^2 = 1 - \frac{b^2}{a^2} \ \Rightarrow \ \frac{b^2}{a^2} = 1 - e^2 = 1 - \frac{1}{4} \Rightarrow \frac{b^2}{a^2} = \frac{3}{4}$$

Also, one of the directrices is $x = 4$

$$\Rightarrow \ \frac{a}{e} = 4 \ \Rightarrow \ a = 4e = 4.\frac{1}{2} = 2;$$

$$b^2 = \frac{3}{4}a^2 = \frac{3}{4}.4 = 3$$

$\therefore$ Required ellipse is $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

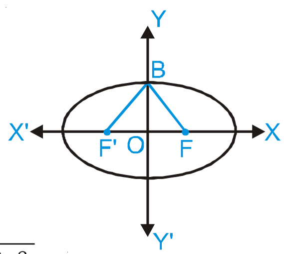
or $3x^2 + 4y^2 = 12$

4. $\angle F'BF = 90°$, $F'B \perp FB$

i.e., slope of $(F'B) \times$ Slope of $(FB) = -1$

$$\Rightarrow \ \frac{b}{ae} \times \frac{b}{-ae} = -1,$$

$b^2 = a^2e^2$ ..... (i)



We know that

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{a^2e^2}{a^2}} = \sqrt{1 - e^2}$$

$$e^2 = 1 - e^2, 2e^2 = 1, e^2 = \frac{1}{2}, e = \frac{1}{\sqrt{2}}$$

5. Distance between foci = 6

$\Rightarrow$ $ae = 3$

Minor axis = 8

$\Rightarrow$ $2b = 8$ $\Rightarrow$ $b = 4$ $\Rightarrow$ $b^2 = 16$

$\Rightarrow$ $a^2(1 - e^2) = 16$ $\Rightarrow$ $a^2 - a^2e^2 = 16$

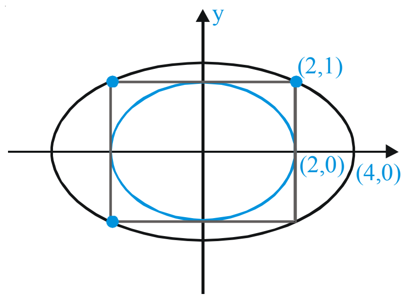
$\Rightarrow$ $a^2 - 9 = 16$

$\Rightarrow$ $a = 5$

Hence $ae = 3$

$\Rightarrow$ $e = \dfrac{3}{5}$

7.



Ellipse $x^2 + 4y^2 = 4$ (Given)

Eq^n. of the ellipse (required)

$$\frac{x^2}{16} + \frac{y^2}{b^2} = 1$$

Ellipse passes through (2, 1)

therefore $\dfrac{4}{16} + \dfrac{1}{b^2} = 1 \ \Rightarrow \ b^2 = \dfrac{4}{3}$

$$\frac{x^2}{16} + \frac{y^2}{4/3} = 1 \ \Rightarrow \ \frac{x^2}{16} + \frac{3y^2}{4} = 1$$

$\Rightarrow$ $x^2 + 12y^2 = 16$

9. Let the equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

from the given conditions

$a = 4$ and $b = 2$

$\therefore$ Eq of ellipse is

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

or $x^2 + 4y^2 = 16$

10. Let equation of any tangent to $y^2 = 16\sqrt{3}\,x$

be $y = mx + \dfrac{4\sqrt{3}}{m}$ ......(i)

and equation of any tangent to $2x^2 + y^2 = 4$

be $y = mx + \sqrt{2m^2 + 4}$ ......(ii)

but (i) and (ii) are same lines

$$\therefore \ \frac{4\sqrt{3}}{m} = \sqrt{2m^2 + 4}$$

$\Rightarrow$ $m^4 + 2m^2 - 24 = 0$ $\Rightarrow$ $m^2 = -6, 4$

$\therefore$ $m = \pm 2$

4. Given tangent is drawn at $(3\sqrt{3}\cos\theta, \sin\theta)$ to

$$\frac{x^2}{27}+\frac{y^2}{1}=1$$

$\Rightarrow$ Equation of tangent is $\dfrac{x\cos\theta}{3\sqrt{3}}+\dfrac{y\sin\theta}{1}=1$

Thus sum of intercepts $=(3\sqrt{3}\sec\theta+\operatorname{cosec}\theta)=f(\theta)$ To minimise $f(\theta)$, $f'(\theta)=0$

$$\Rightarrow\ f'(\theta)=\frac{3\sqrt{3}\sin^3\theta-\cos^3\theta}{\sin^2\theta\cos^2\theta}=0$$

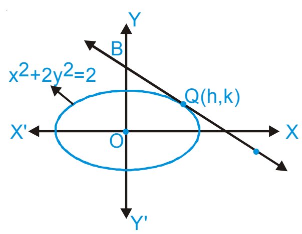
$$\Rightarrow\ \sin^3\theta=\frac{1}{3^{3/2}}\cos^3\theta \text{ or } \tan\theta=\frac{1}{\sqrt{3}},\ \text{i.e. } \theta=\frac{\pi}{6}$$

7. Let the point of contact be

$R \equiv (\sqrt{2}\cos\theta, \sin\theta)$

Equation of tangent AB is

$$\frac{x}{\sqrt{2}}\cos\theta+y\sin\theta=1$$



$\Rightarrow$ $A \equiv (\sqrt{2}\sec\theta, 0)$; $B \equiv (0, \operatorname{cosec}\theta)$

Let the middle point Q of AB be (h, k)

$$\Rightarrow\ h=\frac{\sec\theta}{\sqrt{2}},\ k=\frac{\operatorname{cosec}\theta}{2}$$

$$\Rightarrow\ \cos\theta=\frac{1}{h\sqrt{2}},\ \sin\theta=\frac{1}{2k} \Rightarrow \frac{1}{2h^2}+\frac{1}{4k^2}=1,$$

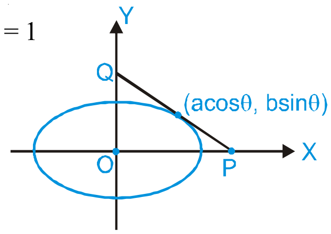
$$\therefore\ \text{Required locus is } \frac{1}{2x^2}+\frac{1}{4y^2}=1$$

Trick : The locus of mid-points of the portion of tangents to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ intercepted between axes is $a^2y^2+b^2x^2=4x^2y^2$

i.e., $\dfrac{a^2}{4x^2}+\dfrac{b^2}{4y^2}=1$ or $\dfrac{1}{2x^2}+\dfrac{1}{4y^2}=1$

8. Equation of tangent at $(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$



$$P=\left(\frac{a}{\cos\theta}, 0\right)$$

$$Q=\left(0, \frac{b}{\sin\theta}\right)$$

$$\text{Area of OPQ}=\frac{1}{2}\left|\left(\frac{a}{\cos\theta}\right)\left(\frac{b}{\sin\theta}\right)\right|=\frac{ab}{|\sin 2\theta|}$$

$\therefore$ $(\text{Area})_{min}=ab$

10. Equation of ellipse is $\dfrac{x^2}{4}+y^2=1$

eccentricity $e=\dfrac{\sqrt{3}}{2}$

so focus are $(\sqrt{3},0)$ & $(-\sqrt{3},0)$

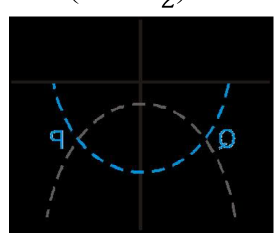
so end points of latus rectum will be

$$\left(\sqrt{3},\frac{1}{2}\right)\left(\sqrt{3},-\tfrac{1}{2}\right),\left(-\sqrt{3},\tfrac{1}{2}\right) \,\&\, \left(-\sqrt{3},-\tfrac{1}{2}\right)$$

$\because$ $y_1<0$ & $y_2<0$

Hence coordinates of P & Q will be

$$P\left(\sqrt{3},-\frac{1}{2}\right) \,\&\, Q\left(-\sqrt{3},-\frac{1}{2}\right).$$



So now equation of parabola taking these points as end points of latus rectum.

Focus will be (0, –1/2)

$$4a=2\sqrt{3} \Rightarrow a=\frac{\sqrt{3}}{2}$$

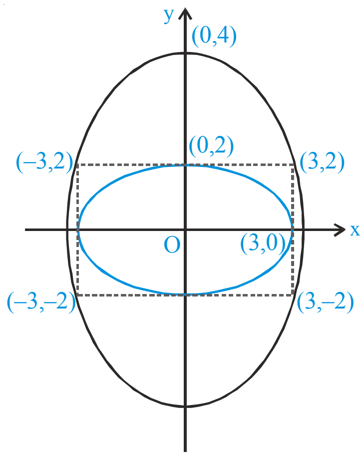
Hence vertex of the parabolas will be

$$\left(0,-\frac{1}{2}+\frac{\sqrt{3}}{2}\right),\ \left(0,-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)$$

so eq. of parabolas will be

$$x^2=-2\sqrt{3}\left(y+\frac{1}{2}-\frac{\sqrt{3}}{2}\right) \,\&$$

16. Let equation of E_2 be



$\frac{x^2}{a^2}+\frac{y^2}{16}=1$ $\quad$ ($\because$ E_2 passes through (0, 4))

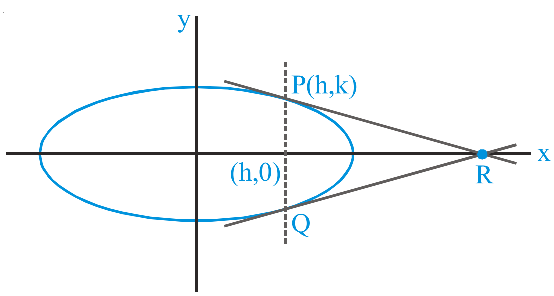
$\because$ E_2 passes through (3,2)

$\therefore \quad \frac{9}{a^2}+\frac{4}{16}=1$

$\Rightarrow \quad a^2 = 12$

$\therefore \quad e^2 = 1-\frac{a^2}{16} = 1-\frac{3}{4} \Rightarrow e = \frac{1}{2}$

17.



Tangent at P(h, k) is $\frac{xh}{4}+\frac{ky}{3}=1$

$\Rightarrow \quad R\left(\frac{4}{h}, 0\right)$

$\Delta PQR = k\left(\frac{4}{h}-h\right)$

$= \sqrt{3\left(1-\frac{h^2}{4}\right)}\left(\frac{4}{h}-h\right)$

which is a decreasing function in $\left[\frac{1}{2}, 1\right]$

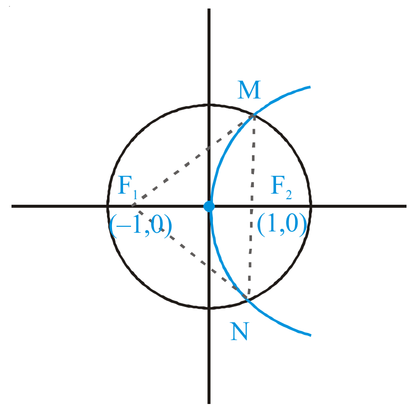
$\Rightarrow \quad \Delta_1 = \sqrt{3\left(1-\frac{1}{16}\right)}\left(8-\frac{1}{2}\right) = \frac{45\sqrt{5}}{8}$

& $\quad \Delta_2 = \sqrt{3\left(1-\frac{1}{4}\right)}(4-1) = \frac{9}{2}$

$\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = 45-36 = 9$

20. $a = 3$

$e = \frac{1}{3}$



$\therefore \quad F_1 \equiv (-1,0)$

$F_2 \equiv (1,0)$

So, equation of parabola is $y^2 = 4x$

Solving simultaneously, we get $\left(\frac{3}{2}, \pm\sqrt{6}\right)$

$\therefore \quad$ Orthocentre is $\left(\frac{-9}{10}, 0\right)$

21. Equation of tangent at M is $\frac{x\times 3}{2\times 9}+\frac{y\sqrt{6}}{8}=1$

Put y = 0 as intersection will be on x-axis.

$\therefore \quad R \equiv (6,0)$

Equation of normal at M is $\sqrt{\frac{3}{2}}x + y = 2\sqrt{\frac{3}{2}}+\left(\sqrt{\frac{3}{2}}\right)^3$

Put $y = 0, x = 2+\frac{3}{2}=\frac{7}{2}$

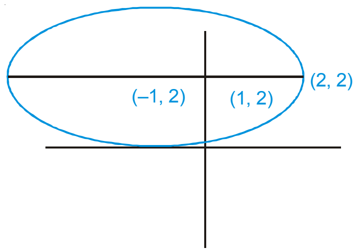
$\therefore Q \equiv \left(\frac{7}{2}, 0\right)$

$\therefore$ Area (Δ MQR) $=\frac{1}{2}\times\left(6-\frac{7}{2}\right)\times\sqrt{6}=\frac{5}{4}\sqrt{6}$

Area of quadrilateral (MF_1NF_2) = 2 × Area (ΔF_1F_2M)

$= 2\times\frac{1}{2}\times 2\times\sqrt{6} = 2\sqrt{6}$ sq. units

$\therefore \quad$ Required Ratio $=\frac{5/4}{2}=\frac{5}{8}$



$$\frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1$$

$\therefore \quad G = 5, L = 1$

S_3 : $\frac{2b^2}{a} = \frac{2a}{3} \quad \Rightarrow \quad 3b^2 = a^2$

$\therefore \quad b^2 = a^2(1-e^2) \quad \Rightarrow \quad 1 = 3(1-e^2)$

$\Rightarrow \quad e = \sqrt{2/3}$

S_4 : The equation is $\left(x-\frac{1}{13}\right)^2 + \left(y-\frac{2}{13}\right)^2$

$$= \frac{1}{a^2}\left(\frac{5x+12y-1}{13}\right)^2$$

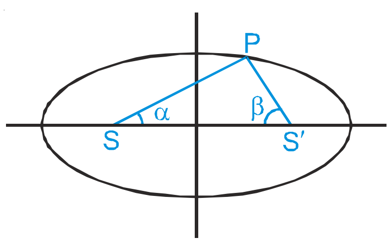
$\therefore \quad \frac{1}{a^2} < 1 \qquad$ i.e. $\quad a^2 > 1$

11. (A,B,C)

Focal property of ellipse

$PS + PS' = 2a \qquad ; \quad \text{if } a > b$

$PS + PS' = 2b \qquad ; \quad \text{if } a < b$



$PS \cos\alpha + PS' \cos\beta = 2ae \qquad$ ......**(i)**

$PS \sin\alpha - PS' \sin\beta = 0 \qquad$ ......**(ii)**

$PS + PS' = 2a \qquad$ ......**(iii)**

from **(i)** and **(ii)**, we get $PS = \frac{2ae\sin\beta}{\sin(\alpha+\beta)}$,

$PS' = \frac{2ae\sin\alpha}{\sin(\alpha+\beta)} \qquad$ ......**(iv)**

from **(iii)** and **(iv)**

$e(\sin\alpha + \sin\beta) = \sin(\alpha+\beta)$

$\therefore \quad e \, . \, 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}$

$\therefore \quad e\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)$

$= \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}$

$\therefore \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1-e}{1+e} = \frac{2a\left(a-\sqrt{a^2-b^2}\right)-b^2}{b^2}$

$\therefore \quad$ C is correct option & D is incorrect

12. (A, B, C)

Equation of tangent at θ is

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

$\Rightarrow \quad \frac{x}{a\sec\theta} + \frac{y}{b\,\mathrm{cosec}\,\theta} = 1$

$\therefore \quad$ Length of intercept,

$d = \sqrt{a^2\sec^2\theta + b^2\,\mathrm{cosec}^2\theta}$

$\therefore \quad d^2 = a^2\sec^2\theta + b^2\,\mathrm{cosec}^2\,\theta$

$\Rightarrow \quad \frac{d(d^2)}{d\theta} = 2a^2\frac{\sin\theta}{\cos^3\theta} - 2b^2\frac{\cos\theta}{\sin^3\theta} = 0$

i.e., $\quad \frac{a^2\sin\theta}{\cos^3\theta} = \frac{b^2\cos\theta}{\sin^3\theta}$

$\therefore \quad \tan^4\theta = \frac{b^2}{a^2}$

$\therefore \quad \tan\theta = \pm\sqrt{\frac{b}{a}}$

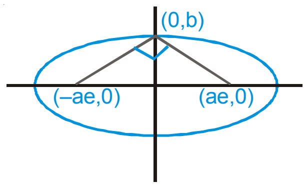
$\Rightarrow \quad \theta = \tan^{-1}\sqrt{\frac{b}{a}}, -\tan^{-1}\sqrt{\frac{b}{a}}$

$\Rightarrow \quad \theta = \pi - \tan^{-1}\sqrt{\frac{b}{a}}, -\pi + \tan^{-1}\sqrt{\frac{b}{a}}$

14. (A, C)

The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P.

So answer are (A) and (C)



$\Rightarrow \quad \dfrac{a^2(1-e^2)}{a^2e^2} = 1 \qquad \Rightarrow \quad 1 - e^2 = e^2$

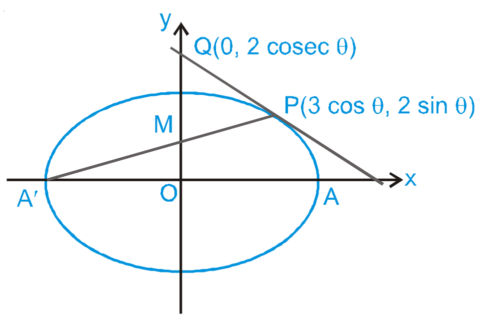
$\Rightarrow \quad 2e^2 = 1 \qquad \Rightarrow \quad e = \dfrac{1}{\sqrt{2}}$

(B) Here $a = 3; b = 2$

and $T : \dfrac{x\cos\theta}{3} + \dfrac{y\sin\theta}{2} = 1$

$x = 0; y = 2 \text{ cosec } \theta$

Equation of chord A′P, $y = \dfrac{2\sin\theta}{3(\cos\theta + 1)}(x + 3)$



Put $x = 0, y = \dfrac{2\sin\theta}{1+\cos\theta} = OM$

Now,

$OQ^2 - MQ^2 = OQ^2 - [OQ - OM]^2$

$= 2(OQ)(OM) - OM^2 = OM\{2(OQ) - (OM)\}$

$= \dfrac{2\sin\theta}{1+\cos\theta}\left[\dfrac{4}{\sin\theta} - \dfrac{2\sin\theta}{1+\cos\theta}\right] = 4$

(C) $\left(y + \dfrac{1}{2}\right)^2 = 8(x-2)$

$\Rightarrow \quad Y^2 = 8X$

for three normal $X > 2a$

$\Rightarrow \quad x - 2 > 4 \therefore x > 6$

(D) Area of parallelogram $= 2ab$

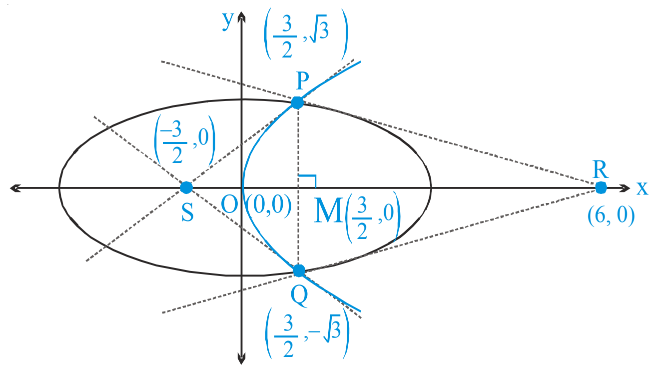
$= 2 \times (2) \times \left(\dfrac{1}{2}\right) = 2$

23. Solving the curves $y^2 = 2x$ and $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ for the points of intersection, we have

$4x^2 + 18x - 36 = 0 \qquad \Rightarrow \quad x = \dfrac{3}{2}, -6$

But from $y^2 = 2x$ we have $x > 0$

$\therefore \quad x = \dfrac{3}{2} \qquad$ at which $y^2 = 2 \cdot \dfrac{3}{2}$



$\Rightarrow \quad y = \pm\sqrt{3}$

$\therefore \quad P\left(\dfrac{3}{2}, \sqrt{3}\right)$ and $Q\left(\dfrac{3}{2}, -\sqrt{3}\right)$

Now equation of tangents at P and Q to ellipse

$\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ is $\dfrac{x}{9}\left(\dfrac{3}{2}\right) + \dfrac{y}{4}\left(\pm\sqrt{3}\right) = 1$

which intersect at R(6, 0)

Equation of tangents at P and Q to parabola $y^2 = 2x$ will be $y\left(\pm\sqrt{3}\right) = x + \dfrac{3}{2}$ which cut x-axis $S\left(\dfrac{-3}{2}, 0\right)$

$\therefore \quad \dfrac{\text{Area }\Delta PQS}{\text{Area }\Delta PQR} = \dfrac{\frac{1}{2}PQ \cdot MS}{\frac{1}{2}PQ \cdot MR} = \dfrac{MS}{MR}$

$= \dfrac{\frac{3}{2} - \left(\frac{-3}{2}\right)}{6 - \frac{3}{2}} = \dfrac{3}{\frac{9}{2}} = \dfrac{2}{3}$

Area of quadrilateral PRQS $= \dfrac{1}{2}PQ(MS + MR) = \dfrac{1}{2} \cdot$

3. (C)

Let the bird is at S and projection of S on the ground is S′

$\therefore \quad OS' = \text{semi minor axis} = \dfrac{1}{\sqrt{b}}$

$\therefore \quad \left(\dfrac{1}{\sqrt{2}}\right)^2 = 1 - \dfrac{\frac{1}{b}}{25} \quad \Rightarrow \quad \dfrac{1}{\sqrt{b}} = \dfrac{5}{\sqrt{2}}$

$\therefore \quad PS' = \sqrt{\left(\dfrac{5}{\sqrt{2}}\right)^2 + (10)^2} = \sqrt{\dfrac{25+200}{2}}$

$= \sqrt{\dfrac{225}{2}} = \dfrac{15}{\sqrt{2}}$ and $SS' = 5\sqrt{3}$

$\therefore \quad \tan\theta = \dfrac{SS'}{PS'} = \sqrt{\dfrac{2}{3}}$

26. (3)

$\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1$

equation of normal at the point $(13\cos\theta, 5\sin\theta)$

$(y - 5\sin\theta) = \dfrac{13\sin\theta}{5\cos\theta}(x - 13\cos\theta)$ it passes through (0, 6)

then $\cos\theta\,(15 + 72\sin\theta) = 0$

or $\quad \cos\theta = 0, \quad \sin\theta = -\dfrac{5}{24}$

$\theta = \dfrac{\pi}{2}, 2\pi - \sin^{-1}\dfrac{5}{24}, \pi + \sin^{-1}\dfrac{5}{24}$

equation has three roots hence three normal can be drawn.

27. Any point on the parabola $y^2 = 4ax$ is $(at^2, 2at)$. Equation of chord of the ellipse $\dfrac{x^2}{2a^2} + \dfrac{y^2}{a^2} = 1$, whose mid-point is $(at^2, 2at)$ is $\dfrac{x \cdot at^2}{2a^2} + \dfrac{y \cdot 2at}{a^2} = \dfrac{a^2t^4}{2a^2} + \dfrac{4a^2t^2}{a^2}$

$\Rightarrow \quad tx + 4y = at^3 + 8at \qquad (\because t \neq 0)$

As it passes through $\left(11a, -\dfrac{a^2}{4}\right)$,

$\Rightarrow \quad 11at - 4\left(\dfrac{a^2}{4}\right) = at^3 + 8at$

$\Rightarrow \quad at^3 - 3at + a^2 = 0$

$\Rightarrow \quad t^3 - 3t + a = 0 \quad (a \neq 0)$

Now, three chords of the ellipse will be bisected by the parabola if the equation (1) has three real and distinct roots.

Let $f(t) = t^3 - 3t + a$

$f'(t) = 3t^2 - 3 = 0 \qquad \Rightarrow \quad t = \pm 1$

So, $f(1)\,f(-1) < 0$

$\Rightarrow \quad a \in (-2, 2)$

But $a \neq 0$, so $a \in (-2, 0) \cup (0, 2)$

$\therefore$ Number of integral values of 'a' = 2.

28. (1/2)

Let line OPQ makes angle θ with x-axis so

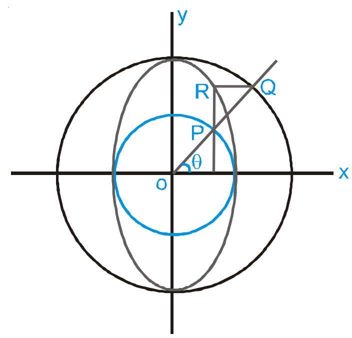
$P \equiv (a\cos\theta, a\sin\theta)$, $Q\,(b\cos\theta, b\sin\theta)$

and Let R (x, y)

So $\quad X = a\cos\theta \quad Y = b\sin\theta$

eliminating θ, we get

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, locus of R is an ellipse.



Also a < b so vertices are (0,b) and (0, – b)

and extremities of minor axis are (±a, 0).

So ellipse touches both inner circle and outer circle if focii are (0, ± a)

$\Rightarrow \quad a = be \quad$ i.e $\quad e = a/b$

Also $\quad e = \sqrt{1-e^2} \quad \Rightarrow \quad e^2 = 1 - e^2 \quad \Rightarrow \quad e = 1/\sqrt{2}$

and ratio of radii is $\dfrac{a}{b} = e = \dfrac{1}{\sqrt{2}}$.

29. Clearly the parabola should pass through (1, 0) and (–1, 0). Let directrix of this parabola be $x\cos\theta + y\sin\theta = 2$. If M (h,k) be the focus of this parabola, then distance of (±1, 0) from 'M' and from the directrix should be same.

$\Rightarrow \quad (h-1)^2 + k^2 = (\cos\theta - 2)^2 \qquad$ ....(i)