ELLIPSE

DEFINITIONS

It is locus of a point which moves in such a way that the ratio of its distance from a fixed point and a fixed line (not passes through fixed point and all points and line lies in same plane) is constant (e), which is less than one.

The fixed point is called - focus

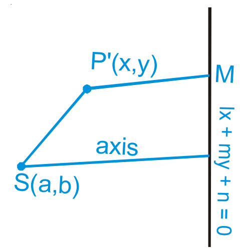
The fixed line is called -directrix.

The constant ratio is called - eccentricity, it is denoted by 'e'.

GENERAL EQUATION OF AN ELLIPSE

Let (a, b) be the focus S, and $lx + my + n = 0$ is the equation of directrix.

Let P(x, y) be any point on the ellipse. Then by definition.

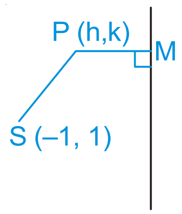


$\Rightarrow$ SP = e PM (e is the eccentricity) $\Rightarrow (x-a)^2 + (y-b)^2 = e^2 \dfrac{(lx+my+n)^2}{(l^2+m^2)}$

$\Rightarrow$ $(l^2+m^2)\{(x-a)^2+(y-b)^2\} = e^2\{lx+my+n\}^2$

Ex. Find the equation to the ellipse whose focus is the point (– 1, 1), whose directrix is the straight line $x - y + 3 = 0$ and eccentricity is $\dfrac{1}{2}$.

Sol. Let $P \equiv (h, k)$ be moving point,



$$e = \frac{PS}{PM} = \frac{1}{2}$$

$\Rightarrow$ $(h+1)^2 + (k-1)^2 = \dfrac{1}{4}\left(\dfrac{h-k+3}{\sqrt{2}}\right)^2$

$\Rightarrow$ locus of P(h, k) is

$8\{x^2 + y^2 + 2x - 2y + 2\} = (x^2 + y^2 - 2xy + 6x - 6y + 9)$

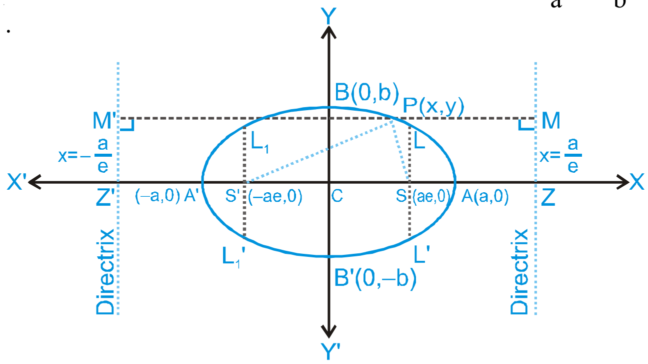
$7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0.$

STANDARD EQUATION & DEFINITION

Standard equation of an ellipse referred to its principal axes along the co-ordinate axes is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. where $a > b$ & $b^2 = a^2(1 - e^2) \Rightarrow a^2 - b^2 = a^2e^2$.

where e = eccentricity ($0 < e < 1$).

FOCI : $S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.



(a) **Equation of directrices :**

$x = \dfrac{a}{e}$ & $x = -\dfrac{a}{e}$.

(b) **Vertices :**

$A' \equiv (-a, 0)$ & $A \equiv (a, 0)$.

(c) **Major axis :** The line segment A' A in which the foci S' & S lie is of length 2a & is called the major axis (a > b) of the ellipse. Point of intersection of major axis with directrix is called the foot of the directrix (z) $\left(\pm\frac{a}{e}, 0\right)$.

(d) **Minor Axis :** The y-axis intersects the ellipse in the points B' $\equiv$ (0,- b) & B $\equiv$ (0, b). The line segment B'B of length 2b (b < a) is called the Minor Axis of the ellipse.

(e) **Principal Axes :** The major & minor axis together are called Principal Axes of the ellipse.

(f) **Centre :** The point which bisects every chord of the conic drawn through it is called the centre of the conic. C $\equiv$ (0,0) the origin is the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(g) **Diameter :** A chord of the conic which passes through the centre is called a diameter of the conic.

(h) **Focal Chord :** A chord which passes through a focus is called a focal chord.

(i) **Double Ordinate :** A chord perpendicular to the major axis is called a double ordinate.

(j) **Latus Rectum :** The focal chord perpendicular to the major axis is called the latus rectum.

(i) Length of latus rectum $(LL')=\frac{2b^2}{a}=\frac{(\text{minor axis})^2}{\text{major axis}}=2a(1-e^2)$

(ii) Equation of latus rectum : x = ± ae.

(iii) Ends of the latus rectum are $L\left(ae, \frac{b^2}{a}\right)$, $L'\left(ae, -\frac{b^2}{a}\right), L_1\left(-ae, \frac{b^2}{a}\right)$ and $L_1'\left(-ae, -\frac{b^2}{a}\right)$.

(k) **Focal radii :** SP = a –ex & S'P = a + ex $\Rightarrow$ SP + S'P = 2a = Major axis.

(l) **Eccentricity :** $e=\sqrt{1-\frac{b^2}{a^2}}$

(i) The sum of the focal distances of any point on the ellipse is equal to the major Axis. Hence distance of focus from the extremity of a minor axis is equal to semi major axis. i.e BS = CA.

(ii) If the equation of the ellipse is given as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & nothing is mentioned, then the rule is to assume that a > b.

Ex. If LR of an ellipse is half of its minor axis, then its eccentricity is -

Sol. As given $\frac{2b^2}{a}=b$ $\Rightarrow$ 2b = a $\Rightarrow$ $4b^2 = a^2$

$\Rightarrow$ $4a^2(1 - e^2) = a^2$ $\Rightarrow$ $1 - e^2 = 1/4$

$\therefore$ $e=\sqrt{3}/2$

Ex. Find the equation of the ellipse whose foci are (2, 3), (–2, 3) and whose semi minor axis is of length $\sqrt{5}$.

Sol. Here S is (2, 3) & S' is (–2, 3) and b = $\sqrt{5}$ $\Rightarrow$ SS' = 4 = 2ae $\Rightarrow$ ae = 2

but $b^2 = a^2(1 - e^2)$ $\Rightarrow$ $5 = a^2 - 4$ $\Rightarrow$ a = 3.

Hence the equation to major axis is y = 3

Centre of ellipse is midpoint of SS' i.e. (0, 3)

$\therefore$ Equation to ellipse is $\dfrac{x^2}{a^2}+\dfrac{(y-3)^2}{b^2}=1$ or $\dfrac{x^2}{9}+\dfrac{(y-3)^2}{5}=1$

Ex. Find the equation of the ellipse whose focii are (4, 0) and (– 4, 0) and eccentricity is $\dfrac{1}{3}$

Sol. Since both focus lies on x-axis, therefore x-axis is major axis and mid point of focii is origin which is centre and a line perpendicular to major axis and passes through centre is minor axis which is y-axis.

Let equation of ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

$\because$ $ae = 4$ and $e=\dfrac{1}{3}$ (Given)

$\therefore$ $a = 12$ and $b^2 = a^2(1-e^2)$

$\Rightarrow$ $b^2 = 144\left(1-\dfrac{1}{9}\right)$

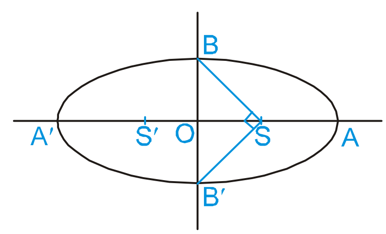
$b^2 = 16 \times 8$

$b = 8\sqrt{2}$

Equation of ellipse is $\dfrac{x^2}{144}+\dfrac{y^2}{128}=1$

Ex. If minor-axis of ellipse subtend a right angle at its focus then find the eccentricity of ellipse.

Sol. Let the equation of ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ $(a > b)$



$\because$ $\angle BSB' = \dfrac{\pi}{2}$

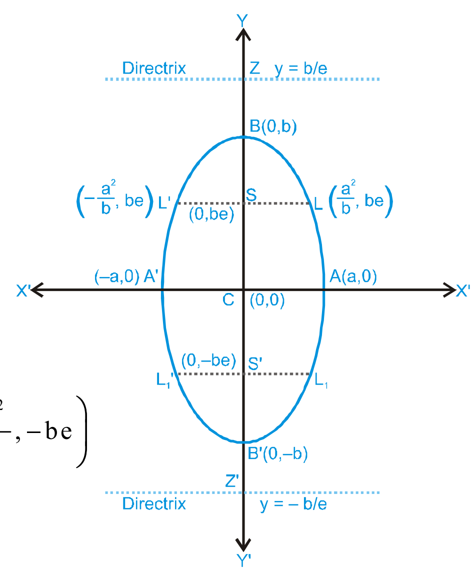
and $OB = OB'$

$\therefore$ $\angle BSO = \dfrac{\pi}{4}$

$\Rightarrow$ $OS = OB$ $\Rightarrow$ $ae = b$

$\Rightarrow$ $e^2 = \dfrac{b^2}{a^2} = 1 - e^2$ $\Rightarrow$ $e = \dfrac{1}{\sqrt{2}}$

Another form of ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $(a<b)$



(a) **AA' = Minor axis = 2a**

(b) **BB' = Major axis = 2b**

(c) $a^2 = b^2(1-e^2)$

(d) **Latus rectum $LL' = L_1L_1'$** $=\dfrac{2a^2}{b}$, equation $y = \pm be$

(e) **Ends of the latus rectum are :**

$$L\left(\frac{a^2}{b}, be\right), L'\left(-\frac{a^2}{b}, be\right), L_1\left(\frac{a^2}{b}, -be\right), L_1'\left(-\frac{a^2}{b}, -be\right)$$

(f) **Equation of directrix $y = \pm b/e$**

(g) **Eccentricity :** $e = \sqrt{1 - \frac{a^2}{b^2}}$

Ex. The equation of the ellipse with respect to coordinate axes whose minor axis is equal to the distance between its foci and whose LR = 10, will be-

Sol. When $a > b$

As given $2b = 2ae \Rightarrow b = ae$ ...... **(i)**

Also $\frac{2b^2}{a} = 10 \Rightarrow b^2 = 5a$ ...... **(ii)**

Now since $b^2 = a^2 - a^2e^2 \Rightarrow b^2 = a^2 - b^2$ [From **(i)**]

$\Rightarrow 2b^2 = a^2$ ...... **(iii)**

(ii), **(iii)** $\Rightarrow a^2 = 100, b^2 = 50$

Hence equation of the ellipse will be $\frac{x^2}{100} + \frac{y^2}{50} = 1 \Rightarrow x^2 + 2y^2 = 100$

Similarly when $a < b$ then required ellipse is $2x^2 + y^2 = 100$

POSITION OF A POINT W.R.T. AN ELLIPSE

The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as ; $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 >< \text{or} = 0$.

Ex. Check whether the point P(3, 2) lies inside or outside of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

Sol. $S_1 \equiv \frac{9}{25} + \frac{4}{16} - 1 = \frac{9}{25} + \frac{1}{4} - 1 < 0$

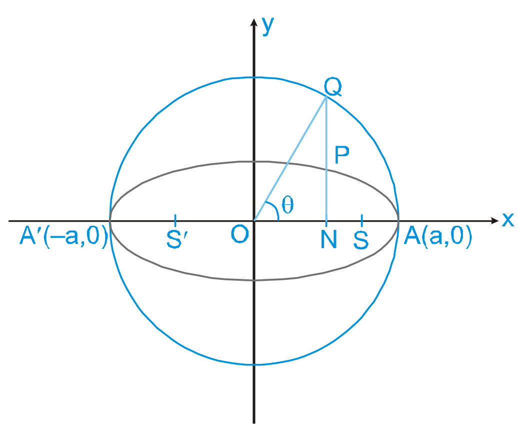
$\therefore$ Point $P \equiv (3, 2)$ lies inside the ellipse.

AUXILIARY CIRCLE / ECCENTRIC ANGLE

A circle described on major axis of ellipse as diameter is called the auxiliary circle.

Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that line through Q perpendicular to the x – axis on the way intersects the ellipse at P, then P & Q are called as the Corresponding Points on the ellipse & the auxiliary circle respectively. 'θ' is called the Eccentric Angle of the point P on the ellipse $(-\pi < \theta \le \pi)$. $Q \equiv (a \cos\theta, a \sin\theta)$

$P \equiv (a \cos\theta, b \sin\theta)$



❖ $\frac{\ell(PN)}{\ell(QN)} = \frac{b}{a} = \frac{\text{Semi minor axis}}{\text{Semi major axis}}$

❖ If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle.

PARAMETRIC REPRESENTATION

The equations x = a cos θ & y = b sin θ together represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Where θ is a parameter. Note that if P(θ) ≡ (a cos θ, b sin θ) is on the ellipse then;

Q(θ) ≡ (a cos θ, a sin θ) is on the auxiliary circle.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$.

Ex. Write the equation of chord of an ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ joining two points $P\left(\frac{\pi}{4}\right)$ and $Q\left(\frac{5\pi}{4}\right)$.

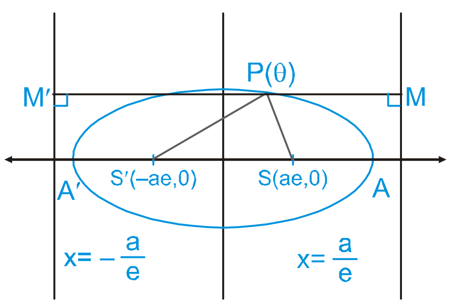
Sol. Equation of chord is $\frac{x}{5}\cos\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2} + \frac{y}{4}\,.\sin\frac{\left(\frac{\pi}{4}+\frac{5\pi}{4}\right)}{2} = \cos\frac{\left(\frac{\pi}{4}-\frac{5\pi}{4}\right)}{2}$

$$\frac{x}{5}\,.\cos\left(\frac{3\pi}{4}\right) + \frac{y}{4}\,.\sin\left(\frac{3\pi}{4}\right) = 0$$

$$-\frac{x}{5} + \frac{y}{4} = 0 \quad \Rightarrow \quad 4x = 5y$$

Ex. Find the focal distance of a point P(θ) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (a > b)

Sol. Let 'e' be the eccentricity of ellipse.



∴ PS = e . PM

$= e\left(\frac{a}{e} - a\cos\theta\right)$

PS = (a – a e cosθ)

and PS′ = e. PM′

$= e\left(a\cos\theta + \frac{a}{e}\right)$

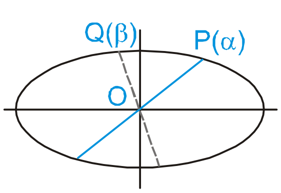
PS′ = a + ae cosθ

∴ focal distance are (a ± ae cosθ)

Note : PS + PS′ = 2a

PS + PS′ = AA′

Ex. Find the angle between two diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Whose extremities have eccentricity angle α and $\beta = \alpha + \frac{\pi}{2}$.



Sol. Let ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Slope of OP = $m_1 = \frac{b\sin\alpha}{a\cos\alpha} = \frac{b}{a}\tan\alpha$

Slope of OQ = $m_2 = \frac{b\sin\beta}{a\cos\beta} = -\frac{b}{a}\cot\alpha$ given $\beta = \alpha + \frac{\pi}{2}$

$\therefore \qquad \tan\theta = \left|\dfrac{m_1 - m_2}{1+m_1m_2}\right| = \left|\dfrac{\frac{b}{a}(\tan\alpha+\cot\alpha)}{1-\frac{b^2}{a^2}}\right| = \left|\dfrac{2ab}{(a^2-b^2)\sin 2\alpha}\right|$

LINE AND AN ELLIPSE

The line y = mx + c meets the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ in two points real, coincident or imaginary according as c^2 is < = or > $a^2m^2 + b^2$.

Hence y = mx + c is tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ if $c^2 = a^2m^2 + b^2$.

Ex. For what value of λ does the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$.

Sol. Equation of ellipse is $\qquad 9x^2+16y^2=144$ or $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$

Comparing this with $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ then we get $a^2 = 16$ and $b^2 = 9$

and comparing the line $y = x + \lambda$ with $y = mx + c$

$\therefore \quad m = 1$ and $c = \lambda$

If the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$, then $c^2 = a^2m^2 + b^2$

$\Rightarrow \qquad \lambda^2 = 16 \times 1^2 + 9$

$\Rightarrow \qquad \lambda^2 = 25 \qquad\qquad \therefore \qquad\qquad \lambda = \pm 5$

Ex. Find the set of value(s) of 'λ' for which the line $3x - 4y + \lambda = 0$ intersect the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$ at two distinct points.

Sol. Solving given line with ellipse, we get $\dfrac{(4y-\lambda)^2}{9\times16}+\dfrac{y^2}{9}=1$

$\dfrac{2y^2}{9} - \dfrac{y\lambda}{18} + \dfrac{\lambda^2}{144} - 1 = 0$

Since, line intersect the parabola at two distinct points,

$\therefore \qquad$ roots of above equation are real & distinct

$\therefore \qquad D > 0$

$\Rightarrow \qquad \dfrac{\lambda^2}{(18)^2} - \dfrac{8}{9}\cdot\left(\dfrac{\lambda^2}{144}-1\right) > 0$

$\Rightarrow \qquad -12\sqrt{2} < \lambda < 12\sqrt{2}$

TANGENT TO THE ELLIPSE $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$:

(a) **Point form :** Equation of tangent to the given ellipse at its point (x_1, y_1) is $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$

Note : For general ellipse replace x^2 by (xx_1), y^2 by (yy_1), 2x by $(x + x_1)$, 2y by $(y + y_1)$, 2xy by $(xy_1 + yx_1)$ & c by (c).

(b) **Slope form :** Equation of tangent to the given ellipse whose slope is 'm', is $\mathbf{y = mx \pm \sqrt{a^2m^2+b^2}}$

Point of contact are $\left(\dfrac{\pm a^2 m}{\sqrt{a^2m^2+b^2}}, \dfrac{\mp b^2}{\sqrt{a^2m^2+b^2}}\right)$

Note that there are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.

(c) **Parametric form :** Equation of tangent to the given ellipse at its point $(a\cos\theta, b\sin\theta)$, is $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$

(i) There are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.These tangents touches the ellipse at extremities of a diameter.

(ii) Point of intersection of the tangents at the point α & β is, $\left(a\dfrac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\dfrac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

(iii) The eccentric angles of the points of contact of two parallel tangents differ by π.

Ex. Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Sol. Let m be the slope of the tangent, since the tangent is perpendicular to the line $y + 2x = 4$.

$\therefore \quad mx - 2 = -1 \quad \Rightarrow \quad m = \dfrac{1}{2}$

Since $3x^2 + 4y^2 = 12$ or $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

Comparing this with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\therefore \quad a^2 = 4$ and $b^2 = 3$

So the equation of the tangent are $y = \dfrac{1}{2}x \pm \sqrt{4 \times \dfrac{1}{4} + 3}$

$\Rightarrow \quad y = \dfrac{1}{2}x \pm 2 \quad$ or $\quad x - 2y \pm 4 = 0.$

Ex. Find the locus of foot of perpendicular drawn from centre to any tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

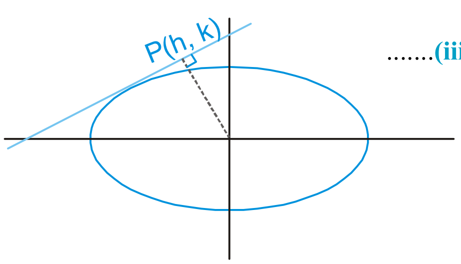
Sol. Let P(h, k) be the foot of perpendicular to a tangent $y = mx + \sqrt{a^2m^2+b^2}$ ....**(i)**

from centre

$\therefore \quad \dfrac{k}{h} . m = -1 \quad \Rightarrow \quad m = -\dfrac{h}{k}$ ....**(ii)**

$\because \quad$ P(h, k) lies on tangent

$\therefore \quad k = mh + \sqrt{a^2m^2+b^2}$ ....**(iii)**

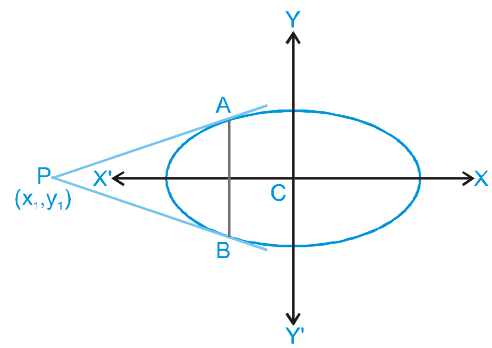


from equation **(ii)** & **(iii)**, we get

$$\left(k+\frac{h^2}{k}\right)^2 = \frac{a^2h^2}{k^2} + b^2$$

$\Rightarrow$ locus is $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$

PAIR OF TANGENTS



If $P(x_1,y_1)$ be any point lies outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

and a pair of tangents PA, PB can be drawn to it from P.

Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$

where $S_1 = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$, $T = \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

i.e. $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right) = \left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

Ex. How many real tangents can be drawn from the point (4, 3) to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. Find the equation of these tangents & angle between them.

Sol. Given point $P \equiv (4, 3)$

ellipse $S \equiv \frac{x^2}{16}+\frac{y^2}{9}-1=0$

$\because$ $S_1 \equiv \frac{16}{16}+\frac{9}{9}-1 = 1 > 0$

$\Rightarrow$ Point $P \equiv (4, 3)$ lies outside the ellipse.

$\therefore$ Two tangents can be drawn from the point P(4, 3).

Equation of pair of tangents is $SS_1 = T^2$

$$\Rightarrow \left(\frac{x^2}{16}+\frac{y^2}{9}-1\right).1 = \left(\frac{4x}{16}+\frac{3y}{9}-1\right)^2$$

$$\Rightarrow \frac{x^2}{16}+\frac{y^2}{9}-1 = \frac{x^2}{16}+\frac{y^2}{9}+1+\frac{xy}{6}-\frac{x}{2}-\frac{2y}{3}$$

$\Rightarrow$ $-xy + 3x + 4y - 12 = 0$ $\Rightarrow$ $(4-x)(y-3) = 0$

$\Rightarrow$ $x = 4$ & $y = 3$

and angle between them $= \frac{\pi}{2}$

NORMAL TO THE ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

(a) **Point form :** Equation of the normal to the given ellipse at (x_1, y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2$.

(b) **Slope form :** Equation of a normal to the given ellipse whose slope is 'm' is $y = mx \mp \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$.

(c) **Parametric form :** Equation of the normal to the given ellipse at the point (acosθ, bsinθ) is $ax\sec\theta - by\text{cosec}\theta = (a^2 - b^2)$.

Ex. Find the condition that the line $\ell x + my = n$ may be a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. Equation of normal to the given ellipse at (a cos θ, b sin θ) is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$ ......**(i)**

If the line $\ell x + my = n$ is also normal to the ellipse then there must be a value of θ for which line **(i)** and line $\ell x + my = n$ are identical. For that value of θ we have

$$\frac{\ell}{\left(\frac{a}{\cos\theta}\right)} = \frac{m}{-\left(\frac{b}{\sin\theta}\right)} = \frac{n}{(a^2 - b^2)} \quad \text{or} \quad \cos\theta = \frac{an}{\ell(a^2 - b^2)} \qquad \text{......(iii)}$$

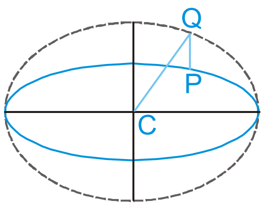
and $\sin\theta = \frac{-bn}{m(a^2 - b^2)}$ ...... **(iv)**

Squaring and adding **(iii)** and **(iv)**, we get $1 = \frac{n^2}{(a^2 - b^2)^2}\left(\frac{a^2}{\ell^2} + \frac{b^2}{m^2}\right)$ which is the required condition.

Ex. P and Q are corresponding points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the auxiliary circles respectively. The normal at P to the ellipse meets CQ in R, where C is the centre of the ellipse. Prove that CR = a + b

Sol. Let $P \equiv (a\cos\theta, b\sin\theta)$

∴ $Q \equiv (a\cos\theta, a\sin\theta)$



Equation of normal at P is

$(a\sec\theta)x - (b\,\text{cosec}\,\theta)y = a^2 - b^2$ ....**(i)**

equation of CQ is $y = \tan\theta \cdot x$ ....**(ii)**

Solving equation **(i)** & **(ii)**, we get $(a - b)x = (a^2 - b^2)\cos\theta$

$x = (a + b)\cos\theta$, & $y = (a + b)\sin\theta$

∴ $R \equiv ((a + b)\cos\theta, (a + b)\sin\theta)$

∴ CR = a + b

Ex. Find the shortest distance between the line x + y = 10 and the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

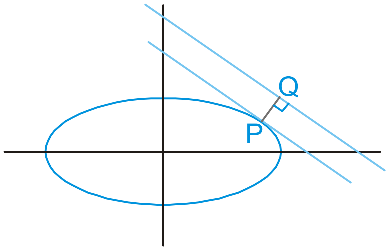
Sol. Shortest distance occurs between two non-intersecting curve always along common normal.
Let 'P' be a point on ellipse and Q is a point on given line for which PQ is common normal.

∴ Tangent at 'P' is parallel to given line

∴ Equation of tangent parallel to given line is ($y = mx \pm \sqrt{a^2m^2 + b^2}$)

$y = -x \pm 5$

⇒ $x + y + 5 = 0$ or $x + y - 5 = 0$



$\therefore$ minimum distance = distance between

$x + y - 10 = 0$ & $x + y - 5 = 0$

$\Rightarrow$ shortest distance $= \dfrac{|10-5|}{\sqrt{1+1}} = \dfrac{5}{\sqrt{2}}$

CHORD OF CONTACT

If PA and PB be the tangents from point $P(x_1,y_1)$ to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The equation of the chord of contact AB is $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$ **or** $T = 0$ (at x_1,y_1).

Ex. If tangents to the parabola $y^2 = 4ax$ intersect the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at A and B, the find the locus of point of intersection of tangents at A and B.

Sol. Let $P \equiv (h, k)$ be the point of intersection of tangents at A & B

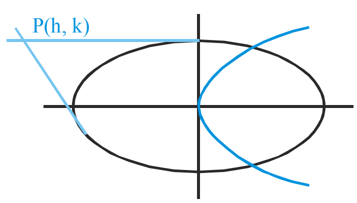
$\therefore$ equation of chord of contact AB is $\dfrac{xh}{a^2}+\dfrac{yk}{b^2}=1$ .... **(i)**

which touches the parabola.

Equation of tangent to parabola $y^2 = 4ax$ is $y = mx + \dfrac{a}{m}$

$\Rightarrow$ $mx - y = -\dfrac{a}{m}$ .... **(ii)**

equation **(i)** & **(ii)** as must be same



$\therefore$ $\dfrac{m}{\left(\dfrac{h}{a^2}\right)}=\dfrac{-1}{\left(\dfrac{k}{b^2}\right)}=\dfrac{-\dfrac{a}{m}}{1}$ $\Rightarrow$ $m = -\dfrac{h}{k}\dfrac{b^2}{a^2}$ & $m = \dfrac{ak}{b^2}$

$\therefore$ $-\dfrac{hb^2}{ka^2}=\dfrac{ak}{b^2}$ $\Rightarrow$ locus of P is $y^2 = -\dfrac{b^4}{a^3}.x$

EQUATION OF CHORD WITH MID POINT (x_1,y_1)

The equation of the chord of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, whose mid-point be (x_1,y_1) is $T = S_1$

where $T = \dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}-1$, $S_1 = \dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-1$,

i.e. $\left(\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}-1\right)=\left(\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-1\right)$

Ex. Find the locus of the mid-point of focal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Sol. Let $P \equiv (h, k)$ be the mid-point

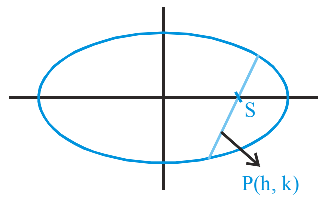
$\therefore$ equation of chord whose mid-point is given $\frac{xh}{a^2}+\frac{yk}{b^2}-1=\frac{h^2}{a^2}+\frac{k^2}{b^2}-1$

since it is a focal chord,

$\therefore$ It passes through focus, either (ae, 0) or (–ae, 0)

If it passes through (ae, 0)

$\therefore$ locus is $\frac{ex}{a}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$



If it passes through (–ae, 0)

$\therefore$ locus is $-\frac{ex}{a}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

DIRECTOR CIRCLE

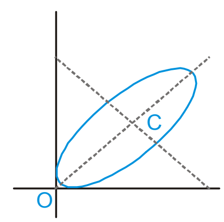
Locus of the point of intersection of the tangents which meet at right angles is called the Director Circle. The equation to this locus is $x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axes.

Ex. An ellipse slides between two perpendicular lines. Show that the locus of its centre is a circle.

Sol. Let length of semi-major and semi-minor axis are 'a' and 'b' and centre is $C \equiv (h, k)$

Since ellipse slides between two perpendicular lines, therefore point of intersection of two perpendicular tangents lies on director circle.

Let us consider two perpendicular lines as x & y axes



$\therefore$ point of intersection is origin $O \equiv (0, 0)$

$\therefore$ OC = radius of director circle

$\therefore$ $\sqrt{h^2+k^2} = \sqrt{a^2+b^2}$

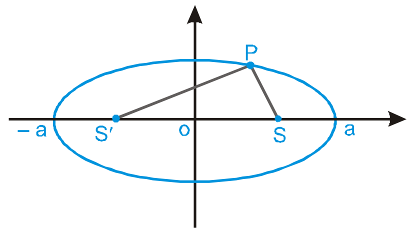
$\Rightarrow$ locus of $C \equiv (h, k)$ is

$\Rightarrow$ $x^2 + y^2 = a^2 + b^2$ which is a circle

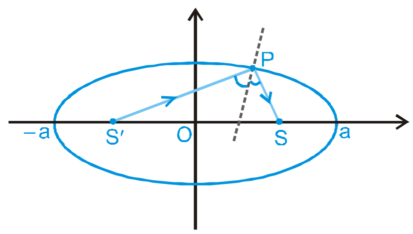
IMPORTANT CONCEPTS

Refering to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

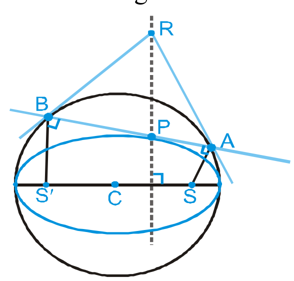
(1) If P be any point on the ellipse with S & S′ as its foci then $\ell(SP) + \ell(S'P) = 2a$.



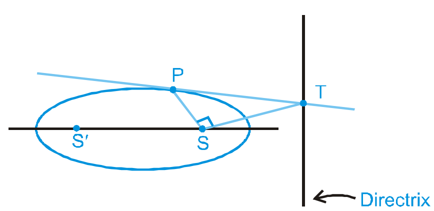
(2) The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice–versa. Hence we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.



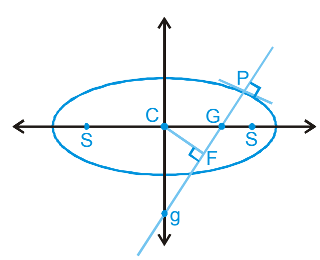
(3) The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b^2 and the feet of these perpendiculars lie on its auxiliary circle and the tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of intersection is a similiar ellipse as that of the original one.



(4) The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.



(5) If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be perpendicular upon this normal, then



(i) PF. PG = b^2

(ii) PF. Pg = a^2

(iii) PG. Pg = SP. S′P

(iv) CG. CT = CS^2

(v) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse.

[where S and S′ are the focii of the ellipse and T is the point where tangent at P meet the major axis]

(6) The circle on any focal distance as diameter touches the auxiliary circle. Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.

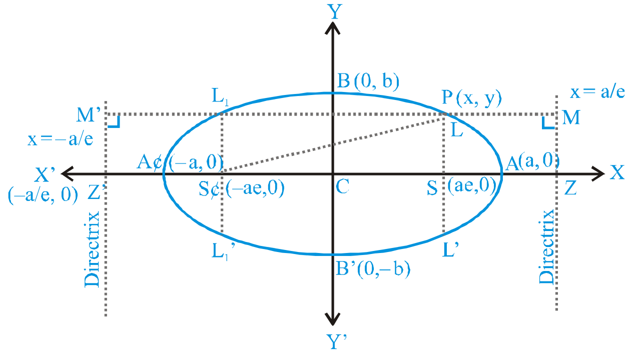
TIPS & FORMULAS

1. Standard Equation & Definition

Standard equation of an ellipse referred to its principal axis along the co–ordinate axis is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a > b$ & $b^2 = a^2(1-e^2)$.

$\Rightarrow \quad a^2 - b^2 = a^2 e^2$.

where e = Eccentricity $(0 < e < 1)$



FOCI : $S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

(a) **Equations of Directrices :** $x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

(b) **Vertices :** $A' \equiv (-a, 0)$ & $A \equiv (a, 0)$.

(c) **Major Axis :** The line segment A′A in which the foci S′ & S lie is of length 2a & is called the **major axis** (a > b) of the ellipse. Point of intersection of major axis with directrix is called **the foot of the directrix (Z)** $\left(\pm\frac{a}{e}, 0\right)$.

(d) **Minor Axis :** The y–axis intersects the ellipse in the points $B' \equiv (0, -b)$ & $B \equiv (0, b)$. The line segment B′B is of length 2b (b < a) is called the Minor Axis of the ellipse.

(e) **Principal Axis :** The major & minor axis together are called Principal Axis of the ellipse.

(f) **Centre :** The point which bisects every chord of the conic drawn through it, is called the centre of the conic. $C \equiv (0, 0)$ the origin is the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(g) **Diameter :** A chord of the conic which passes through the centre is called a diameter of the conic.

(h) **Focal Chord :** A chord which passes through a focus is called a focal chord.

(i) **Double Ordinate :** A chord perpendicular to the major axis is called a double ordinate with respect to major axis as diameter.

(j) **Latus Rectum :** The focal chord perpendicular to the major axis is called the latus rectum.

(i) Length of latus rectum (LL′) $= \frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1-e^2)$

(ii) Equation of latus rectum : $x = \pm ae$.

(iii) Ends of the latus rectum are $L\left(ae,\frac{b^2}{a}\right)$, $L'\left(ae,-\frac{b^2}{a}\right)$, $L_1\left(-ae,\frac{b^2}{a}\right)$ and $L_1'\left(-ae,-\frac{b^2}{a}\right)$

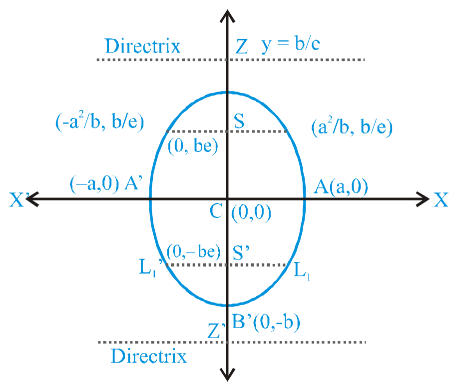
(k) **Focal radii :** SP = a – ex and S'P = a + ex

$\Rightarrow$ SP + S'P = 2a = Major axis

(l) **Eccentricity :** $e=\sqrt{1-\frac{b^2}{a^2}}$

2. Another form of Ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ (a<b)$$



(a) AA' = Minor axis = 2a

(b) BB' = Major axis = 2b

(c) $a^2 = b^2(1-e^2)$

(d) latus rectum

$$LL'=L_1L_1'=\frac{2a^2}{b}.$$

equation $y=\pm be$

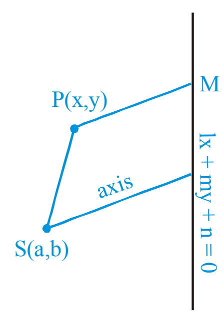
(e) Ends of the latus rectum are :

$$L\left(\frac{a^2}{b},be\right),\ L'\left(-\frac{a^2}{b},be\right),\ L_1\left(\frac{a^2}{b},-be\right),\ L_1'\left(-\frac{a^2}{b},-be\right)$$

(f) Equation of directrix $y=\pm\frac{b}{e}$. (g) Eccentricity : $e=\sqrt{1-\frac{a^2}{b^2}}$

3. General Equation of an Ellipse

Let (a,b) be the focus S, and lx + my + n = 0 is the equation of directrix. Let P (x, y) be any point on the ellipse. Then by definition.



$\Rightarrow$ SP = e PM (e is the eccentricity)

$$\Rightarrow \quad (x-a)^2+(y-b)^2=e^2\frac{(lx+my+n)^2}{(l^2+m^2)}$$

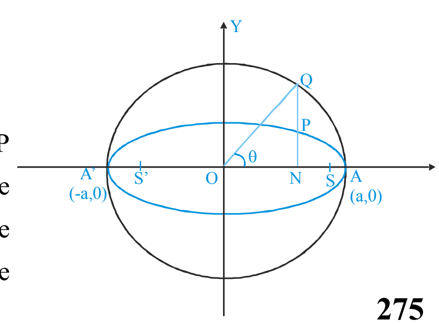
$$\Rightarrow \quad (l^2+m^2)\{(x-a)^2+(y-b)^2\}=e^2\{lx+my+n\}^2$$

4. Position of a Point W.R.T. an Ellipse

The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as :

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 >< \text{or} = 0$$

5. Auxilliary Circle/eccentric Angle



A circle described on major axis as diameter is called the auxilliary circle. Let Q be a point on the auxilliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P & Q are called as the CORRESPONDING POINTS on the ellipse & the auxilliary circle respectively. 'q' is called ECCENTRIC ANGLE of the point P on the ellipse $(0 \le 0 < 2\pi)$.

Note that $\frac{l(PN)}{l(QN)} = \frac{b}{a} = \frac{\text{Semi minor axis}}{\text{Semi major axis}}$

Hence "If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxilliary circle."

6. Parametric Representation

The equations x = a cos θ & y = b sin θ together represent the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where θ is a parameter (eccentric angle).

Note that if P(θ) = (a cos θ, b sin θ) is on the ellipse then ;

Q(θ) = (a cos θ, a sin θ) is on the auxilliary circle.

7. Line and an Ellipse

The line y = mx + c meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two real points, coincident or imaginary according as c^2 is

< = or > $a^2m^2 + b^2$.

Hence y = mx + c is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}.$$

8. Tangent to the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:

(a) Point form :

Equation of tangent to the given ellipse at its point (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

(b) Slope form :

Equation of tangent to the given ellipse whose slope is 'm', is $y = mx \pm \sqrt{a^2m^2 + b^2}$

Point of contact are $\left(\frac{\pm a^2m}{\sqrt{a^2m^2+b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2+b^2}}\right)$

(c) Parametric form :

Equation of tangent to given ellipse at its point (a cos θ, b sin θ), is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

9. Normal to the Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(a) Point form : Equation of the normal to the given ellipse at

(x_1, y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y}=a^2-b^2=a^2e^2$.

(b) Slope form : Equation of a normal to the given ellipse whose slope is 'm' is

$$y=mx\mp\frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$$

(c) Parametric form : Equation of the normal to the given ellipse at the point (a cosθ, b sinθ) is

ax . sec θ – by . cosec θ = $(a^2 – b^2)$.

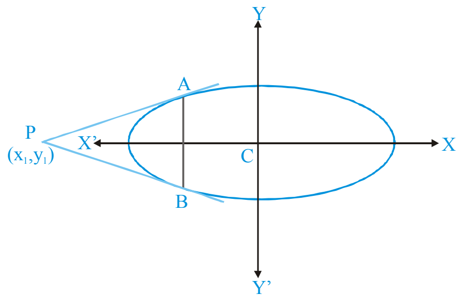
10. Chord of Contact

If PA and PB be the tangents from point $P(x_1,y_1)$ to the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then the equation of the chord of contact AB is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ or T = 0 at (x_1,y_1)

11. Pair of Tangents

If $P(x_1, y_1)$ be any points lies outside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and a pair of tangents PA, PB can be drawn to it from P.



Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$

where $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$, $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

i.e. $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$

12. Director Circle

Locus of the point of intersection of the tangents which meet at right angles is called the Director Circle. The equation to this locus is $x^2 + y^2 = a^2 + b^2$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axis.

13. Equation of Chord with Mid Point (x_1, y_1)

The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose mid-point be (x_1, y_1) is $T = S_1$

where $T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$, $S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

i.e. $\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)=\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)$