HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

1. Let point of intersection is (x_1, y_1).

So $\sqrt{3}\,x_1 - y_1 = 4\sqrt{3}\,K$ .....(i)

$\sqrt{3}\,K x_1 + K y_1 = 4\sqrt{3}$ .....(ii)

Multiply (i) and (ii), we get $3x_1^2 - y_1^2 = 48$.

3. Centre of hyperbola is (5, 0), so equation is

$$\frac{(x-5)^2}{a^2} - \frac{y^2}{b^2} = 1$$

$a = 5,\ ae - a = 8 \Rightarrow e = \frac{13}{5}$

$b^2 = 144.$

So equation is $\frac{(x-5)^2}{25} - \frac{y^2}{144} = 1.$

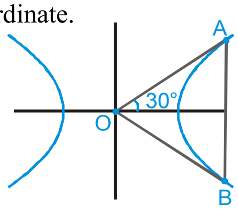
4. The equation of the hyperbola is

$$\frac{\{(2x-y+4)/\sqrt{5}\}^2}{1/2} = \frac{\{(x+2y-3)/\sqrt{5}\}^2}{1/3}$$

or $\frac{2}{5}(2x-y+4)^2 - \frac{3}{5}(x+2y-3)^2 = 1$

5. Let ℓ be the length of double ordinate.
Co-ordinate of point A is

$$\left(\ell\frac{\sqrt{3}}{2}, \frac{\ell}{2}\right)$$



So . $\frac{3\ell^2}{4a^2} - \frac{\ell^2}{4b^2} = 1$

$\Rightarrow \frac{\ell^2}{4}\left(\frac{3}{a^2} - \frac{1}{b^2}\right) = 1 \Rightarrow \frac{3}{a^2} > \frac{1}{b^2}$

$\Rightarrow \frac{b^2}{a^2} > \frac{1}{3} \Rightarrow e^2 - 1 > \frac{1}{3}$

$\Rightarrow e^2 > \frac{4}{3}$

7. Equation of tangents to two hyperbolas are

$y = mx \pm \sqrt{a^2m^2 - b^2}$ .....(i)

$y = mx \pm \sqrt{-b^2m^2 + a^2}$ .....(ii)

Solving (i) & (ii) we get $m = \pm 1$

$\therefore$ equation of common tangent is

$y = \pm x \pm \sqrt{a^2 - b^2}$.

8. Let $y = mx \pm \sqrt{m^2a^2 - a^2}$ be two tangents that pass through (h, k). Then,

$(k - mh)^2 = m^2a^2 - a^2$

or $m^2(h^2 - a^2) - 2khm + k^2 + a^2 = 0$

or $m_1 + m_2 = \frac{2kh}{h^2 - a^2}$

and $m_1m_2 = \frac{k^2 + a^2}{h^2 - a^2}$

Now, $\tan 45° = \frac{m_1 - m_2}{1 + m_1m_2}$

or $1 = \frac{(m_1 + m_2)^2 - 4m_1m_2}{(1 + m_1m_2)^2}$

or $\left(1 + \frac{k^2 + a^2}{h^2 - a^2}\right)^2 = \left(\frac{2kh}{h^2 - a^2}\right)^2 - 4\left(\frac{k^2 + a^2}{h^2 - a^2}\right)$

or $(h^2 + k^2)^2 = 4h^2k^2 - 4(k^2 + a^2)(h^2 - a^2)$

or $(x^2 + y^2)^2 = 4(a^2y^2 - a^2x^2 + a^4)$

or $(x^2 + y^2)^2 + 4a^2(x^2 - y^2) = 4a^2$

9. Let the slope of common tangent be m.
Equation of tangent to parabola is

$y = mx + \frac{2}{m}$ .....(i)

Equation of tangent to hyperbola is

$y = mx \pm \sqrt{m^2 - 3}$ .....(ii)

By comparing (i) & (ii), we get $m = \pm 2$.

$\therefore$ Equation of common tangent is $y = \pm(2x + 1)$

i.e. $2x \pm y + 1 = 0$.

11. Let equation of asymptotes be $xy - 3x - 2y + \lambda = 0$.

Then $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow \frac{3}{2} - \frac{\lambda}{4} = 0$

$\Rightarrow \lambda = 6$

$\therefore$ Equation of asymptotes is $xy - 3x - 2y + 6 = 0$

i.e., $(x - 2)(y - 3) = 0$.

13. Equation of normal of rectangular hyperbola $xy = c^2$ at $P(ct,\ c/t)$ will be

$$y - \frac{c}{t} = t^2(x - ct)$$

as it also passes through t_1

$$\Rightarrow \quad c\left(\frac{1}{t_1} - \frac{1}{t}\right) = ct^2(t_1 - t)$$

$$\Rightarrow \quad t^3 t_1 = -1$$

15. Normal at θ, ϕ are

$$\begin{cases} ax\cos\theta + by\cot\theta = a^2 + b^2 \\ ax\cos\phi + by\cot\phi = a^2 + b^2 \end{cases}$$

where $\phi = \frac{\pi}{2} - \theta$ and these passes through (h, k).

$\therefore \quad ah\cos\theta + bk\cot\theta = a^2 + b^2 \qquad$ ......(i)

$\quad ah\sin\theta + bk\tan\theta = a^2 + b^2 \qquad$ ......(ii)

Multiply (i) by $\sin\theta$ & (ii) by $\cos\theta$ & subtract them, we get

$\Rightarrow \quad (bk + a^2 + b^2)(\sin\theta - \cos\theta) = 0$

$k = -(a^2 + b^2)/b$

18. The equation of tangent at point $P(\alpha\cos\theta, \sin\theta)$ is

$$\frac{x}{\alpha}\cos\theta + \frac{y}{1}\sin\theta = 1$$

Let it cut the hyperbola at points P and Q.

Homogenizigin the hyperbola $\alpha^2x^2 - y^2 = 1$ with the help of the above the equation, we get

$$\alpha^2x^2 - y^2 = \left(\frac{x}{\alpha}\cos\theta + y\sin\theta\right)^2$$

This is a pair of straight lines OP and OQ.

Given $\angle POQ = \pi/2$.

Coefficient of x^2 + Coefficient of $y^2 = 0$

or $\quad \alpha^2 - \frac{\cos^2\theta}{\alpha^2} - 1 - \sin^2\theta = 0$

or $\quad \alpha^2 - \frac{\cos^2\theta}{\alpha^2} - 1 - 1 + \cos^2\theta = 0$

or $\quad \cos^2\theta = \frac{\alpha^2(2-\alpha^2)}{\alpha^2 - 1}$

Now, $0 \le \cos^2\theta \le 1$

or $\quad 0 \le \frac{\alpha^2(2-\alpha^2)}{\alpha^2-1} \le 1$

Solving, we get $\alpha^2 \in \left[\frac{\sqrt{5}+1}{2}, 2\right]$

EXERCISE - 2

Part # I : Multiple Choice

2. We have,

$$\left|\sqrt{x^2 + (y-1)^2} - \sqrt{x^2 + (y+1)^2}\right| = K$$

which is equivalent to $|S_1P - S_2P|$ = constant, where

$S_1 \equiv (0, 1)$, $S_2 \equiv (0, -1)$, and $P \equiv (x, y)$

The above equation represents a hyperbola. So, we have

$\quad 2a = K$

and $2ae = S_1S_2 = 2$

where 2a is the transverse axis and e is the eccentricity.

Dividing, we have

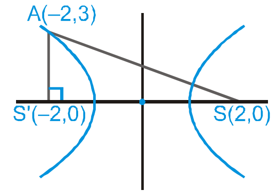
$$e = \frac{2}{K}$$

Since $e > 1$ for a hyperbola, $K < 2$.

Also, K must be a positive quantity.

So, we have $K \in (0, 2)$

4. $S \equiv (2, 0)$, $S' \equiv (-2, 0)$



Using reflection property of hyperbola,

$\quad$ S'A is incident ray.

Equation of incident ray

$\quad$ S'A is $x = -2$

Equation of reflected ray

$\quad$ SP is $3x + 4y = 6$.

Now $2ae = 4 \quad \Rightarrow \quad ae = 2 \qquad$ ......(i)

Point (–2, 3) lies on hyperbola,

$$\therefore \quad \frac{4}{a^2} - \frac{9}{b^2} = 1 \quad \Rightarrow \quad \frac{4}{a^2} - \frac{9}{4-a^2} = 1$$

on solving it we get $a = 4$ (reject), $a = 1$ .....(ii)

$\therefore \quad$ Using (i) & (ii), we get $e = 2$

length of latus rectum $= 2a(e^2 - 1) = 6$

6. Let A(5, 12) and B(24, 7) be two fixed points.

So, |OA – OB| = 12 and |OA + OB| = 38.

It the conic is an ellipse, then

$$e = \frac{\sqrt{386}}{38} \quad (\because 2ea = \sqrt{386} \text{ and } a = 19)$$

If the conic is a hyperbola, then

$$e = \frac{\sqrt{386}}{12} \quad (\because 2ae = \sqrt{386} \text{ and } a = 6)$$

8. $\tan\frac{\theta}{2} = \frac{b}{a} \Rightarrow e^2 - 1 = \tan^2\frac{\theta}{2} \Rightarrow \sec\frac{\theta}{2} = e$

or $e^2 - 1 = \cot^2\frac{\theta}{2} \Rightarrow \operatorname{cosec}\frac{\theta}{2} = e$

$\Rightarrow \sec\frac{\theta}{2} = \frac{e}{\sqrt{e^2 - 1}}$.

11. The locus of the point of intersection of perpendicular tangents is director circle $x^2 + y^2 = a^2 - b^2$. Now,

$$e^2 = 1 + \frac{b^2}{a^2}$$

If $a^2 > b^2$, the there are infinite (or more than 1) points on the circle, i.e., $e^2 < 2$ or $e < \sqrt{2}$.

If $a^2 < b^2$, there does not exist any point on the plane, i.e., $e^2 > 2$ or $e > \sqrt{2}$

If $a^2 = b^2$, there is exactly one point (center of the hyperbola),

i.e., $e = \sqrt{2}$.

12. Given equation will represent hyperbola if

$\lambda^2 > (\lambda + 2)(\lambda - 1) \quad [\therefore\ h^2 > ab]$

$\Rightarrow \lambda < 2$

Also $\Delta \neq 0$

$\Rightarrow -2(\lambda^2 + \lambda - 2) - 4(\lambda - 1) + 2\lambda^2 \neq 0$

$\Rightarrow \lambda \neq \frac{4}{3}$.

Part # II : Assertion & Reason

3. Let equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Hyperbola xy = 4 cut the circle at four points then

$$x^2 + \frac{16}{x^2} + 2gx + \frac{8f}{x} + c = 0$$

$$x^4 + 2gx^3 + cx^2 + 8fx + 16 = 0$$

$\Rightarrow x_1 x_2 x_3 x_4 = 16$

$\Rightarrow 2.4.6.1/4 = 12$

$\Rightarrow$ statement I is false

statement II is true.

4. We have

$$\sqrt{(\lambda - 3)^2 + 16} - 4 = 1$$

i.e., $\lambda = 0$ or 6

EXERCISE - 3

Part # I : Matrix Match Type

1. **a → p, s ; b → q, r ; c → r ; d → p, s**

a. We must have

$e_1 < 1 < e_2$

or $f(1) < 0$

or $1 - a + 2 < 0$

or $a > 3$

b. We must have both the roots greater than 1.

$D > 0$ or $a^2 - 4 > 0$ or $a \in (-\infty, -2) \cup (2, \infty)$ ....(i)

$1 . f(1) > 0$ or $1 - a + 2 > 0$ or $a < 3$ ....(ii)

$a/b \geq 1$ or $a > 2$ ....(iii)

From (i), (ii) and (iii), we have $a \in (2, 3)$

c. We must have

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$$

or $$\frac{(e_1 + e_2)^2 - 2e_1e_2}{e_1^2 e_2^2} = 1$$

or $$\frac{a^2 - 4}{4} = 1$$

or $a = \pm 2\sqrt{2}$

d. We must have

$e_2 < \sqrt{2} < e_2$

or $f(\sqrt{2}) < 0$

or $2 - a\sqrt{2} + 2 < 0$

or $a > 2\sqrt{2}$

4. **(A)** Tangent to the given hyperbola at $P\left(\frac{\pi}{6}\right)$ is

$$\frac{2x}{\sqrt{3}a} - \frac{1}{\sqrt{3}}\frac{y}{b} = 1 \Rightarrow 2xb - ya = \sqrt{3}ab$$

It cuts x-axis at $\left(\frac{\sqrt{3}a}{2}, 0\right)$ & y-axis at $\left(0, -\sqrt{3}b\right)$

$\therefore$ area of triangle $= \frac{3}{4}ab$

$\Rightarrow 3a^2 = \frac{3}{4}ab \Rightarrow \frac{b}{a} = 4$

$\therefore e^2 = 17.$

(B) $e_1^2 = 1 + \frac{5\cos^2\theta}{5}$ & $e_2^2 = 1 - \frac{25\cos^2\theta}{25}$

According to question $e_1^2 = 3e_2^2$,

$1 + \cos^2\theta = 3 - 3\cos^2\theta \Rightarrow \cos^2\theta = \frac{1}{2}$

Smallest possible value of $\theta = \frac{\pi}{4}$.

Hence $p = 24$.

(C) Angle between asymptotes is

$$2\tan^{-1}\left(\pm\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

$\therefore \frac{\pi}{3} = \frac{\ell\pi}{24} \Rightarrow \ell = 8.$

or $\frac{2\pi}{3} = \frac{\ell\pi}{24} \Rightarrow \ell = 16.$

(D) Equation of tangents on hyperbola at $P(x_1, y_1)$ is

$xy_1 + yx_1 = 16$

$\therefore$ It cuts the co-ordinate axes at

$A\left(\frac{16}{y_1}, 0\right)$ & $B\left(0, \frac{16}{x_1}\right)$

$\therefore \Delta = 16.$ $(\because x_1y_1 = 8)$

Part # II : Comprehension

Comprehension # 2

1. Tangent of $xy = c^2$ at t_1 & t_2 are

$x + t_1^2 y = 2ct_1$ .....(i)

and $x + t_2^2 y = 2ct_2$ .....(ii)

on solving (i) & (ii) we get

$$y = \frac{2c}{t_1 + t_2} = \frac{2c}{4}, x = \frac{2ct_1t_2}{t_1 + t_2} = \frac{4c}{4}$$

$\therefore$ point of intersection is $\left(c, \frac{c}{2}\right)$.

2. $e_1 = \sqrt{2}$, $e_2 = \sqrt{2}$

$\Rightarrow$ $(\sqrt{2}, \sqrt{2})$ is the point on the circle.

$\Rightarrow$ radius of $C_1 = 2$.

$\Rightarrow$ radius of director circle of $C_1 = 2\sqrt{2}$.

$\therefore$ $(\text{radius})^2 = 8$

3. Equation of normal of $xy = c^2$ at t_1 is

$$y - \frac{c}{t_1} = t_1^2 (x - ct_1)$$

As it also passes through t_2,

$$\frac{c}{t_2} - \frac{c}{t_1} = t_1^2 (ct_2 - ct_1)$$

$\Rightarrow$ $t_1 t_2 = -t_1^{-2}$.

Comprehension # 3

1. (b), 2. (c) 3. (d)

1. **(b)** Perpendicular tangents intersect at the center of rectangular hyperbola. Hence, the center of the hyperbola is (1, 1) and the equations of asymptotes are $x - 1 = 0$ and $y - 1 = 0$.

2. **(c)** Let the equation of the hyperbola be

$xy - x - y + 1 + \lambda = 0$

It passes through (3, 2). Hence, $\lambda = -2$.

So, the equation of hyperbola is

$xy = x + y + 1$

3. **(d)** From the center of the hyperbola, we can draw two real tangents to the rectangular hyperbola.

EXERCISE - 4

Subjective Type

1. Point of intersection of lines

$7x + 13y - 87 = 0$ & $5x - 8y + 7 = 0$ is (5, 4).

Then $\frac{25}{a^2} - \frac{16}{b^2} = 1$ .....(i)

Also latus rectum $LR = \frac{2b^2}{a} = \frac{32\sqrt{2}}{5}$

$\Rightarrow$ $b^2 = \frac{16\sqrt{2}a}{5}$ .....(ii)

From (i) & (ii) $a^2 = \frac{25}{2}$, $b^2 = 16$.

2. Equation of tangent of given hyperbola at point

(h, k) is $\frac{hx}{a^2} - \frac{ky}{b^2} = 1$ .....(i)

Equation of auxillary circle is $x^2 + y^2 = a^2$ .....(ii)

from (i) & (ii)

$$\left[\left(1 + \frac{ky}{b^2}\right)\frac{a^2}{h}\right]^2 + y^2 - a^2 = 0$$

$\Rightarrow$ $y^2 (k^2a^4 + b^4h^2) + 2kb^2a^4y + b^4a^2 (a^2 - h^2) = 0$

Now $\frac{y_1 + y_2}{y_1 y_2} = -\frac{2kb^2a^4}{b^4a^2(a^2 - h^2)} = \frac{-2ka^2}{b^2a^2\left(1 - \frac{h^2}{a^2}\right)}$

$$= \frac{-2k}{b^2\left(\frac{-k^2}{b^2}\right)} = \frac{2}{k}.$$

3. Given hyperbola can be written as

$$\frac{(x + 1)^2}{9} - \frac{(y - 2)^2}{16} = 1$$

so $e = \frac{5}{3}$, centre is (–1, 2)

foci = (–1 ± 5, 2) = (–6, 2) & (4, 2)

directrix is $x + 1 = \pm\frac{9}{5} \Rightarrow x = -1 \pm \frac{9}{5}$

$L.R. = \frac{32}{3}$, Length of axes is 8 and 6,

Equation of axis is $y - 2 = 0$ and $x + 1 = 0$.

4. Let mid point of chord of given hyperbola is (h, k)

Also let $\left(ct_1, \frac{c}{t_1}\right)$ & $\left(ct_2, \frac{c}{t_2}\right)$ be the end points of the chord

then $2h = c(t_1 + t_2)$ and $2k = c\left(\frac{1}{t_1} + \frac{1}{t_2}\right)$

According to question

$$c^2 (t_1 - t_2)^2 + c^2 \left(\frac{1}{t_1} - \frac{1}{t_2}\right)^2 = 4d^2$$

$$\Rightarrow \quad c^2[(t_1 + t_2)^2 - 4t_1t_2]\left[1 + \frac{1}{(t_1t_2)^2}\right] = 4d^2$$

$$\Rightarrow \quad c^2\left[\frac{4h^2}{c^2} - \frac{4h}{k}\right]\left[1 + \frac{k^2}{h^2}\right] = 4d^2$$

$$\Rightarrow \quad (xy - c^2)(x^2 + y^2) = d^2 xy.$$

5. Given conic can be written as

$$\frac{(x-2)^2}{16} - \frac{(y-2)^2}{16} = -1$$

so eccentricity is $\sqrt{2}$.

7. Equation of normal of given hyperbola at P is

$ax \cos\theta + by \cot\theta = a^2 + b^2$

As it cut x-axis at G, so G $(ae^2 \sec\theta, 0)$

Now $SG = ae^2 \sec\theta - ae$

$= e(ae \sec\theta - a) = e\, SP$

8. Let any point on circle be $(r\cos\theta, r\sin\theta)$

Then equation of chord of contact is

$$\frac{x}{a^2} r\cos\theta - \frac{y}{b^2} r\sin\theta = 1 \qquad .....(i)$$

Let mid point of chord of contact is (h, k)

Then equation of chord of contact is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2} \qquad .....(ii)$$

On comparing (i) & (ii)

$$\frac{r\cos\theta}{h} = \frac{r\sin\theta}{k} = \frac{1}{\frac{h^2}{a^2} - \frac{k^2}{b^2}}$$

On solving we get required locus i.e.

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 = \frac{x^2 + y^2}{r^2}.$$

9. If (h, k) be mid point of any chord of hyperbola $x^2 - y^2 = a^2$, then its equation is

$hx - ky = h^2 - k^2 \qquad .....(i)$

But (i) is normal to hyperbola, then its equation is

$x\cos\theta + y\cot\theta = 2a \qquad .....(ii)$

Comparing (i) & (ii)

$$\frac{h}{\cos\theta} = \frac{-k}{\cot\theta} = \frac{h^2 - k^2}{2a}$$

on solving it we get $(y^2 - x^2)^3 = 4a^2x^2y^2$

10. Equation of tangent to parabola $x^2 = 4ay$

is $\quad y - mx + am^2 = 0 \qquad .....(i)$

Let mid point of PQ is (x_1, y_1).

Then equation of PQ is

$xy_1 + yx_1 = 2k^2 \qquad .....(ii)$

On comparing (i) & (ii)

$$\frac{x_1}{1} = \frac{y_1}{-m} = \frac{2k^2}{am^2}$$

$$\Rightarrow \quad x_1 = \frac{2k^2}{am^2} \qquad .....(iii)$$

$$y_1 = \frac{-2k^2}{am} \qquad .....(iv)$$

using (iii) & (iv) eliminate m.

11. Let equation of asymptotes are

$2x^2 - 3xy - 2y^2 + 3x - y + 8 + \lambda = 0$

As it represents two straight lines

$$\therefore \quad -4(8+\lambda) + \frac{9}{4} - \frac{1}{2} + \frac{9}{2} - (8+\lambda)\frac{9}{4} = 0$$

$$\Rightarrow \quad \lambda = -7$$

So asymptotes are $2x^2 - 3xy - 2y^2 + 3x - y + 1 = 0$

$\Rightarrow$ $2y - x - 1 = 0$ & $2x + y + 1 = 0$

and the equation of conjugate hyperbola will be

$$2x^2 - 3xy - 2y^2 + 3x - y + 8 - 14 = 0.$$

12. For the first hyperbola,

$$(y - mx)\left(m\frac{dy}{dx} + 1\right) + (my + x)\left(\frac{dy}{dx} - m\right) = 0$$

or $\dfrac{dy}{dx} = \dfrac{-y + m^2y + 2mx}{2my + x - m^2x} = m_1$

For the second hyperbola,

$$(m^2 - 1)\left(2y\frac{dy}{dx} - 2x\right) + 4m\left(x\frac{dy}{dx} + y\right) = 0$$

or $\dfrac{dy}{dx} = \dfrac{-2my + m^2x - x}{m^2y - y + 2mx} = m_2$

$\therefore$ $m_1m_2 = -1$

The angle between the hyperbolas is $\pi/2$.

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

5. $2ae = 4$

$ae = 2$

$a(2) = 2$

$a = 1$

$b^2 = a^2(e^2 - 1)$

$= 1(4 - 1) = 3$

equation $\dfrac{x^2}{1} - \dfrac{y^2}{3} = 1$

$3x^2 - y^2 = 3$

6. $\dfrac{2b^2}{a} = 8$

$2b = ae$

$4b^2 = a^2e^2$

$4a^2(e^2 - 1) = a^2e^2$

$3e^2 = 4$

$e = \dfrac{2}{\sqrt{3}}$

Part # II : IIT-JEE ADVANCED

1. Any point on $y^2 = 8x$ is $(2t^2, 4t)$ where the tangent is $yt = x + 2t^2$

Solving it with $xy = -1$, $y(yt - 2t^2) = -1$

or $ty^2 - 2t^2y + 1 = 0$

For common tangent, it should have equal roots

$\therefore$ $4t^2 - 4t = 0$

$\Rightarrow$ $t = 0, 1$

$\therefore$ The common tangent is $y = x + 2$,

(when $t = 0$, it is $x = 0$ which can touch $xy = -1$ at infinity only)

2. The given equation of hyperbola is

$$\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$$

$\Rightarrow$ $a = \cos\alpha$, $b = \sin\alpha$

$\Rightarrow$ $e = \sqrt{1 + \dfrac{b^2}{a^2}} = \sqrt{1 + \tan^2\alpha} = \sec\alpha$

$\Rightarrow$ ae = 1

$\therefore$ foci ($\pm 1, 0$)

$\therefore$ foci remain constant with respect to α.

5. Eccentricity of ellipse = 3/5

Eccentricity of hyperbola = 5/3 and it passes through (± 3, 0)

$\Rightarrow$ its equation $\frac{x^2}{9} - \frac{y^2}{b^2} = 1$

where $1 + \frac{b^2}{9} = \frac{25}{9}$

$\Rightarrow$ $b^2 = 16$

$\Rightarrow$ $\frac{x^2}{9} - \frac{y^2}{16} = 1$

$\Rightarrow$ $\frac{x^2}{9} - \frac{y^2}{16} = 1$

and its foci are (± 5, 0)

6. Given $3x^2 + 4y^2 = 12$ an ellipse

$\therefore$ $a^2 = 4$ $b^2 = 3$

$\therefore$ $e = \sqrt{1 - \frac{3}{4}}$

$\Rightarrow$ $e = \frac{1}{2}$

$\therefore$ It's focus will be (± 1, 0)

Since hyperbola is confocal to given ellipse, therefore $\pm ae = \pm 1$, but a = $\sin\theta$ given

$\therefore$ $e = \frac{1}{\sin\theta}$, Now $b^2 = a^2(e^2 - 1)$

$b^2 = \sin^2\theta \frac{\cos^2\theta}{\sin^2\theta}$ $\Rightarrow$ $b^2 = \cos^2\theta$

Hence required equation will be,

$\frac{x^2}{\sin^2\theta} - \frac{y^2}{\cos^2\theta} = 1$

$\Rightarrow$ $x^2\operatorname{cosec}^2\theta - y^2\sec^2\theta = 1$

8. $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$

either $x^2 - 5xy + 6y^2 = 0$

$\Rightarrow$ two straight lines passing through origin.

or $ax^2 + by^2 + c = 0$

(A) If c = 0, and a and b are of same sign then it will represent a point.

(B) If a = b, c is of sign opposite to a then it will represent circle.

(C) When a & b are of same sign and c is of sign opposite to a then it will represent ellipse.

(D) This is clearly incorrect.

9. The given equation is

$(x - \sqrt{2})^2 - 2(y + \sqrt{2})^2 = 4$

$\frac{(x-\sqrt{2})^2}{4} - \frac{(y+\sqrt{2})^2}{2} = 1$

$a = 2$, $b = \sqrt{2}$

hence eccentricity $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{\frac{3}{2}}$

Area $= \frac{1}{2} a(e - 1) \times \frac{b^2}{a}$

$= \left(\sqrt{\frac{3}{2}} - 1\right)$ sq. units.

10. $x^2 - y^2 = \frac{1}{2}$ .....**(i)** $\rightarrow$ its $e = \sqrt{2}$

e of ellipse is $\frac{1}{\sqrt{2}}$

$\frac{x^2}{2} + \frac{y^2}{1} = b^2$ .....**(ii)**

add **(i)** & **(ii)** $\frac{3x^2}{2} = \frac{1}{2} + b^2$

$3x^2 = 1 + 2b^2$

$y^2 = \frac{1}{3} + \frac{2h^2}{3} - \frac{1}{6}$

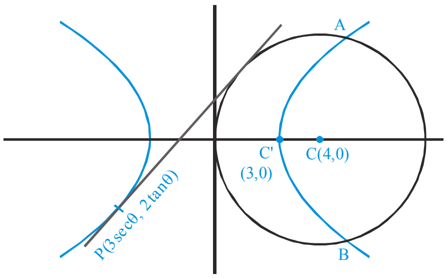
$y^2 = \frac{1}{6}(4b^2 - 1)$

$m_1 . m_2 = -1$ $\Rightarrow$ $\frac{1+2b^2}{3} = \frac{2(4b^2-1)}{6}$

$b^2 = 1$ $\Rightarrow$ $x^2 + 2y^2 = 2$.

Paragraph for Question 11 and 12

11. Let the point on the hyperbola P(3secθ, 2tanθ)



Equation of tangent $\frac{x\sec\theta}{3} - \frac{y\tan\theta}{2} = 1$

$|p| = r$

$$\frac{\left|\frac{4}{3}\sec\theta - 1\right|}{\sqrt{\frac{\sec^2\theta}{9} + \frac{\tan^2\theta}{4}}} = 4$$

$$\Rightarrow \frac{16}{9}\sec^2\theta + 1 - \frac{8}{3}\sec\theta = 16\left(\frac{4\sec^2\theta + 9\tan^2\theta}{4\times 9}\right)$$

$16\sec^2\theta + 9 - 24\sec\theta = 52\sec^2\theta - 36$

$\Rightarrow 36\sec^2\theta + 24\sec\theta - 45 = 0$

$\Rightarrow 12\sec^2\theta + 8\sec\theta - 15 = 0$

$\Rightarrow 12\sec^2\theta + 18\sec\theta - 10\sec\theta - 15 = 0$

$\Rightarrow (6\sec\theta - 5)(2\sec\theta + 3) = 0$

$\sec\theta = \frac{5}{6}$ (not possible), $\sec\theta = -\frac{3}{2}$

$$\tan\theta = \pm\sqrt{\frac{9}{4} - 1} = \pm\frac{\sqrt{5}}{2}$$

($\because$ slope is positive $\Rightarrow \tan\theta = -\frac{\sqrt{5}}{2}$)

Hence the required equation be $-\frac{3x}{2\times 3} + \frac{y\sqrt{5}}{2\times 2} = 1$

$\Rightarrow 2x - \sqrt{5}y + 4 = 0$

12. Solving (a) & (b) for x, we get

$x = 6$

$y = \pm 2\sqrt{3}$

$(x - 6)^2 + y^2 - 12 = 0$

$x^2 + y^2 - 12x + 24 = 0$

Option (A) is correct

13. As directrix cut the x–axis at $(\pm a/e, 0)$

Hence, $\frac{2a}{e} + 0 = 1$ (for nearer directrix)

$\Rightarrow 2a = e$ .....(i)

Now, $b^2 = a^2(e^2 - 1) = a^2(4a^2 - 1)$

$\Rightarrow \frac{b^2}{a^2} = 4a^2 - 1$ .....(ii)

Given line $y = -2x + 1$ is a tangent to the hyperbola condition of tangency is $c^2 = a^2m^2 - b^2$

$\Rightarrow 1 = 4a^2 - b^2$

$\Rightarrow 4a^2 - 1 = b^2$ .....(iii)

from (ii) & (iii), $a^2 = 1$

$\Rightarrow$ from (ii), $b^2 = 3$

$\Rightarrow e = \sqrt{\frac{1+3}{1}} = 2$

14. Given hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

ellipse is $\frac{x^2}{2^2} + \frac{y^2}{1} = 1$

eccentricity of ellipse $= \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

eccentricity of hyperbola $= \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{\frac{4}{3}}$

$\Rightarrow \frac{b^2}{a^2} = \frac{1}{3}$ $\Rightarrow 3b^2 = a^2$ .....(i)

also hyperbola passes through foci of ellipse $(\pm\sqrt{3}, 0)$

$\frac{3}{a^2} = 1 \Rightarrow a^2 = 3$ .....(ii)

from (i) & (ii)

$b^2 = 1$

equation of hyperbola is $\frac{x^2}{3} - \frac{y^2}{1} = 1$

$\Rightarrow x^2 - 3y^2 = 3$

eccentricity of hyperbola $= \sqrt{1 + \frac{1}{3}} = \sqrt{\frac{4}{3}}$

focus of hyperbola $= \left(\pm\sqrt{3}.\frac{2}{\sqrt{3}}, 0\right) \equiv (\pm 2, 0)$

15. Equation of normal at P(6, 3) on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{a^2x}{6}+\frac{b^2y}{3}=a^2e^2$$

It intersects x-axis at (9, 0)

$$\Rightarrow \quad a^2\frac{9}{6}=a^2e^2 \quad \Rightarrow \quad e=\sqrt{\frac{3}{2}}$$

16. Let parametric coordinates be P(3secθ, 2 tanθ)

Equation of tangent at point P will be

$$\frac{x\sec\theta}{3}-\frac{y\tan\theta}{2}=1$$

$\because$ tangent is parallel to $2x - y = 1$

$$\Rightarrow \quad \frac{2\sec\theta}{3\tan\theta}=2 \quad \Rightarrow \quad \sin\theta=\frac{1}{3}$$

$\therefore$ coordinates are $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{9}{2\sqrt{2}},-\frac{1}{2}\right)$

MOCK TEST

1. **(B)**

origin lies in acute angle of asymptotes

P(1, 2) lies in obtuse angle of asymptotes

acute angle between the asymptotes is $\frac{\pi}{3}$

$$\therefore \quad e=\sec\frac{\theta}{2}=\sec\frac{\pi}{6}=\frac{2}{\sqrt{3}}$$

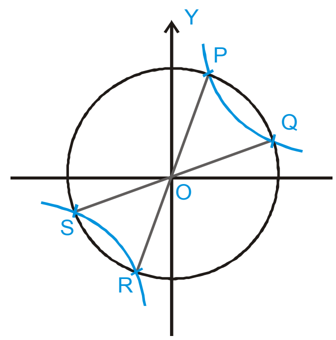
3. **(B)**

$$x-2=m$$

$$y+1=\frac{4}{m}$$

$\therefore \quad (x-2)(y+1)=4$

$\Rightarrow \quad XY=4$, where $X=x-2,\ Y=y+1$



and $S\equiv(x-2)^2+(y+1)^2=25$

$\Rightarrow \quad X^2+Y^2=25$

Curve 'C' & circle S both are concentric

$\therefore \quad OP^2+OQ^2+OR^2+OS^2=4r^2=4.25=100$

5. **(A)** Mid point is $M\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

$\therefore$ equation of the chord to the hyperbola $xy = c^2$

whose midpoint is M, is $\frac{x}{\frac{x_1+x_2}{2}}+\frac{y}{\frac{y_1+y_2}{2}}=2$

$$\Rightarrow \quad \frac{x}{x_1+x_2}+\frac{x}{y_1+y_2}=1$$

7. **(C)**

Equation of director circles of ellipse and hyperbola are respectively.

$x^2+y^2=a^2+b^2$ and $x^2+y^2=a^2-b^2$

$a^2+b^2=4r^2$ ......**(i)**

$a^2 - b^2 = r^2$ ......**(ii)**

$$a^2 = \frac{5r^2}{2}, \quad b^2 = \frac{3r^2}{2}$$

$$e_e^2 = 1 - \frac{b^2}{a^2}$$

$$\Rightarrow e_e^2 = 1 - \frac{3r^2}{2} \times \frac{2}{5r^2} = 1 - \frac{3}{5} = \frac{2}{5}$$

$$e_h^2 = 1 + \frac{b^2}{a^2}$$

$$\Rightarrow e_h^2 = 1 + \frac{3}{5} = \frac{8}{5}$$

So $4e_h^2 - e_e^2 = 4 \times \frac{8}{5} - \frac{2}{5} = \frac{30}{5} = 6$

9. (A)

S_1 : Equation of hyperbola $(x-3)(y-2) = c^2$

$xy - 2x - 3y + 6 = c^2$

$\therefore$ it passes through (4, 6), then

$4 \times 6 - 2 \times 4 - 3 \times 6 + 6 = c^2$

$c^2 = 4$

$c = 2$

Latus rectum $(\ell) = 2\sqrt{2}\,c = 2\sqrt{2} \times 2 = 4\sqrt{2}$

S_2 : Let the equation to the rectangular hyperbola be $x^2 - y^2 = a^2$ ....**(i)**

As the asymptotes of this are the axes of the other and vice-versa, hence the equation of the other hyperbola may be written as $xy = c^2$ ......**(ii)**

Let **(i)** and **(ii)** meet at some point whose coordinates are $(a \sec\alpha, a \tan\alpha)$.

then the tangent at the point $(a \sec\alpha, a \tan\alpha)$ to equation on **(i)** is

$x - y \sin\alpha = a \cos\alpha$ ....**(iii)**

and the tangent at the point $(a \sec\alpha, a \tan\alpha)$ to equation on **(ii)** is

$$y + x \sin\alpha = \frac{2c^2}{a}\cos\alpha \quad \text{.......(iv)}$$

So, the slopes of the tangents given by **(iii)** and **(iv)** are respectively $\frac{1}{\sin\alpha}$ and $-\sin\alpha$ and their product is

$$-\sin\alpha \times \frac{1}{\sin\alpha} = -1$$

Hence the tangents are a right angle.

S_3 : Hyperbola $xy = 16$

$\Rightarrow c = 4$

equation of directrices

$$x + y = \pm\sqrt{2}\,c$$

$$x + y = \pm 4\sqrt{2}$$

distance b/w directrices of hyperbola is

$$\Rightarrow \left|\frac{8\sqrt{2}}{\sqrt{1^2+1^2}}\right| \Rightarrow \left|\frac{8\sqrt{2}}{\sqrt{2}}\right| = 8$$

S_4 : Let point (h, k) on the parabola. then equation of tangent is $\frac{x}{h} + \frac{y}{k} = 2$ ....**(i)**

Equation of line $\frac{x}{x_1} + \frac{y}{y_1} = 1$

$\therefore h = \frac{x_1}{2}$ and $k = \frac{y_1}{2}$

$\therefore$ point of contact is $\left(\frac{x_1}{2}, \frac{y_1}{2}\right)$

11. (A, B, D)

For the ellipse : $e = \sqrt{\frac{25-9}{25}} = \frac{4}{5}$

$\therefore$ focii are (–4, 0) and (4, 0)

For the hyperbola

$ae = 4, e = 2$

$\therefore a = 2$

$b^2 = 4(4-1) = 12$

$b = \sqrt{12}$

13. (A, B)

equation of tangent

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$$

compare this with eqution of tangent

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow \frac{b}{a}\frac{\sec\theta}{\tan\theta} = m \Rightarrow \frac{b}{a\sin\theta} = m$$

$$\sin\theta = \frac{b}{ma}$$

$$\theta = \sin^{-1}\left(\frac{b}{ma}\right) \text{ and } \pi + \sin^{-1}\left(\frac{b}{ma}\right) \quad m > 0$$

15. (B, C)

Equation of chord joining θ and ϕ

$$\frac{x}{a}\cos\frac{\theta-\varphi}{2}-\frac{y}{b}\sin\frac{\theta+\varphi}{2}=\cos\frac{\theta+\varphi}{2}$$

it passes through (ae, 0)

$$\therefore\quad e\cos\frac{\theta-\varphi}{2}=\cos\frac{\theta+\varphi}{2}$$

$$\therefore\quad \frac{\cos\frac{\theta-\varphi}{2}}{\cos\frac{\theta+\varphi}{2}}=\frac{1}{e}$$

$$\frac{\cos\frac{\theta-\varphi}{2}-\cos\frac{\theta+\varphi}{2}}{\cos\frac{\theta-\varphi}{2}+\cos\frac{\theta+\varphi}{2}}=\frac{1-e}{1+e}$$

$$\frac{2\sin\frac{\theta}{2}\sin\frac{\varphi}{2}}{2\cos\frac{\theta}{2}\cos\frac{\varphi}{2}}=\frac{1-e}{1+e}$$

$$\Rightarrow\quad \tan\frac{\theta}{2}\tan\frac{\varphi}{2}=\frac{1-e}{1+e}$$

since the chord may also passes through (–ae, 0)

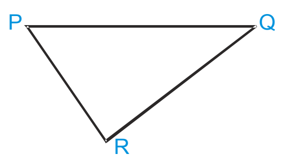
similarly as above, we get $\tan\frac{\theta}{2}\tan\frac{\varphi}{2}=\frac{1+e}{1-e}$

16. (A)

Let P be the position of the gun

and Q be the position of the target.

Let u be the velocity of sound, v be the velocity of bullet



and R be the position of the man

then we have

$t_1 = t + t_2$

$t_1 - t_2 = t$ ('t' represent time)

i.e. $\frac{PR}{u}-\frac{QR}{u}=\frac{PQ}{v}$

i.e. $PR - QR = \frac{u}{v}.PQ$ = constant and $\frac{u}{v}PQ < PQ$

∴ locus of R is a hyperbola

17. (D)

(5, 0) is a focus of the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$

and $3y \pm 4x = 0$ are assymptotes.

the auxiliary circle is $x^2 + y^2 = 9$

∴ the feet lie on $x^2 + y^2 = 9$

∴ Statement-1 is false

Statement -2 is true.

19. (D)

Statement -2 is true

for the point (2, 2), $t_1 = 1$

for the point (4, 1), $t_2 = 2$

for the point (6, 2/3), $t_3 = 3$

for the point (1/4, 16), $t_4 = \frac{1}{8}$

Now $t_1 . t_2 . t_3 . t_4 = \frac{3}{4} \neq 1$

∴ statement -1 is false

20. (A)

Statement -2 is true

Since $\left(\frac{15}{4}, 3\right)$ and $\left(-\frac{15}{4}, -3\right)$ are extremities of a diameter

∴ tangents at the points are parallel.

21. (A) Very important property of ellipse and hyperbola ($p_1p_2 = b^2$) ⇒ **(R), (S)**

(B) $y\frac{dy}{dx}=2$ ⇒ $\frac{y^2}{2}=2x+C$

$x=1, y=2$ ⇒ $C=0$

⇒ $y^2=4x$ ⇒ parabola

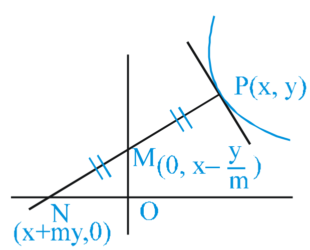
⇒ **(Q)**

(C) Equation of normal at P

$$Y-y=-\frac{1}{m}(X-x)$$

$Y=0,\quad X=x+my$

$X=0,\quad Y=y-\frac{x}{m}$



hence $x + my + x = 0 \Rightarrow 2x + y\frac{dy}{dx} = 0$

$2x\,dx + y\,dy = 0$

$x^2 + \frac{y^2}{2} = C$ passes through (1, 4)

$1 + 8 = C$

hence $x^2 + \frac{y^2}{2} = 9 \Rightarrow \frac{x^2}{9} + \frac{y^2}{18} = 1$

$\Rightarrow$ ellipse $\Rightarrow$ **(R)**

(D) length of normal

$(x + my - x)^2 + y^2 = 4$

$m^2y^2 + y^2 = 4$

$m^2 = \frac{4 - y^2}{y^2}$; $\frac{dy}{dx} = \frac{\sqrt{4-y^2}}{y}$; $\int \frac{y\,dy}{\sqrt{4-y^2}} = \int dx$

$-\sqrt{4-y^2} = x + C$

$x = 1, y = 4 \Rightarrow C = -1$

$\therefore (x-1)^2 = 4 - y^2$

$(x-1)^2 + y^2 = 4 \Rightarrow$ circle $\Rightarrow$ **(P)**

22. (A)→(r), (B)→(s), (C)→(s), (D)→(p)

(A) Since $3x^2 - 5xy - 2y^2 + 5x + 11y + c = 0$ are asympotes

$\therefore$ it represents a pair of a straight lines

$\therefore 3(-2)\,c + 2 \cdot \frac{11}{2}\left(\frac{5}{2}\right)\left(\frac{-5}{2}\right) - 3\left(\frac{11}{2}\right)^2 - (-2)\left(\frac{5}{2}\right)^2 - c\left(-\frac{5}{2}\right)^2 = 0$

i.e. $-6c - \frac{275}{4} - \frac{363}{4} + \frac{25}{2} - \frac{25}{4}c = 0$

i.e. $-24c - 275 - 363 + 50 - 25c = 0$

i.e. $49c = -588$

i.e. $c = -12$

(B) Let the point be (h, k). Then equation of the chord of contact is $hx + ky = 4$

Since $hx + ky = 4$ is tangent to $xy = 1$

$\therefore x\left(\frac{4 - hx}{k}\right) = 1$ has two equal roots

i.e. $hx^2 - 4x + k = 0$

i.e. $hk = 4$

$\therefore$ locus of (h, k) is $xy = 4$

i.e. $c^2 = 4$

(C) Equation of the hyperbola is $\frac{x^2}{c/a} - \frac{y^2}{c/b} = 1$

eccentricity $e = \sqrt{\frac{a+b}{b}}$

$\therefore \sqrt{\frac{c}{b}} = \frac{5}{2}$ and $\frac{13}{2} = \sqrt{\frac{c}{a}} \cdot \sqrt{\frac{a+b}{b}}$

$\Rightarrow \frac{13}{2} = \frac{5}{2}\sqrt{1 + \frac{b}{a}} \Rightarrow \frac{b}{a} = \frac{144}{25}$

$\therefore \frac{c}{a} = 36$

$\therefore$ the hyperbola is $25x^2 - 144y^2 = 900$

$\therefore a = 25, b = 144, c = 900$

$\therefore \frac{ab}{c} = 4$

(D) Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

then $2a = ae$ i.e. $e = 2$

$\therefore \frac{b^2}{a^2} = e^2 - 1 = 3$

$\therefore \frac{(2b)^2}{(2a)^2} = 3$

23.

1. (C)

$2a = 3$

Distance between the focii (1, 2) and (5, 5) is 5

$2ae = 5 \quad \therefore e = \frac{5}{3}$

$\frac{1}{e^2} + \frac{1}{e'^2} = 1 \Rightarrow e' = \frac{5}{4}$

2. (D)

Director circle $(x - h)^2 + (y - k)^2 = a^2 - b^2$, where (h, k) is centre

centre is $\left(\frac{1+5}{2}, \frac{2+5}{2}\right) \equiv \left(3, \frac{7}{2}\right)$

$b^2 = a^2(e^2 - 1) = \left(\frac{3}{2}\right)^2\left(\left(\frac{5}{3}\right)^2 - 1\right) = 4$

Director circle $(x-3)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{9}{4} - 4$

$$(x-3)^2 + \left(y-\frac{7}{2}\right)^2 = -\frac{7}{4}$$

this does not represent any real point

3. (B)

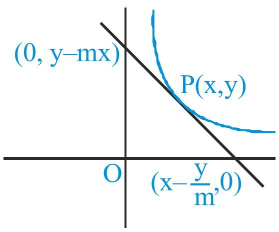
Slope of transverse axis is $\frac{3}{4}$

$\therefore$ angle of rotation $= \theta = \tan^{-1}\frac{3}{4}$

24. $Y - y = m(X - x)$; if $Y = 0$ then

$X = x - \frac{y}{m}$ and if $X = 0$ then $Y = y - mx$.

Hence $x - \frac{y}{m} = 2x \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

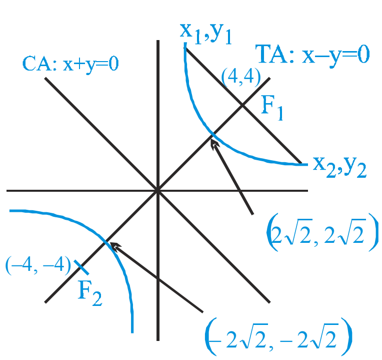


$\int \frac{dy}{y} + \int \frac{dx}{x} = c \Rightarrow xy = c$

passes through (2, 4)

$\Rightarrow$ equation of conic is $xy = 8$

which is a rectangular hyperbola with $e = \sqrt{2}$.



Hence the two vertices are $(2\sqrt{2}, 2\sqrt{2})$, $(-2\sqrt{2}, -2\sqrt{2})$ focii are $(4, 4)$ & $(-4, 4)$

$\therefore$ Equation of S is $x^2 + y^2 = 32$

25.

1. (A)

Let centre of rectangular hyperbola (H) be P(h, k) then

centroid of quadrilateral can be given by

$$G\left(\frac{h+0}{2}, \frac{k+0}{2}\right)$$

{G is same as midpoint of centres of circle and rectangular hyperbola (H)}

Now $G\left(\frac{h}{2}, \frac{k}{2}\right)$

lies on $3x - 4y + 1 = 0$

$\therefore \frac{3h}{2} - \frac{4k}{2} + 1 = 0$

$\Rightarrow 3h - 4k + 2 = 0 \Rightarrow 3x - 4y + 2 = 0$

2. (B)

Let centre of circle and hyperbola are (α, β) and (h, k) respectively and points are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$, then

$$\frac{h+\alpha}{2} = \frac{x_1 + x_2 + x_3 + x_4}{4} \quad \text{........(i)}$$

and $$\frac{k+\beta}{2} = \frac{y_1 + y_2 + y_3 + y_4}{4} \quad \text{........(ii)}$$

As any chord passing through centre of hyperbola is bisected at the centre.

$\therefore$ AB is bisected at (h, k)

$$\Rightarrow \frac{x_1 + x_2}{2} = h \quad \text{........(iii)}$$

and $$\frac{y_1 + y_2}{2} = k \quad \text{........(iv)}$$

From (i) and (iii) $\frac{x_1 + x_2}{2} + \alpha = \frac{x_1 + x_2 + x_3 + x_4}{2}$

$\Rightarrow \alpha = \frac{x_3 + x_4}{2}$

From (ii) and (iv) $\beta = \frac{y_3 + y_4}{2}$

$\Rightarrow$ (α, β) is mid-point of CD

$\Rightarrow$ (α, β) is lies on CD

$\Rightarrow$ centre of circle lies on CD.

3. (C)

Let the four concylic points at which normals to rectangular hyperbola are concurrent are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ and centre of circle be (h, k)

$$\therefore \quad \frac{x_1+x_2+x_3+x_4}{4} = \frac{h+o}{2}$$

$$\text{and} \quad \frac{y_1+y_2+y_3+y_4}{4} = \frac{k+o}{2}$$

$$\Rightarrow \quad x_1+x_2+x_3+x_4 = 2h \qquad \text{........(i)}$$

$$\text{and} \quad y_1+y_2+y_3+y_4 = 2k \qquad \text{........(ii)}$$

Normal to rectangular hyperbola $xy = c^2$ at $\left(ct, \frac{c}{t}\right)$

$ct^4 - xt^3 + yt - c = 0$

As all normal pass through (α, β)

$\therefore \quad ct^4 - \alpha t^3 + \beta t - c = 0$

$$\Rightarrow \quad x_1+x_2+x_3+x_4 = c(t_1+t_2+t_3+t_4)$$

$$= c\left(\frac{\alpha}{c}\right) = \alpha \qquad \text{........(iii)}$$

$$\text{and} \quad y_1+y_2+y_3+y_4$$

$$= c\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{1}{t_4}\right) = c\left(\frac{\sum t_1t_2t_3}{t_1t_2t_3t_4}\right)$$

$$= c\left(\frac{-\beta\,|\,c}{-c\,|\,c}\right) = \beta \qquad \text{........(iv)}$$

From (i) and (iii), $2h = \alpha$

From (ii) and (iv), $2k = \beta$

$$\Rightarrow \quad (h, k) = \left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$$

27. (4)

Let (h, k) be the mid-point of the chord of the circle $x^2 + y^2 = 4$

so, by mid point form, equation is $T = S_1$

$$hx + ky = h^2 + k^2 \text{ or } \quad y = -\frac{h}{k}x + \frac{h^2+k^2}{k}$$

$\Rightarrow \quad y = mx + c$

it will touch the hyperbola if $c^2 = a^2m^2 - b^2$

$$\Rightarrow \quad \left(\frac{h^2+k^2}{k}\right)^2 = 4\left(\frac{-h}{k}\right)^2 - 16$$

$$\Rightarrow \quad (x^2 + y^2)^2 = 4x^2 - 16y^2$$

29. (2)

Since $\frac{e}{2}$ and $\frac{e'}{2}$ are eccentricities of a hyperbola and its conjugate

$$\therefore \quad \frac{4}{e^2} + \frac{4}{e'^2} = 1$$

$$\text{i.e.} \quad 4 = \frac{e^2e'^2}{e'^2 + e'^2}$$

equation of variable line is $\frac{x}{e} + \frac{y}{e'} = 1$

$e'x + ey - ee' = 0$

it is tangent to the circle $x^2 + y^2 = r^2$

$$\therefore \quad \frac{ee'}{\sqrt{e^2+e'^2}} = r$$

$$\therefore \quad r^2 = \frac{e^2e'^2}{e^2+e'^2} = 4 \qquad \therefore \quad r = 2$$