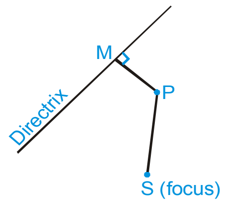
HYPERBOLA

DEFINITION

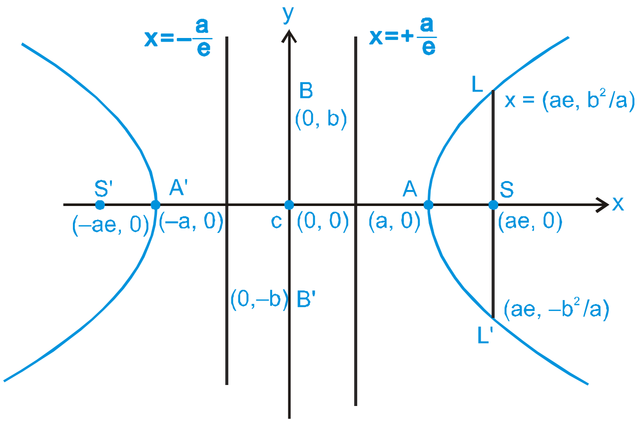
A hyperbola is defined as the locus of a point moving in a plane in such a way that the ratio of its distance from a fixed point to that from a fixed line (the point does not lie on the line) is a fixed constant greater than 1.



$$\frac{PS}{PM} = e > 1, \qquad e - \text{eccentricity}$$

STANDARD EQUATION & DEFINITION(S)

Standard equation of the hyperbola is



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2(e^2 - 1)$$

$$\text{or } a^2e^2 = a^2 + b^2 \text{ i.e. } e^2 = 1 + \frac{b^2}{a^2}$$

$$= 1 + \left(\frac{\text{Conjugate Axis}}{\text{Transverse Axis}}\right)^2$$

(a) Foci

$S \equiv (ae, 0) \quad \& \quad S' \equiv (-ae, 0)$.

(b) Equations of directrices

$$x = \frac{a}{e} \;\&\; x = -\frac{a}{e}.$$

(c) Vertices

$A \equiv (a, 0) \;\&\; A' \equiv (-a, 0)$.

(d) Latus rectum

(i) Equation : $x = \pm ae$

(ii) Length $= \dfrac{2b^2}{a} = \dfrac{(\text{Conjugate Axis})^2}{(\text{Transverse Axis})} = 2a(e^2 - 1) = 2e$ (distance from focus to directrix)

(iii) Ends : $\left(ae, \dfrac{b^2}{a}\right), \left(ae, \dfrac{-b^2}{a}\right); \left(-ae, \dfrac{b^2}{a}\right), \left(-ae, \dfrac{-b^2}{a}\right)$

(e) (i) Transverse Axis

The line segment A′A of length 2a in which the foci S′ & S both lie is called the Transverse Axis of the Hyperbola.

(ii) Conjugate Axis :

The line segment B′B between the two points $B' \equiv (0, -b)$ & $B \equiv (0, b)$ is called as the Conjugate Axis of the Hyperbola.

The Transverse Axis & the Conjugate Axis of the hyperbola are together called the Principal axes of the hyperbola.

(f) Focal Property

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e. $\left| |PS| - |PS'| \right| = 2a$. The distance SS′ = focal length.

(g) Focal distance

Distance of any point P(x, y) on Hyperbola from foci $PS = ex - a$ & $PS' = ex + a$.

(i) Length of latus rectum = 2 e × (distance of focus from corresponding directrix)

(ii) End points of latus rectum are $L \equiv \left(ae, \frac{b^2}{a}\right)$, $L' \equiv \left(ae, -\frac{b^2}{a}\right)$, $M \equiv \left(-ae, \frac{b^2}{a}\right)$, $M' \equiv \left(-ae, -\frac{b^2}{a}\right)$

(iii) Centre :

The point which bisects every chord of the conic, drawn through it, is called the centre of the conic.

$C \equiv (0,0)$ the origin is the centre of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(iv) Since the fundamental equation to hyperbola only differs from that to ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for hyperbola are derived from those for ellipse by simply changing the sign of b^2.

Ex. Find the equation of the hyperbola whose directrix is $2x + y = 1$, focus (1, 2) and eccentricity $\sqrt{3}$.

Sol. Let P(x, y) be any point on the hyperbola and PM is perpendicular from P on the directrix.

Then by definition $SP = e\,PM$

$\Rightarrow \quad (SP)^2 = e^2 (PM)^2 \quad \Rightarrow \quad (x-1)^2 + (y-2)^2 = 3\left\{\frac{2x + y - 1}{\sqrt{4+1}}\right\}^2$

$\Rightarrow \quad 5(x^2 + y^2 - 2x - 4y + 5) = 3(4x^2 + y^2 + 1 + 4xy - 2y - 4x)$

$\Rightarrow \quad 7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$

which is the required hyperbola.

Ex. The eccentricity of the hyperbola $4x^2 - 9y^2 - 8x = 32$ is -

Sol. $4x^2 - 9y^2 - 8x = 32$

$\Rightarrow \quad 4(x-1)^2 - 9y^2 = 36$

$\Rightarrow \quad \frac{(x-1)^2}{9} - \frac{y^2}{4} = 1$

Here $a^2 = 9, b^2 = 4$

$\therefore$ eccentricity $e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{4}{9}} = \frac{\sqrt{13}}{3}$

Ex. If foci of a hyperbola are foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$. If the eccentricity of the hyperbola be 2, then its equation

Sol. For ellipse $e = \frac{4}{5}$, so foci $= (\pm 4, 0)$

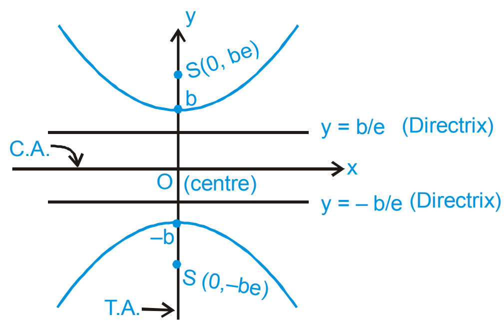
For hyperbola $e = 2$, so $a = \frac{ae}{e} = \frac{4}{2} = 2$, $b = 2\sqrt{4-1} = 2\sqrt{3}$

Hence equation of the hyperbola is $\frac{x^2}{4}-\frac{y^2}{12}=1$

Conjugate Hyperbola

Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called conjugate hyperbolas of each other.

eg. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ & $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are conjugate hyperbolas of each other.



Equation : $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$

$a^2 = b^2(e^2-1)$ $\Rightarrow$ $e = \sqrt{1+\frac{a^2}{b^2}}$

Vertices $(0, \pm b)$; $\ell(\text{L.R.}) = \frac{2a^2}{b}$

(i) If e_1 & e_2 are the eccentrcities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

(ii) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

(iii) Two hyperbolas are said to be similar if they have the same eccentricity.

(iv) Two similar hyperbolas are said to be equal if they have same latus rectum.

(v) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

Ex. The eccentricity of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is -

Sol. Equation of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is

$$-x^2 + 3y^2 = 1 \quad \Rightarrow \quad -\frac{x^2}{1} + \frac{y^2}{1/3} = 1$$

Here $a^2 = 1, b^2 = 1/3$

$\therefore$ eccentricity $e = \sqrt{1 + a^2/b^2} = \sqrt{1+3} = 2$

Ex. Find the lengths of transverse axis and conjugate axis, eccentricity, the co-ordinates of foci, vertices, length of the latus-rectum and equations of the directrices of the following hyperbola $16x^2 - 9y^2 = -144$.

Sol. The equation $16x^2 - 9y^2 = -144$ can be written as $\frac{x^2}{9} - \frac{y^2}{16} = -1$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

$\therefore$ $a^2 = 9, b^2 = 16$

$\Rightarrow$ $a = 3, b = 4$

Length of transverse axis : The length of transverse axis = 2b = 8

Length of conjugate axis : The length of conjugate axis = 2a = 6

Eccentricity : $e = \sqrt{\left(1 + \frac{a^2}{b^2}\right)} = \sqrt{\left(1 + \frac{9}{16}\right)} = \frac{5}{4}$

Foci : The co-ordinates of the foci are $(0, \pm be)$ i.e., $(0, \pm 5)$

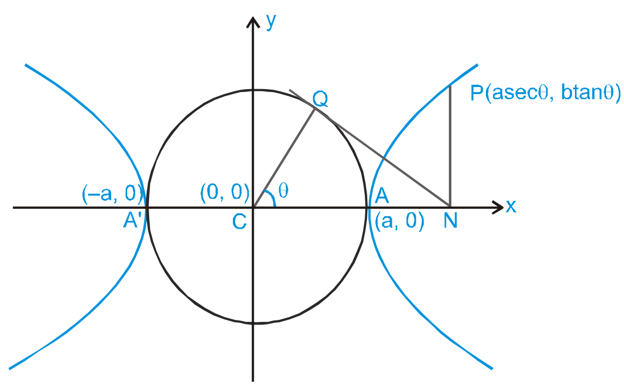
Vertices : The co–ordinates of the vertices are $(0, \pm b)$ i.e., $(0, \pm 4)$

Length of latus–rectum : The length of latus–rectum = $\frac{2a^2}{b} = \frac{2(3)^2}{4} = \frac{9}{2}$

Equation of directrices : The equation of directrices are

$$y = \pm \frac{b}{e} \quad \Rightarrow \quad y = \pm \frac{4}{(5/4)} \quad \Rightarrow \quad y = \pm \frac{16}{5}$$

AUXILIARY CIRCLE



A circle drawn with centre C & T.A. as a diameter is called the Auxiliary Circle of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Note from the figure that P & Q are called the "Corresponding Points" on the hyperbola & the auxiliary circle. 'θ' is called the eccentric angle of the point 'P' on the hyperbola. ($0 \le \theta < 2\pi$).

Parametric Equation :

The equations $x = a \sec\theta$ & $y = b\tan\theta$ together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter.

The parametric equations ; $x = a \cos h\,\phi$, $y = b \sin h\,\phi$ also represents the same hyperbola.

Parametric Representation

The equations $x = a \sec\theta$ & $y = b \tan\theta$ together represent the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter.

Note that if $P(\theta) \equiv (a \sec\theta, b\tan\theta)$ is on the hyperbola then,

$Q(\theta) \equiv (a\cos\theta, a\sin\theta)$ is on the auxiliary circle.

The equation to the chord of the hyperbola joining the two points $P(\alpha)$ & $Q(\beta)$ is given by

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}.$$

POSITION OF A POINT 'P' W.R.T. A HYPERBOLA

The quantity $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ is positive , zero or negative according as the point (x_1, y_1) lies within , upon or outside the curve.

LINE AND A HYPERBOLA

The straight line $y = mx + c$ is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as :

$c^2 >$ or $=$ or $< a^2m^2 - b^2$, respectively.

Ex. Find the position of the point (5, – 4) relative to the hyperbola $9x^2 - y^2 = 1$.

Sol. Since $9(5)^2 - (-4)^2 - 1 = 225 - 16 - 1 = 208 > 0$,

So the point (5,–4) lies inside the hyperbola $9x^2 - y^2 = 1$.

Ex. Show that the line $x\cos\alpha + y\sin\alpha = p$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$.

Sol. The given line is $x\cos\alpha + y\sin\alpha = p$

$\Rightarrow$ $y\sin\alpha = -x\cos\alpha + p$

$\Rightarrow$ $y = -x\cot\alpha + p\,\mathrm{cosec}\,\alpha$

Comparing this line with $y = mx + c$

$m = -\cot\alpha$, $c = p\,\mathrm{cosec}\,\alpha$

Since the given line touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then

$c^2 = a^2m^2 - b^2$ $\Rightarrow$ $p^2\,\mathrm{cosec}^2\alpha = a^2\cot^2\alpha - b^2$ or $p^2 = a^2\cos^2\alpha - b^2\sin^2\alpha$

TANGENT TO THE HYPERBOLA $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

(a) Point form : Equation of the tangent to the given hyperbola at the point (x_1, y_1) is $\frac{x x_1}{a^2}-\frac{y y_1}{b^2}=1$.

Note : In general two tangents can be drawn from an external point $(x_1 y_1)$ to the hyperbola and they are $y - y_1 = m_1 (x - x_1)$ & $y - y_1 = m_2 (x - x_1)$, where m_1 & m_2 are roots of the equation $(x_1^2 - a^2) m^2 - 2 x_1 y_1 m + y_1^2 + b^2 = 0$. If $D < 0$, then no tangent can be drawn from $(x_1 y_1)$ to the hyperbola.

(b) Slope form : The equation of tangents of slope m to the given hyperbola is $y = m x \pm \sqrt{a^2m^2 - b^2}$. Point of contact are $\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$

Note that there are two parallel tangents having the same slope m.

(c) Parametric form : Equation of the tangent to the given hyperbola at the point $(a \sec\theta, b \tan\theta)$ is $\frac{x \sec\theta}{a}-\frac{y \tan\theta}{b}=1$.

(i) Point of intersection of the tangents at $P(\theta_1)$ & $Q(\theta_2)$ is $\left(a\frac{\cos\frac{\theta_1-\theta_2}{2}}{\cos\frac{\theta_1+\theta_2}{2}},\ b\tan\left(\frac{\theta_1+\theta_2}{2}\right)\right)$

(ii) If $|\theta_1 + \theta_2| = \pi$, then tangents at these points (θ_1 & θ_2) are parallel.

(iii) There are two parallel tangents having the same slope m. These tangents touches the hyperbola at the extremities of a diameter.

Ex. Find the locus of the point of intersection of two tangents of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if the product of their slopes is c^2.

Sol. Equation of any tangent of the hyperbola with slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$

If it passes through (x_1, y_1) then

$(y_1 - mx_1)^2 = a^2m^2 - b^2$

$\Rightarrow \quad (x_1^2 - a^2) m^2 - 2x_1y_1m + (y_1^2 + b^2) = 0$

If $m = m_1, m_2$ then as given $m_1m_2 = c^2$

$\Rightarrow \quad \frac{y_1^2 + b^2}{x_1^2 - a^2} = c^2$

Hence required locus will be : $y^2 + b^2 = c^2(x^2 - a^2)$

Ex. Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Sol. Let m be the slope of the tangent. Since the tangent is perpendicular to the line $x - y = 0$

$\therefore$ $m \times 1 = -1$ $\Rightarrow$ $m = -1$

Since $x^2 - 4y^2 = 36$

or $\dfrac{x^2}{36} - \dfrac{y^2}{9} = 1$

Comparing this with $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$\therefore$ $a^2 = 36$ and $b^2 = 9$

So the equation of tangents are $y = (-1)\,x \pm \sqrt{36 \times (-1)^2 - 9}$

$y = -x \pm \sqrt{27}$ $\Rightarrow$ $x + y \pm 3\sqrt{3} = 0$

Pair of Tangents

The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is given by: $SS_1 = T^2$ where :

$$S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 \;; \qquad S_1 = \frac{x_1^{\,2}}{a^2} - \frac{y_1^{\,2}}{b^2} - 1 \;; \qquad T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1.$$

Ex. How many real tangents can be drawn from the point (4, 3) to the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$. Find the equation of these tangents & angle between them.

Sol. Given point $P \equiv (4, 3)$

Hyperbola $S \equiv \dfrac{x^2}{16} - \dfrac{y^2}{9} - 1 = 0$

$\because$ $S_1 \equiv \dfrac{16}{16} - \dfrac{9}{9} - 1 = -1 < 0$

$\Rightarrow$ Point $P \equiv (4, 3)$ lies outside the hyperbola.

$\therefore$ Two tangents can be drawn from the point P(4, 3).

Equation of pair of tangents is $SS_1 = T^2$

$$\Rightarrow \quad \left(\frac{x^2}{16} - \frac{y^2}{9} - 1\right) . (-1) = \left(\frac{4x}{16} - \frac{3y}{9} - 1\right)^2$$

$$\Rightarrow \quad -\frac{x^2}{16} + \frac{y^2}{9} + 1 = \frac{x^2}{16} + \frac{y^2}{9} + 1 - \frac{xy}{6} - \frac{x}{2} + \frac{2y}{3}$$

$$\Rightarrow \quad 3x^2 - 4xy - 12x + 16y = 0$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$

NORMAL TO THE HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(a) **Point form :** The equation of the normal to the given hyperbola at the point P (x_1, y_1) on it is

$$\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2 = a^2e^2.$$

(b) **Slope form :** The equation of normal of slope m to the given hyperbola is $y = mx \mp \frac{m(a^2+b^2)}{\sqrt{(a^2 - m^2b^2)}}$ foot of normal are $\left(\pm\frac{a^2}{\sqrt{(a^2 - m^2b^2)}}, \mp\frac{mb^2}{\sqrt{(a^2 - m^2b^2)}}\right)$

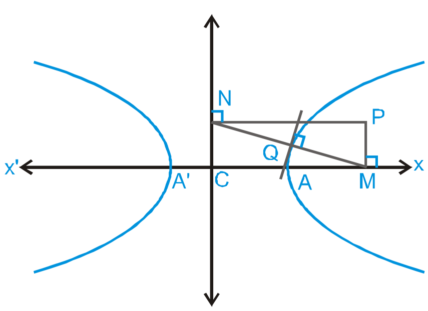
(c) **Parametric form :** The equation of the normal at the point P $(a\sec\theta, b\tan\theta)$ to the given hyperbola is

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2.$$

Ex. The normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes in M and N, and lines MP and NP are drawn at right angles to the axes. Prove that the locus of P is hyperbola $(a^2x^2 - b^2y^2) = (a^2 + b^2)^2$.

Sol. Equation of normal at any point Q is $ax\cos\theta + by\cot\theta = a^2 + b^2$

$\therefore$ $M \equiv \left(\frac{a^2+b^2}{a}\sec\theta, 0\right)$, $N \equiv \left(0, \frac{a^2+b^2}{b}\tan\theta\right)$



$\therefore$ Let $P \equiv (h, k)$

$\Rightarrow$ $h = \frac{a^2+b^2}{a}\sec\theta$, $k = \frac{a^2+b^2}{b}\tan\theta$

$\Rightarrow$ $\frac{a^2h^2}{(a^2+b^2)} - \frac{b^2k^2}{(a^2+b^2)^2} = \sec^2\theta - \tan^2\theta = 1$

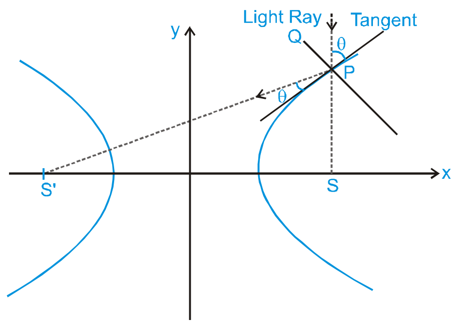
$\therefore$ locus of P is $(a^2x^2 - b^2y^2) = (a^2 + b^2)$.

IMPORTANT POINTS ON TANGENT AND NORMAL

(a) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of lengths to these perpendiculars is b^2 (semi Conjugate Axis)^2

(b) The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

(c) The tangent & normal at any point of a



hyperbola bisect the angle between the focal radii. This spells the reflection property of the hyperbola as "An incoming light ray" aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.

Note that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & the hyperbola $\frac{x^2}{a^2-k^2}-\frac{y^2}{k^2-b^2}=1$ $(a>k>b>0)$ are confocal and therefore orthogonal.

(d) The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.

DIRECTOR CIRCLE

The locus of the intersection of tangents which are at right angles is known as the Director Circle of the hyperbola. The equation to the director circle is: $x^2 + y^2 = a^2 - b^2$.

If $b^2 < a^2$, this circle is real; if $b^2 = a^2$ the radius of the circle is zero & it reduces to a point circle at the origin. In this case the centre is the only point from which the tangents at right angles can be drawn to the curve. If $b^2 > a^2$, the radius of the circle is imaginary, so that there is no such circle & so no tangents at right angle can be drawn to the curve.

❖ Equations of chord of contact, chord with a given middle point, pair of tangents from an external point are to be interpreted in the similar way as in ellipse.

CHORD OF CONTACT

Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is, $T = 0$, where $T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

Ex. If tangents to the parabola $y^2 = 4ax$ intersect the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at A and B, then find the locus of point of intersection of tangents at A and B.

Sol. Let $P \equiv (h, k)$ be the point of intersection of tangents at A & B

$\therefore$ equation of chord of contact AB is $\frac{xh}{a^2}-\frac{yk}{b^2}=1$ ....**(i)**

which touches the parabola

equation of tangent to parabola $y^2 = 4ax$

$y = mx + \frac{a}{m}$ $\Rightarrow$ $mx - y = -\frac{a}{m}$ ....**(ii)** equation **(i)** & **(ii)** as must be same

$$\therefore \quad \frac{m}{\left(\frac{h}{a^2}\right)} = \frac{-1}{\left(-\frac{k}{b^2}\right)} = \frac{-\frac{a}{m}}{1} \qquad \Rightarrow \qquad m = \frac{h}{k}\,\frac{b^2}{a^2} \ \& \ m = -\frac{ak}{b^2}$$

$$\therefore \quad \frac{hb^2}{ka^2} = -\frac{ak}{b^2} \qquad \Rightarrow \qquad \text{locus of P is } y^2 = -\frac{b^4}{a^3}\,.\,x$$

Chord with a given middle point

Equation of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose middle point is (x_1, y_1) is $\mathbf{T = S_1}$,

where $S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$; $\qquad T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1.$

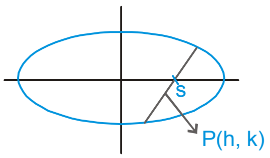
Ex. Find the locus of the mid - point of focal chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

Sol. Let $P \equiv (h, k)$ be the mid-point

$\therefore$ equation of chord whose mid-point is given is $\frac{xh}{a^2} - \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} - \frac{k^2}{b^2} - 1$

since it is a focal chord,

$\therefore$ it passes through focus, either (ae, 0) or (–ae, 0)



If it passes through (ae, 0)

$\therefore$ locus is $\frac{ex}{a} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

If it passes through (–ae, 0)

$\therefore$ locus is $-\frac{ex}{a} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

DIAMETER

The locus of the middle points of a system of parallel chords with slope 'm' of hyperbola is called its diameter. It is a straight line passing through the centre of the hyperbola and has the equation $y = -\frac{b^2}{a^2 m}x.$

❖ All diameters of the hyperbola pass through its centre.

ASYMPTOTES

Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the Asymptote of the Hyperbola.

To find the asymptote of the hyperbola :

Let $y = mx + c$ is the asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solving these two we get the quadratic as $(b^2 - a^2m^2)\,x^2 - 2a^2\,mcx - a^2(b^2 + c^2) = 0$ ....**(1)**

In order that $y = mx + c$ be an asymptote, both roots of equation (1) must approach infinity, the conditions for which are : coefficient of

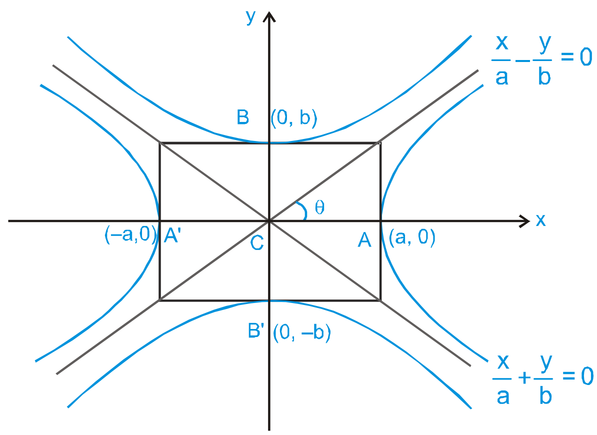
$x^2 = 0$ & coefficient of $x = 0$.

$\Rightarrow$ $b^2 - a^2m^2 = 0$ or $m = \pm\frac{b}{a}$ &

$a^2\,mc = 0 \Rightarrow c = 0.$

$\therefore$ equations of asymptote are $\frac{x}{a} + \frac{y}{b} = 0$

and $\frac{x}{a} - \frac{y}{b} = 0$.



combined equation to the asymptotes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

Particular Case :

When $b = a$ the asymptotes of the rectangular hyperbola.

$x^2 - y^2 = a^2$ are $y = \pm x$ which are at right angles.

(i) Equilateral hyperbola $\Leftrightarrow$ rectangular hyperbola.

(ii) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

(iii) A hyperbola and its conjugate have the same asymptote.

(iv) The equation of the pair of asymptotes differ the hyperbola & the conjugate hyperbola by the same constant only.

(v) The asymptotes pass through the centre of the hyperbola & the bisectors of the angles between the asymptotes are the axes of the hyperbola.

(vi) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(vii) Asymptotes are the tangent to the hyperbola from the centre.

(viii) A simple method to find the co–ordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as : Let $f(x, y) = 0$ represents a hyperbola.

Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ gives the centre of the hyperbola.

(ix) No tangent to the hyperbola can be drawn from its centre.

(x) Only one tangent to the hyperbola can be drawn from a point lying on its asymptotes other than centre

(xi) Two tangents can be drawn to the hyperbola from any of its external points which does not lie at its asymptotes

IMPORTANT POINTS ON ASYMPTOTES

(a) If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.

(b) Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.

(c) The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with centre C, meets the asymptotes in Q and R and cuts off a Δ CQR of constant area equal to ab from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact . This implies that locus of the centre of the circle circumscribing the Δ CQR in case of a rectangular hyperbola is the hyperbola itself.

(d) If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ then the eccentricity of the hyperbola is $\sec\theta$.

Ex. Find the asymptotes of $xy - 3y - 2x = 0$.

Sol. Since equation of a hyperbola and its asymptotes differ in constant terms only,

$\therefore$ Pair of asymptotes is given by $xy - 3y - 2x + \lambda = 0$

where λ is any constant such that it represents two straight lines.

$\therefore$ $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\Rightarrow \quad 0 + 2 \times -\frac{3}{2} \times -1 \times \frac{1}{2} - 0 - 0 - \lambda\left(\frac{1}{2}\right)^2 = 0$$

$\therefore$ $\lambda = 6$

From (1), the asymptotes of given hyperbola are given by

$xy - 3y - 2x + 6 = 0$ or $(y - 2)(x - 3) = 0$

$\therefore$ Asymptotes are $x - 3 = 0$

and $y - 2 = 0$

Ex. Find the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$. Find also the general equation of all the hyperbolas having the same set of asymptotes.

Sol. Let $2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0$ be asymptotes. This will represent two straight line so

$$4\lambda + 25 - \frac{25}{2} - 8 - \frac{25}{4}\lambda = 0$$

$\Rightarrow$ $\lambda = 2$

$\Rightarrow$ $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$ are asymptotes

$\Rightarrow$ $(2x + y + 2) = 0$ and $(x + 2y + 1) = 0$ are asymptotes

and $2x^2 + 5xy + 2y^2 + 4x + 5y + c = 0$ is general equation of hyperbola.

RECTANGULAR HYPERBOLA (EQUILATERAL HYPERBOLA)

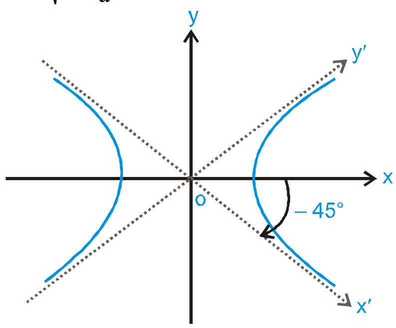
The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an Equilateral Hyperbola. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$.

Since $a = b$

equation becomes $x^2 - y^2 = a^2$

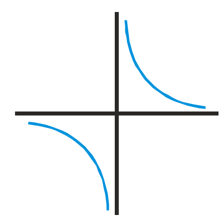
whose asymptotes are $y = \pm x$.

$$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+1} = \sqrt{2}$$



Rotation of this system through an angle of 45° in clockwise direction gives another form to the equation of rectangular hyperbola.

which is $xy = c^2$ where $c^2 = \frac{a^2}{2}$.



Rectangular Hyperbola ($xy = c^2$) :

It is referred to its asymptotes as axes of co–ordinates.

Vertices : (c, c) & $(-c, -c)$;

Foci : $\left(\sqrt{2}\,c, \sqrt{2}\,c\right)$ & $\left(-\sqrt{2}\,c, -\sqrt{2}\,c\right)$,

Directrices : $x + y = \pm\sqrt{2}\,c$

Latus Rectum (l) : $\ell = 2\sqrt{2}\,c$ = T.A. = C.A.

Parametric equation $x = ct$, $y = c/t$, $t \in R - \{0\}$

Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is $x + t_1t_2y = c(t_1 + t_2)$.

Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at $P(t)$ is $\frac{x}{t} + ty = 2c$.

Equation of the normal at $P(t)$ is $xt^3 - yt = c(t^4 - 1)$.

Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

Ex. A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.

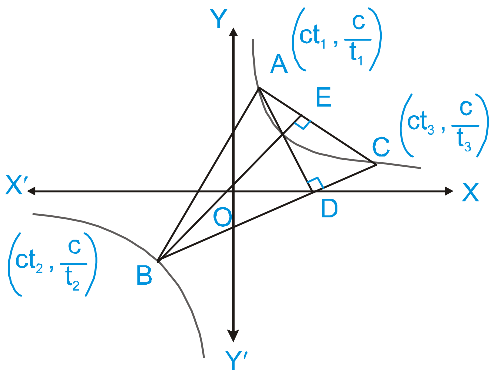
Sol. Let "t_1", "t_2" and "t_3" are the vertices of the triangle ABC, described on the rectangular hyperbola $xy = c^2$.

$\therefore$ Co–ordinates of A,B and C are $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ respectively

Now slope of BC is $\dfrac{c(t_3 - t_2)}{c(t_2 - t_3)t_2 t_3} = -\dfrac{1}{t_2 t_3}$

$\therefore$ Slope of AD is $t_2 t_3$

Equation of Altitude AD is $y - \dfrac{c}{t_1} = t_2 t_3(x - ct_1)$



$\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_3, \frac{c}{t_3}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$

or $\quad t_1 y - c = x\, t_1 t_2 t_3 - c t_1^2 t_2 t_3$ .....**(1)**

Similarly equation of altitude BE is

$t_2 y - c = x\, t_1 t_2 t_3 - c t_1 t_2^2 t_3$ ......**(2)**

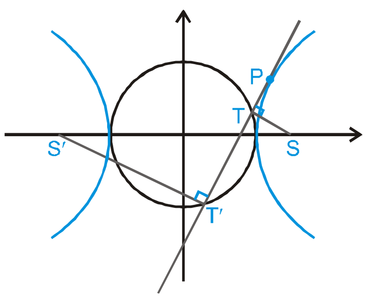
Solving **(1)** and **(2)**, we get the orthocentre $\left(-\dfrac{c}{t_1 t_2 t_3}, -ct_1 t_2 t_3\right)$

Which lies on $xy = c^2$.

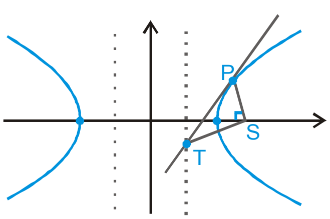
IMPORTANT RESULTS

(I) Difference of focal distances is a constant, i.e. $|PS - PS'| = 2a$

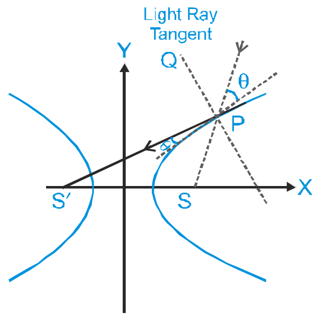
(II) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of these perpendiculars is b^2.



(III) The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

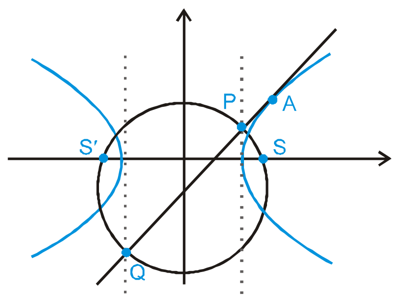


(IV) The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This explains the reflection property of the hyperbola as **"An incoming light ray "** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.

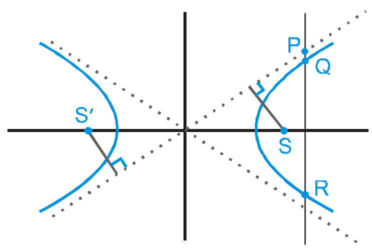


Note that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & the hyperbola $\frac{x^2}{a^2-k^2}-\frac{y^2}{k^2-b^2}=1$ $(a > k > b > 0)$ are confocal and therefore orthogonal.

(V) The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.



(VI) If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.



$(PQ)\,(PR) = b^2$

(VII) Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.

(VIII) The tangent at any point P on a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with centre C, meets the asymptotes in Q and R and cuts off a Δ CQR of constant area equal to ab from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact. This implies that locus of the centre of the circle circumscribing the Δ CQR in case of a rectangular hyperbola is the hyperbola itself & for a standard hyperbola the locus would be the curve, $4\,(a^2x^2-b^2y^2) = (a^2+b^2)^2$.

(IX) If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ then the eccentricity of the hyperbola is $\sec\theta$.

(X) A rectangular hyperbola circumscribing a triangle also passes through the orthocentre of this triangle. If $\left(ct_i, \frac{c}{t_i}\right)$ $i = 1, 2, 3$ be the angular points P, Q, R then orthocentre is $\left(\frac{-c}{t_1\,t_2\,t_3}, -c\,t_1\,t_2\,t_3\right)$.

(XI) If a circle and the rectangular hyperbola $xy = c^2$ meet in the four points t_1, t_2, t_3 & t_4, then

(a) $t_1\,t_2\,t_3\,t_4 = 1$

(b) the centre of the mean position of the four points bisects the distance between the centres of the two curves.

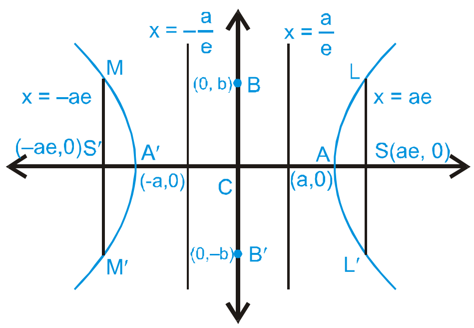
(c) the centre of the circle through the points t_1, t_2 & t_3 is :

$$\left\{\frac{c}{2}\left(t_1 + t_2 + t_3 + \frac{1}{t_1\,t_2\,t_3}\right), \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1\,t_2\,t_3\right)\right\}$$

TIPS & FORMULAS

The Hyperbola is a conic whose eccentricity is greater than unity (e > 1)

1. Standard Equation & Definition(s)



Standard equation of hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$,

where $b^2 = a^2(e^2 - 1)$.

or $a^2 e^2 = a^2 + b^2$ i.e. $e^2 = 1+\frac{b^2}{a^2} = 1+\left(\frac{\text{Conjugate Axis}}{\text{Transverse Axis}}\right)^2$

(a) Foci :

$S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.

(b) Equations of directrices :

$x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

(c) Vertices :

$A \equiv (a, 0)$ & $A' \equiv (-a, 0)$

(d) Latus rectum :

(i) Equation : $x = \pm ae$

(ii) Length $= \frac{2b^2}{a} = \frac{(\text{Conjugate Axis})^2}{(\text{Transverse Axis})} = 2a(e^2-1)$

(iii) Ends : $\left(ae, \frac{b^2}{a}\right), \left(ae, \frac{-b^2}{a}\right); \left(-ae, \frac{b^2}{a}\right), \left(-ae, \frac{-b^2}{a}\right)$

(e) (i) Transverse Axis :

The line segment A′A of length 2a in which the foci S′ & S both lie is called the Transverse Axis of the Hyperbola.

(ii) Conjugate Axis :

The line segment B′B between the two points $B' \equiv (0, -b)$ & $B \equiv (0, b)$ is called as the Conjugate Axis of the Hyperbola.

The Transverse Axis & the Conjugate Axis of the hyperbola are together called the Principal axis of the hyperbola.

(f) Focal Property :

The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e. $||PS| - |PS'|| = 2a$. The distance SS' = focal length.

(g) Focal distance :

Distance of any point P(x,y) on hyperbola from foci PS = ex – a & PS' = ex + a & PS' = ex + a

2. Conjugate Hyperbola

Two hyperbolas such that transverse & conjugate axis of one hyperbola are respectively the conjugate & the transverse axis of the other are called Conjugate Hyperbolas of each other.

eg. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are conjugate hyperbolas of each other.

Note that :

(i) If e_1 & e_2 are the eccentrcities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.

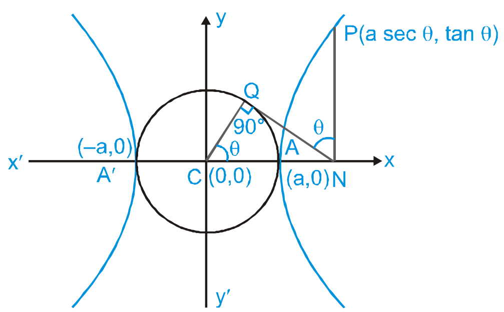
(ii) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

(iii) Two hyperbolas are said to be similar if they have the same eccentricity.

3. Rectangular or Equilateral Hyperbola

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an Equilateral Hyperbola. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$ and the length of it's latus rectum is equal to it's transverse or conjugate axis.

4. Auxiliary Circle



A circle drawn with centre C and transverse axis as a diameter is called the Auxiliary Circle of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Note from the following figure that P & Q are called the "Corresponding Points" of the hyperbola & the auxiliary circle. 'θ' is called the eccentric angle of the point 'P' on the hyperbola. $(0 \le \theta < 2\pi)$.

Parametric Equation :

The equations $x = a \sec\theta$ & $y = b \tan\theta$ together represent the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter.

5. Position of a Point 'P' w.r.t. a Hyperbola

The quantity $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$ is positive, zero or negative according as the point (x_1, y_1) lies within, upon or outside the curve.

6. Line and a Hyperbola

The straight line y = mx + c is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as : $c^2 >$ or $=$ or $< a^2m^2 - b^2$.

Equation of a chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ joining the two points P(α) & Q(β) is

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$$

7. Tangents to the Hyperbola

(i) Point form : Equation of tangent to the given hyperbola at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

Note : In general two tangents can be drawn from an external point (x_1, y_1) to the hyperbola and they are $y - y_1 = m_1(x - x_1)$ & $y - y_1 = m_2(x - x_2)$, where m_1 & m_2 are roots of the equation $(x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 + b^2 = 0$, then no tangent can be drawn from $(x_1\ y_1)$ to the hyperbola.

(ii) Slope form : The equation of tangents of slope m to given hyperbola is $y = mx \pm \sqrt{a^2m^2 - b^2}$.

Point of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \mp\frac{mb^2}{\sqrt{a^2m^2-b^2}}\right)$$

Note that there are two parallel tangents having the same slope m.

(iii) Parametric form : Equation of the tangent to the given hyperbola at the point (a sec θ, b tan θ) is

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1.$$

Note : Point of intersection of the tangents at θ_1 & θ_2 is $x = a\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}, y = b\tan\left(\frac{\theta_1+\theta_2}{2}\right)$]

8. Normal to the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(a) Point form : Equation of the normal to the given hyperbola at the point P (x_1, y_1) on it is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 = a^2e^2.$$

(b) Slope form : The equation of normal of slope m to the given hyperbola is $y = mx \mp \frac{(a^2+b^2)m}{\sqrt{a^2-b^2m^2}}$

foot of normal are $\left(\pm\frac{a^2}{\sqrt{a^2-m^2b^2}}, \mp\frac{mb^2}{\sqrt{a^2-m^2b^2}}\right)$

(c) Parametric form : The equation of the normal at the point P (a sec θ, b tan θ) to the given hyperbola

$$\text{is } \frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2.$$

9. Director Circle

The locus of the point of intersection of the tangents which are at right angle is known as the Director Circle of the hyperbola. The equation to the director circle is : $x^2 + y^2 = a^2 - b^2$.

If $b^2 < a^2$, then the director circle is real ; If $b^2 = a^2$ the radius of the circle is zero and it reduces to a point circle at the origin. In this case centre is the only point from which the tangents at right angles can be drawn to the curve.

If $b^2 > a^2$, then the radius of the circle is imaginary, so that there is no such circle and so no tangents at right 10 angle can be drawn to the curve.

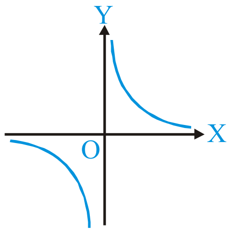
10. Asymptotes

Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the Asymptote of the Hyperbola.

Combined equation of asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ will be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

11. Rectangular Hyperbola

Rectangular hyperbola reffered to its asymptotes as axis of coordinates.



(a) Equation is $xy = c^2$ with parametric representation $x = ct$, $y = c/t$, $t \in R - \{0\}$

(b) Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is

$x + t_1t_2y = c(t_1 + t_2)$ with slope $m = \frac{-1}{t_1t_2}$.

(c) Equation of the tangent at $P(x_1, y_1)$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at $P(t)$ is

$\frac{x}{t} + ty = 2c$.

(d) Equation of normal is $y - \frac{c}{t} = t^2(x - ct)$

(e) Chord with a given middle point as (h, k) is $kx + hy = 2hk$.