HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

1. **Hint :** Distance between directrix and focus is 2a

2. The coordinates of the focus and vertex of required parabola are $S(a_1, 0)$ and $A(a, 0)$, respectively. Therefore, the distance between the vertex and the focus is $AS = a_1 - a$. So, the length of the latus rectum is $4(a_1 - a)$.

Thus, the equation of the parabola is

$$y^2 = 4(a_1 - a)(x - a)$$

4. $x = 3 \cot t$, $y = 4 \sin t$

Eliminating t, we have

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

which is an ellipse. Therefore,

$$x^2 - 2 = 2 \cos t \text{ and } y = 4 \cos^2 \frac{t}{2}$$

or $y = 2(1 + 2 \cos t)$

and $y = 2\left(1 + \dfrac{x^2 - 2}{2}\right)$

which is a parabola.

$\sqrt{x} = \tan t$; $\sqrt{y} = \sec t$

Eliminating t, we have

$y - x = 1$

which is a straight line.

$x = \sqrt{1 - \sin t}$

$y = \sin\dfrac{t}{2} + \cos\dfrac{t}{2}$

Eliminating t, we have

$x^2 + y^2 = 1 - \sin t + 1 + \sin t = 2$

which is a circle.

5. Given $(t^2, 2t)$ be one end of focal chord then other end be $\left(\dfrac{1}{t^2}, \dfrac{-2}{t}\right)$

length of focal chord

$$= \sqrt{\left(t^2 - \frac{1}{t^2}\right)^2 + \left(2t + \frac{2}{t}\right)^2} = \left(t + \frac{1}{t}\right)^2$$

8. $(\sqrt{3h}, \sqrt{3k+2})$ lie on the line $x - y - 1 = 0$. Therefore,

$(\sqrt{3h})^2 = (\sqrt{3k+2} + 1)^2$

or $3h = 3k + 2 + 1 + 2\sqrt{3k+2}$

or $3^2(h - k - 1)^2 = 2^2(\sqrt{3k+2})^2$

or $9(h^2 + k^2 + 1 - 2hk - 2h + 2k) = 4(3k + 2)$

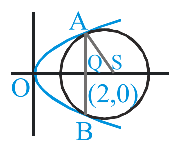
or $9(x^2 + y^2) - 18xy - 18x + 6y + 1 = 0$

Now, $h^2 = ab$ and $\Delta \neq 0$

Therefore, the locus is a parabola.

9. Focus of parabola $y^2 = 8x$ is $(2, 0)$. Equation of circle with centre $(2, 0)$ is

$(x - 2)^2 + y^2 = r^2$



AB is common chord

Q is mid point i.e. $(1, 0)$

$AQ^2 = y^2$ where $y^2 = 8 \times 1 = 8$

$\therefore$ $r^2 = AQ^2 + QS^2 = 8 + 1 = 9$

so circle is $(x - 2)^2 + y^2 = 9$

10. Let the point $P(h, k)$ on the parabola divides the line joining $A(4, -6)$ and $B(3, 1)$ in the ratio λ.

Then, we have

$$(h, k) \equiv \left(\frac{3\lambda + 4}{\lambda + 1}, \frac{\lambda - 6}{\lambda + 1}\right)$$

This point lies on the parabola. Therefore,

$$\left(\frac{\lambda - 6}{\lambda + 1}\right)^2 = 4\left(\frac{3\lambda + 4}{\lambda + 1}\right)$$

or $(\lambda - 6)^2 = 4(3\lambda + 4)(\lambda + 1)$

or $11\lambda^2 + 40\lambda - 20 = 0$

or $\lambda = \dfrac{-20 \pm 2\sqrt{155}}{11} : 1$

12. Since QR is focal chord so vertex of Q is $(at_1^2, 2at_1)$ and R is $(at_2^2, 2at_2)$

$$\text{area of } \Delta PQR = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \end{vmatrix}$$

$$A = \frac{1}{2}|2a^2t_1^2 t_2 - 2a^2t_1t_2^2|$$

$$A = \frac{a}{2}|2at_1 - 2at_2| \qquad [t_1t_2 = -1]$$

15. Let the point be (h, k)

Now equation of tangent to the parabola $y^2 = 4ax$ whose slope is m is

$$y = mx + \frac{a}{m}$$

as it passes through (h, k)

$$\therefore k = mh + \frac{a}{m} \Rightarrow m^2h - mk + a = 0$$

It has two roots m_1, $2m_1$

$$\therefore \quad m_1 + 2m_1 = \frac{k}{h}, \ 2m_1.m_1 = \frac{a}{h}$$

$$m_1 = \frac{k}{3h} \qquad .....(i)$$

$$m_1^2 = \frac{a}{2h} \qquad .....(ii) \qquad \text{from (i) \& (ii)}$$

$$\Rightarrow \frac{k^2}{(3h)^2} = \frac{a}{2h} \Rightarrow \qquad k^2 = \frac{9a}{2}h$$

Thus locus of point is $y^2 = \frac{9}{2}ax$.

16. Let $P(x_1\, y_1)$ be point of contact of two parabola. Tangents at P of the two parabolas are

$yy_1 = 2a(x + x_1) - 4a\ell_1$ and

$xx_1 = 2a(y + y_1) - 4a\ell_2$

$\Rightarrow \quad 2ax - yy_1 = 2a\,(2\ell_1 - x_1) \qquad .....(i)$

and $xx_1 - 2ay = 2a\,(y_1 - 2\ell_2) \qquad .....(ii)$

clearly (i) and (ii) represent same line

$$\therefore \frac{2a}{x_1} = \frac{y_1}{2a} \Rightarrow x_1y_1 = 4a^2$$

Hence locus of P is $xy = 4a^2$

17. Let x_1, x_2 and x_3 be the abscissae of the points on the parabola whose ordinates are y_1, y_2 and y_3, respectively. Then $y_1^2 = 4ax_1$, $y_2^2 = 4ax_2$, and $y_3^2 = 4ax_3$. Therefore, the area of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} y_1^2/4a & y_1 & 1 \\ y_2^2/4a & y_2 & 1 \\ y_3^2/4a & y_3 & 1 \end{vmatrix} = \frac{1}{8a}\begin{vmatrix} y_1^2 & y_1 & 1 \\ y_2^2 & y_2 & 1 \\ y_3^2 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$$

18. Let slope of tangent be m

So equation of tangent is

$$y = mx + \frac{1}{m}$$

Now tangent passes through (–1, 2) so

$\Rightarrow \quad m^2 + 2m - 1 = 0$

$\Rightarrow \quad m = -1 \pm \sqrt{2}$

equation of tangents are

$$y = (-1 + \sqrt{2})x + \frac{1}{-1 + \sqrt{2}} \qquad .....(i)$$

$$y = (-1 - \sqrt{2})x - \frac{1}{1 + \sqrt{2}} \qquad ......(ii)$$

intercept of tangent (i) & (ii) on line x = 2 is

$y_1 = 3\sqrt{2} - 1$ & $y_2 = -3\sqrt{2} - 1$ respectively.

Now $y_1 - y_2$ is $6\sqrt{2}$

21. Equation of directrix of parabola will be the required locus.

26. We know that area of triangle so formed

$$= \frac{\left(y_1^2 - 4ax_1\right)^{3/2}}{2a} = \left(\frac{36 - 32}{4}\right)^{3/2} = 2$$

30. Equation of tangent to $y^2 = 4ax$ at $P(x_1, y_1)$ is

$yy_1 = 2a(x + x_1)$

$\Rightarrow \quad 2ax - yy_1 + 2ax_1 = 0 \qquad .....(i)$

Let (h, k) be mid point of chord QR.

Then equation of QR is

$$ky - 2a(x + h) - 4ab = k^2 - 4a(h + b)$$

$\Rightarrow$ $-2ax + ky + 2ah - k^2 = 0$ .....(ii)

Clearly (i) and (ii) represents same line.

$$\frac{2a}{-2a} = \frac{-y_1}{k} = \frac{2ax_1}{2ah - k^2}$$

$y_1 = k$ and $2ax_1 = k^2 - 2ah$

$2ax_1 = y_1^2 - 2ah$

$2ax_1 = 4ax_1 - 2ah$

$\Rightarrow$ $x_1 = h$

$\therefore$ mid point of QR is (x_1, y_1)

EXERCISE - 2

Part # I : Multiple Choice

2. The line $y = 2x + c$ is a tangent to $x^2 + y^2 = 5$.

If $c^2 = 25$, then $c = \pm 5$

Let the equation of the parabola be $y^2 = 4ax$. Then

$$\frac{a}{2} = \pm 5$$

or $a = \pm 10$

So, the equation of the parabola is $y^2 = \pm 40x$.

Also, the equation of the directrices are $x = \pm 10$.

3. Let $(x_1, y_1) \equiv (at^2, 2at)$

Tangent at this point is $ty = x + at^2$.

Any point on this tangent is $(h, (h + at^2)/t)$.

The chord of contact of this point with respect to the circle $x^2 + y^2 = a^2$ is

$$hx + \left(\frac{h + at^2}{t}\right)y = a^2$$

or $(aty - a^2) + h\left(x + \frac{y}{t}\right) = 0$

which is a family of straight lines passing through the point of intersection of

$$ty - a = 0 \text{ and } x + \frac{y}{t}\ 0$$

So, the fixed point is $(-a/t^2, a/t)$. Therefore,

$$x_2 = -\frac{a}{t^2},\ y_2 = \frac{a}{t}$$

Clearly, $x_1x_2 = -a^2$, $y_1y_2 = 2a^2$

Also, $\frac{x_1}{x_2} = -t^4$

and $\frac{y_1}{y_2} = 2t^2$

or $4\frac{x_1}{x_2} + \left(\frac{y_1}{y_2}\right)^2 = 0$

6. $P \equiv (\alpha, \alpha + 1)$, where $\alpha \neq 0, -1$

or $P \equiv (\alpha, \alpha - 1)$, where $\alpha \neq 0, 1$

The point $(\alpha, \alpha + 1)$ is on $y^2 = 4x + 1$. Therefore,

$$(\alpha + 1)^2 = 4\alpha + 1$$

or $\alpha^2 - 2\alpha = 0$

or $\alpha = 2$ $(\because \alpha \neq 0)$

Therefore, the ordinate of P is 3.

The point $(\alpha, \alpha - 1)$ is on $y^2 = 4x + 1$. Therefore,

$(\alpha - 1)^2 = 4\alpha + 1$

or $\alpha^2 - 6\alpha = 0$

or $\alpha = 6$ $(\because \alpha \neq 0)$

Therefore, the ordinate of P is 5.

7. Here, $x^2 = -\lambda\left(y + \frac{\mu}{\lambda}\right)$

$\therefore$ Vertex $\equiv \left(0, -\frac{\mu}{\lambda}\right)$

Also, the directrix is

$\left(y + \frac{\mu}{\lambda}\right) + \frac{-\lambda}{4} = 0$

Comparing with the given data, we get

$-\frac{\mu}{\lambda} = 1$

and $\frac{\mu}{\lambda} - \frac{\lambda}{4} = -2$

$\therefore$ $-1 - \frac{\lambda}{4} = -2$

or $\lambda = 4$ or $\mu = 4$.

8. Given that the extremities of the latus rectum are (1, 1) and (1, –1). Then,

$4a = 2$ or $a = \frac{1}{2}$

So, the focus of the parabola is (1, 0).

Hence, the vertex can be (1/2, 0) or (3/2, 0).

Therefore, the equations of the parabola can be

$y^2 = 2\left(x - \frac{1}{2}\right)$

or $y^2 = 2\left(x - \frac{3}{2}\right)$

or $y^2 = 2x - 1$

or $y^2 = 2x - 3$

9. Equation of tangent and normal at P $(at^2, 2at)$ on $y^2 = 4ax$ are

$ty = x + at^2$ .....**(i)**

$y + tx = 2at + at^3$ .....**(ii)**

So $T(-at^2, 0)$ & $G(2a + at^2, 0)$

equation of circle passing P, T & G is

$(x + at^2)(x - (2a + at^2)) + (y - 0)(y - 0) = 0$

$x^2 + y^2 - 2ax - at^2(2a + at^2) = 0$

equation of tangent on the above circle at $P(at^2, 2at)$ is $at^2x + 2aty - a(x + at^2) - at^2(2a + at^2) = 0$

slope of line which is tangent to circle at P

$m_1 = \frac{a(1 - t^2)}{2at} = \frac{1 - t^2}{2t}$

slope of tangent at P, $m_2 = \frac{1}{t}$

$\therefore \tan\theta = \frac{\frac{1 - t^2}{2t} - \frac{1}{t}}{1 + \frac{(1 - t^2)}{2t^2}}$ $\Rightarrow$ $\tan\theta = t$

$\Rightarrow$ $\theta = \tan^{-1} t = \sin^{-1}\frac{t}{\sqrt{1 + t^2}}$

11. Let the possible point be $(t^2, 2t)$. The equation of tangent at this point is $yt = x + t^2$.

It must pass through (6, 5). Since the normal to the circle always passes through its center. Therefore,

$t^2 - 5t + 6 = 0$

or $t = 2, 3$

So, the possible points are (4, 4) and (9, 6).

12. $t_2 = -t_1 - \frac{2}{t_1}$

Also, $\frac{2at_1}{at_1^2} \times \frac{2at_2}{at_2^2} = -1$

or $t_1 t_2 = -4$

$\therefore$ $\frac{-4}{t_1} = -t_1 - \frac{2}{t_1}$

or $t_1^2 + 2 = 4$ and $t_1 = \pm\sqrt{2}$

So, the point can be $(2a, \pm 2\sqrt{2}\, a)$.

15. Any point on the parabola is $P(at^2, 2at)$

Therefore, the midpoint of $S(a, 0)$ and $P(at^2, 2at)$ is

$R\left(\frac{a + at^2}{2}, at\right) \equiv (h, k)$

$\therefore$ $h = \frac{a + at^2}{2}$, $k = at$

Eliminate, t, i.e.,

$$2x = a\left(1+\frac{y^2}{a^2}\right) = a + \frac{y^2}{a}$$

i.e., $2ax = a^2 + y^2$

i.e., $y^2 = 2a\left(x - \frac{a}{2}\right)$

It is a parabola with vertex at (a/2, 0) and latus rectum 2a.

The directrix is

$$x - \frac{a}{2} = -\frac{a}{2}$$

i.e., $x = 0$

The focus is

$$x - \frac{a}{2} = \frac{a}{2}$$

i.e., $x = a$

i.e., (a, 0)

18. Any point on $x + y = 1$ can be taken as $(t, 1 - t)$.

The equation of chord with this as midpoint is

$$y(1 - t) - 2a(x + t) = (1 - t^2) - 4at$$

It passes through (a, 2a). So,

$$t^2 - 2t + 2a^2 - 2a + 1 = 0$$

This should have two distinct real roots. So,

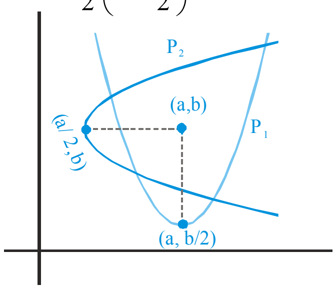
Discriminant > 0 i.e., $a^2 - a < 0$

$$0 < a < 1$$

So, length of latus rectum < 4

and $0 < \lambda < 1$

17. $P_1 \equiv (x - a)^2 = 4.\frac{b}{2}\left(y - \frac{b}{2}\right)$



$\Rightarrow$ $x^2 - 2ax + a^2 - 2yb + b^2 = 0$

Similarly

$$P_2 \equiv y^2 - 2ax - 2by + a^2 + b^2 = 0$$

Common chord is $P_1 - P_2 = 0$

$\Rightarrow$ $x^2 - y^2 = 0$

$\Rightarrow$ $(x + y)(x - y) = 0$

slope will be 1, –1

Part # II : Assertion & Reason

1. Any tangent having slope m is

$$y = m(x + a) + \frac{a}{m}$$

or $y = mx + am + \frac{a}{m}$

This is tangent to the given parabola for all $m \in R - \{0\}$.

Hence, statement 2 is false

However, statement 1 is true as when m = 1, the tangent is $y = x + 2a$.

3. Let $P_1\,(at_1^2, 2at_1)$ & $Q_1\left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right)$

$P_2\,(at_2^2, 2at_2)$ & $Q_2\left(\frac{a}{t_2^2}, \frac{-2a}{t_2}\right)$

on $y^2 = 4ax$

equation of P_1P_2 :

$$(t_1 + t_2)y = 2x + 2at_1t_2 \qquad .....(i)$$

equation of Q_1Q_2

$$-(t_1 + t_2)y = 2x\, t_1t_2 + 2a \qquad .....(ii)$$

add **(i)** & **(ii)**

$x = -a$ which is directrix of $y^2 = 4ax$

Locus of point of intersection of tangent is directrix. In case of parabola director circle is directrix

4. Statement 2 is true as it is the definition of parabola.

From statement 1, we have

$$\sqrt{(x-1)^2 + (y+2)^2} = \frac{|3x + 4y + 5|}{5}$$

which is not a parabola as the point (1, –2) lie on the line $3x + 4y + 5 = 0$.

Hence, statement 1 is false.

6. Given C : $(y - 1)^2 = 8(x + 2)$ (which is a parabola)

Clearly, P(–4, 1) lies on the directrix x = –4.

Also, P(–4, 1) lies on the axis of the parabola,

i.e., at y = 1.

So, from any point on the directrix of the parabola, if two tangents are drawn to the parabola, then these two tangents will be mutually perpendicular.

EXERCISE - 3

Part # I : Matrix Match Type

1. **(a) → r ; (b) → s ; (c) → p ; (d) → q**

The locus of the point of intersection of perpendicular tangent is directrix, which is $12x - 5y + 3 = 0$.

The parabola is symmetrical about its axis, which is a line passing through the focus (1, 2) and perpendicular to the directrix, which has equation $5x + 12y - 29 = 0$.

The minimum length of focal chord occurs along the latus rectum line, which is a line passing through the focus and parallel to the directrix, i.e., $12x - 5y - 2 = 0$.

The locus of the foot of perpendicular from the focus upon any tangent is tangent at the vertex, which is parallel to the directrix and equidistant from the directrix and latus rectum line, i.e.,

$12x - 5y + \lambda = 0$

where $\frac{|\lambda - 3|}{\sqrt{12^2 + 5^2}} = \frac{|\lambda + 2|}{\sqrt{12^2 + 5^2}}$ or $\lambda = \frac{1}{2}$

Hence, the equation of tangent at vertex is $24x - 10y + 1 = 0$

2. **(A)** Equation of normal at $(t^2, 2t)$ on $y^2 = 4x$

$y + tx = 2t + t^3$ using homogenization

$$y^2 = \frac{4x(y + tx)}{(2t + t^3)}$$

for making 90°, coeff. x^2 + coeff. $y^2 = 0$

$$1 - \frac{4}{2 + t^2} = 0$$

$t^2 = 2$

(B) Point on $y^2 = 4x$

whose parameter are 1, 2, 4

(1, 2), (4, 4), (16, 8)

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 4 & 4 & 1 \\ 16 & 8 & 1 \end{vmatrix} = 6$$

(C) Equation of normal is $y = mx - 2am - am^3$

since it passes through $\left(\frac{11}{4}, \frac{1}{4}\right)$.

$\therefore$ so we get $4m^3 - 3m + 1 = 0$.

Value of m are –1, 1/2, 1/2, so 2 normals can be drawn.

(D) Equation of normal at $(at_1^2, 2at_1)$ to $y^2 = 4ax$

$y + t_1x = 2at_1 + at_1^3$ ......(i)

If it again meet the curve again at $(at_2^2, 2at_2)$

then $t_2 = -t_1 - \frac{2}{t_1}$

So $t_1 = 1$, & $t_2 = t$

$\Rightarrow$ $t = -1 - 2 = -3$

$|t - 1| = |-3 - 1| = 4$

4. **(A)** Required area $= \frac{S_1^{3/2}}{2|a|} = \frac{(4)^{3/2}}{2} = 4$

(B) $(x - 2)^2 + (y - 3)^2 = \left(\frac{3x + 4y - 6}{5}\right)^2$

$$\sqrt{(x-2)^2 + (y-3)^2} = \frac{3x + 4y - 6}{5}$$

focus, is (2,3) & directrix is $3x + 4y - 6 = 0$

distance between focus and directrix is

$$2a = \frac{6 + 12 - 6}{5} = \frac{12}{5}$$

$\Rightarrow$ Length of Latus Rectum $= 4a = \frac{24}{5}$

(C) $x^2 = y + 4$ $\therefore$ its focus $(0, \frac{-15}{4})$

Let point on $x^2 = y + 4$ is

$(x_1, x_1^2 - 4)$

$$x_1^2 + (x_1^2 - 4 + \frac{15}{4})^2 = \frac{625}{16}$$

$$x_1^2 + x_1^4 + \frac{1}{16} - \frac{x_1^2}{2} = \frac{625}{16}$$

$$x_1^4 + \frac{x_1^2}{2} = 39$$

$2x_1^4 + x_1^2 - 78 = 0$

$(x_1^2 - 6)(2x_1^2 + 13) = 0$

$x_1 = \pm\sqrt{6}$

$x_1^2 = 6$ $\Rightarrow$ $x_1^2 - 4 = 2$

so point are $(\pm\sqrt{6}, 2)$

& $a + b = 6 + 2 = 8$

(D) $(y - 1)^2 = 2(x + 2)$

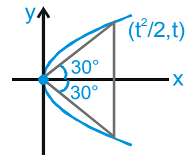
vertex is $(-2, 1)$

so equation is $(y - 1)^2 = 2(x + 2)$

$\Rightarrow$ $Y^2 = 2X$

Let point on $Y^2 = 2X$ is $\left(\frac{1}{2}t^2, t\right)$

From fig. $\tan 30° = \frac{2}{t}$



$\Rightarrow t = 2\sqrt{3}$

so point on parabola is $(6, 2\sqrt{3})$.

But when vertex change, distance (or length of side of equilateral triangle) remain same

$\therefore$ length of side $= \sqrt{(6)^2 + (2\sqrt{3})^2} = 4\sqrt{3}$.

Part # II : Comprehension

Comprehension # 1

1. (B), 2. (C) 3. (D)

1. **(B)** Since no point of the parabola is below the x-axis,

$D = a^2 - 4 \le 0$

Therefore, the maximum value of a is 2.

The equation of the parabola when a = 2 is

$y = x^2 + 2x + 1$

It intersects the y-axis at (0, 1)

The equation of the tangent at (0, 1) is

$y = 2x + 1$

Since $y = 2x + 1$ touches the circle $x^2 + y^2 = r^2$, we get

$r = \frac{1}{\sqrt{5}}$

2. **(C)** The equation of the tangent at (0, 1) to the parabola

$y = x^2 + ax + 1$

is $y - 1 = a(x - 0)$

or $ax - y + 1 = 0$

As it touches the circle, we get

$y = \frac{1}{\sqrt{a^2 + 1}}$

The radius is maximum when a = 0.

Therefore, the equation of the tangent is y = 1.

Therefore, the slope of the tangent is 0.

3. **(D)** The equation of tangent is $y = ax + 1$

The intercepts are $-1/a$ and 1.

Therefore, the area of the triangle bounded by the tangent and the axes is

$\frac{1}{2}\left|-\frac{1}{a}, 1\right| = \frac{1}{2|a|}$

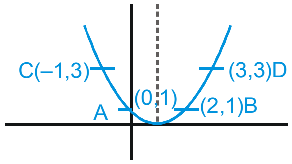
It is minimum when a = 2. Therefore,

Minimum area $= \frac{1}{4}$

Comprehension # 2

Axis of parabola is bisector of parallel chord A B & CD are parallel chord.

so axis x = 1



equation of parabola is

$(x - 1)^2 = ay + b$

It passing (0, 1) & (3, 3)

So $1 = a + b$ .....(i)

$4 = 3a + b$ .....(ii)

from (i) & (ii)

$a = \frac{3}{2}$ & $b = -\frac{1}{2}$

$(x - 1)^2 = \frac{3}{2}\left(y - \frac{1}{3}\right)$

1. Vertex $\left(1, \frac{1}{3}\right)$

2. $a = \frac{3}{8}$

directrix of $x^2 = 4ay$ is $y = -a$

$y - \frac{1}{3} = -\frac{3}{8}$

$\Rightarrow$ $y = \frac{1}{3} - \frac{3}{8}$

$y + \frac{1}{24} = 0$

3. Let parametric point on $y^2 = 4ax$ are $A(t_1)$, $B(t_2)$, $C(t_3)$ and $D(t_4)$

So $t_1 + t_2 = 2 = t_3 + t_4$

Equation of circle passing through OAB is

$x^2 + y^2 + 2gx + 2fy + c = 0$

fourth point $M(t_5)$ putting the value $(t^2, 2t)$ in circle we get four degree equation. In this equation

$t_1 + t_2 + t_5 + 0 = 0 \quad \Rightarrow \quad t_5 = -2$

Similarly circle passing through OCD & fourth point $N(t_6)$ we have $t_1 + t_2 + t_6 + 0 = 0 \quad \Rightarrow \quad t_6 = -2$

It mean both point M and N are same

so common point $(at^2, 2at) \quad \Rightarrow \quad (4, 4)$

Comprehension # 3

1. (a), 2. (b), 3. (c)

Any parabola whose axes is parallel to x-axis will be of the form

$(y - a)^2 = 4b(x - c)$ ....**(i)**

Now, $lx + my = 1$ can be rewritten as

$$y - a = -\frac{l}{m}(x - c) + \frac{1 - am - lc}{m} \quad \text{....(ii)}$$

equ. **(ii)** will touch **(i)** if

$$\frac{1 - am - lc}{m} = \frac{b}{-l/m}$$

or $$-\frac{l}{m} = \frac{bm}{1 - am - lc}$$

or $cl^2 - bm^2 + alm - l = 0$ ....**(iii)**

but given that

$5l^2 + 6m^2 - 4lm + 3l = 0$ ....**(iv)**

comparing **(iii)** and **(iv)**, we get

$$\frac{c}{5} = \frac{-b}{6} = \frac{a}{-4} = \frac{-1}{3}$$

or $c = \dfrac{-5}{3}$, $b = 2$, and $a = \dfrac{4}{3}$

So, the parabola is

$$\left(y - \frac{4}{3}\right)^2 = 8\left(x + \frac{5}{3}\right)$$

whose focus is (1/3, 4/3) and directrix is $3x + 11 = 0$.

EXERCISE - 4
Subjective Type

1. Parabola $y^2 = 4ax$

$P(t_1) = (at_1^2, 2at_1)$ & $Q(t_2) = (at_2^2, 2at_2)$

Given $t_1t_2 = K$

equation of chord PQ

$(t_1 + t_2)y = 2x + 2at_1t_2$

So $(t_1 + t_2)y = 2x + 2ak$

$$\left(\frac{t_1 + t_2}{2}\right) y = x + ak$$

$[L_2 = \lambda L_1 \text{ Type}]$

So $y = 0$ & $x = -ak$

fixed point $(-ak, 0)$

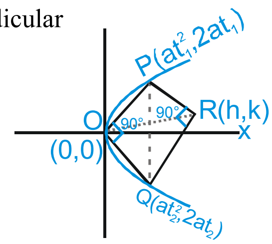
2. Let parabola $y^2 = 4ax$

Let $P(at_1^2, 2at_1)$, & $Q(at_2^2, 2at_2)$

OP & OQ are perpendicular

$$\Rightarrow \frac{2}{t_1} \cdot \frac{2}{t_2} = -1$$

$t_1t_2 = -4$



Now diagonals of a rectangle bisect each other

$$\frac{h}{2} = \frac{at_1^2 + at_2^2}{2} \Rightarrow h = a(t_1^2 + t_2^2) \quad \text{.....(i)}$$

$$\frac{k}{2} = \frac{2at_1 + at_2}{2} \Rightarrow k = 2a(t_1 + t_2) \quad \text{.....(ii)}$$

$$\frac{k^2}{4a^2} = t_1^2 + t_2^2 + 2t_1t_2$$

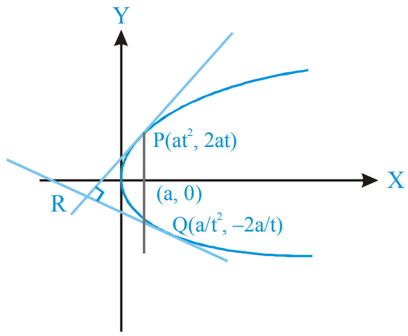
$$\frac{k^2}{4a^2} = \frac{h}{a} - 8$$

Required locus is $y^2 = 4a(x - 8a)$

3. Let the parabola be $y^2 = 4ax$.

ΔPQR is right-angled at R as the tangents at the extremities of the focal chord meet on the directrix at right angle.

Also, the coodinates of points P and Q are $P(at^2, 2at)$ and $(a/t^2, -2a/t)$, respectively.



Hence, the point of intersection of tangent at point P(t) and Q(–1/t) is (–a, a{t – (1/t)}) and the coordinates of the centroid (G) is ((a/3) {t^2 – (1/t^2) – 1}, a{t – (1/t)}). Hence, the slope of line RG is 0 (R is the orthocenter).

4. Equation of tangent of

$y^2 = 4ax$ in slope form at(x_1, y_1) is

$y_1 = mx_1 + \frac{a}{m}$ .....**(i)**

equation of normal at $(2bt_1, bt_1^2)$ on $x^2 = 4by$

$x + t_1y = 2bt_1 + bt_1^3$

It passes through (x_1, y_1)

$\therefore\ x_1 + t_1y_1 = 2bt_1 + bt_1^3$ .....**(ii)**

(i) & **(ii)** are same equation so compare

$$\frac{1}{t_1} = -\frac{m}{1} = \frac{a}{m(2bt_1 + bt_1^3)}$$

$t_1m = -1$

$-m^2t_1(2b + bt_1^2) = a$

$\Rightarrow$ $m(2b + bt_1^2) = a$ .....**(iii)**

Put $m = -\frac{1}{t_1}$ in equation **(iii)**

$2b + bt_1^2 = -at_1$

$bt_1^2 + at_1 + 2b = 0$

t_1 will be real

$a^2 > 8b^2$

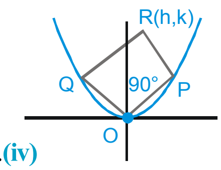
5. $x^2 = y$ .....**(i)**

Let equation of OP y = mx .....**(ii)**

equation of OQ $y = \frac{-1}{m}x$ .....**(iii)**

from (1) & (2) we get P(m, m^2)

from (1) & (3) we get Q $\left(\frac{-1}{m}, \frac{1}{m^2}\right)$



equation of PR

$y - m^2 = -\frac{1}{m}(x - m)$

$y + \frac{1}{m}x = m^2 + 1$ .....**(iv)**

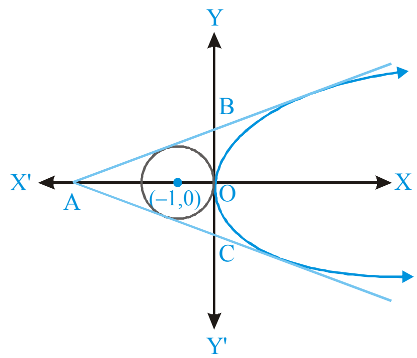
equation of QR is

$y - \frac{1}{m^2} = m(x + \frac{1}{m})$

$y - mx = 1 + \frac{1}{m^2}$ .....**(v)**

Locus of R solving (4) & (5) & eliminating m we get $x^2 = y - 2$

6.



As the circle is $(x + 1)^2 + y^2 = 1$, one of the common tangent is along the y-axis.

Let the other common tangent has slope m.

Then, its equation is

$y = mx + \frac{1}{m}$

Solving it with the equation of circle, we get

$x^2 + \left(mx + \frac{1}{m}\right)^2 + 2x = 0$

or $(1 + m^2)x^2 + 4x + \frac{1}{m^2} = 0$

As the line touches the circle,

D = 0

or $16 - \frac{1}{m^2}(1 + m^2) = 0$

or $4m^2 = 1 + m^2$

or $m = \pm\frac{1}{\sqrt{3}}$

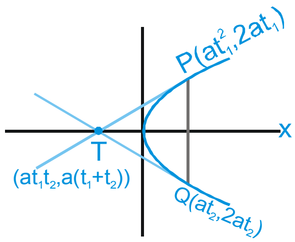
i.e., $\angle BOA = \angle OAC = \frac{\pi}{6}$

Hence, the triangle is equilateral.

7. Let parabola $y^2 = 4ax$

point on parabola $P(at_1^2, 2at_1)$ & $Q(at_2^2, 2at_2)$

Point of intersection of tangent at P & Q is T $(at_1t_2, a(t_1 + t_2))$



Normal at P & Q meet again in the parabola so relation between t_1t_2

$$-t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2}$$

$t_1t_2 = 2$

equation of line perpendicular to TP & passing through mid point of TP is

$2y - a(3t_1 + t_2) = -t_1(2x - a(2 + t_1^2))$ .....**(i)**

$2y + 2xt_1 = a(3t_1 + t_2) + at_1(2 + t_1^2)$

similar equation of passing mid point of TQ and $\perp$ to TQ

$2y + 2xt_2 = a(3t_2 + t_1) + at_2(2 + t_2^2)$ .....**(ii)**

from (1) & (2) & using $t_1t_2 = 2$

Eliminating t_1 & t_2 we get the locus of circumcentre

$2y^2 = a(x - a)$

8. $\alpha \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

point $(\sin\alpha, \cos\alpha)$ not lie out side

$2y^2 + x - 2 = 0$

$\Rightarrow$ $2\cos^2\alpha + \sin\alpha - 2 \le 0$

$2 - 2\sin^2\alpha + \sin\alpha - 2 \le 0$

$\sin\alpha(2\sin\alpha - 1) \ge 0$

$\sin\alpha \le 0$ or $\sin\alpha \ge \frac{1}{2}$

$\alpha \in \left[\pi, \frac{3\pi}{2}\right]$ or $\alpha \in \left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$

10. $y = mx + c$ touch $y^2 = 8(x + 2)$

$\therefore$ $(mx + c)^2 = 8(x + 2)$

$m^2x^2 + x(2mc - 8) + c^2 - 16 = 0$ ...**(i)**

line touch the parabola so D = 0 of equation **(i)**

$4(mc - 4)^2 - 4m^2(c^2 - 16) = 0$

$m^2c^2 - 8mc + 16 - m^2c^2 + 16m^2 = 0$

$2m^2 - mc + 2 = 0$

Since m is real $D \ge 0$

$c^2 - 16 \ge 0$

$c \in (-\infty, -4] \cup [4, \infty)$

11. Let $P(at_1^2, 2at_1)$ & $Q(at_2^2, 2at_2)$

on $y^2 = 4ax$

equation of chord of PQ

$(t_1 + t_2)\,y = 2x + 2at_1t_2$ .....**(i)**

Point on x-axis is $K(-at_1t_2, 0)$

$PK^2 = (at_1^2 + at_1t_2)^2 + 4a^2t_1^2$

$= a^2t_1^2((t_1 + t_2)^2 + 4)$

$QK^2 = a^2t_2^2((t_1 + t_2) + 4)$

$$\frac{1}{PK^2} + \frac{1}{QK^2} = \frac{1}{a^2t_1^2((t_1+t_2)^2+4)} + \frac{1}{a^2t_2^2((t_1+t_2)^2+4)}$$

$$= \frac{t_2^2 + t_1^2}{a^2t_1^2t_2^2((t_1+t_2)^2+4)}$$

$$= \frac{t_1^2 + t_2^2}{a^2t_1^2t_2^2((t_1^2+t_2^2+2t_1t_2+4)}$$

$= \frac{1}{PK^2} + \frac{1}{QK^2}$ will be independent of K

$\Rightarrow$ $\frac{t_1^2 + t_2^2}{a^2t_1^2t_2^2(t_1^2+t_2^2+2t_1t_2+4)}$ $\Rightarrow$ $t_1t_2 = -2$

so fixed point K(2a, 0)

12. Let the fixed parabola be

$y^2 = 4ax$ ....**(i)**

and the moving parabola be

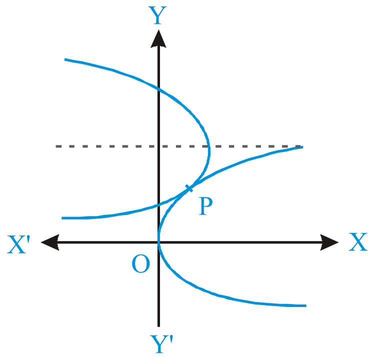
$(y - k)^2 = -4a(x - h)$ ....**(ii)**

on solving (i) and (ii)

$$(y-k)^2 = -4a\left(\frac{y^2}{4a} - h\right)$$

or $y^2 - 2ky + k^2 = -y^2 + 4ah$

or $2y^2 - 2ky + k^2 - 4ah = 0$



Since the two parabola touch each other, D = 0

i.e., $4k^2 - 8(k^2 - 4ah) = 0$

or $-4k^2 + 32ah = 0$

or $k^2 = 8ah$

Therefore, the locus of the vertex of the moving parabola is $y^2 = 8ax$.

13. Let parabola is $y^2 = 4ax$

equation of normal at $(am^2, 2am)$

$y + mx = 2am + am^3$

it passes through (h, k)

$am^3 + m(2a - h) - k = 0$

its root are m_1 m_2 & m_3

$\Sigma m_1 = 0, \Sigma m_1 m_2 = \frac{2a-h}{a}$

$m_1\, m_2\, m_3 = \frac{k}{a}$

let equation of circle be

$x^2 + y^2 + 2gx + 2fy + C = 0$

It passes $(am^2, 2am)$

$a^2m^4 + 4a^2m^2 + 2agm^2 + 4afm + C = 0$

$a^2m^4 + m^2(4a^2 + 2ag) + 4afm + C = 0$

its roots m_1, m_2, m_3 & m_4

$m_1 + m_2 + m_3 + m_4 = 0,$

$\because$ $m_1 + m_2 + m_3 = 0$

$\Rightarrow$ $m_4 = 0$ $\Rightarrow$ circle passes (0, 0)

$m_1m_2 + m_2m_3 + m_3m_4 + m_4m_1 + m_1m_3 + m_2m_4$

$$= \frac{4a^2 + 2ag}{a^2}$$

$$\Rightarrow \frac{2a-h}{a} = \frac{4a^2+2ag}{a^2}$$

$\Rightarrow$ $2a - h = 4a + 2g$

$$\Rightarrow g = \frac{-h-2a}{2}$$

$$m_1m_2m_3 + m_2m_3m_4 + m_3m_4m_1 + m_4m_1m_2 = \frac{-4af}{a^2}$$

$$\Rightarrow \frac{k}{a} = \frac{-4af}{a^2}$$

$$\Rightarrow f = \frac{-k}{4}$$

equation of circle

$$x^2 + y^2 - (h+2a)x + \frac{k}{2}y = 0$$

14. Let point on $y^2 = 4ax$ be $P(at^2, 2at)$

equation of tangent of P

$ty = x + at^2$ .....(i)

It intersect the directrix $x = -a$ .....(ii)

point of intersection of (1) & (2)

is $A(-a, a(t - \frac{1}{t}))$

Let mid point of PA is (h, k)

$2h = at^2 - a$ .....(iii)

$2k = 2at + a(t - \frac{1}{t})$ .....(iv)

from (3) & (4) eliminating t & replace $h \to x$ & $y \to k$ we get

$y^2(2x + a) = a(3x + a)^2$

16. Normal at $P(am^2, 2am)$ on $y^2 = 4ax$

$y + mx = 2am + am^3$ .....(i)

$G(2a + am^2, 0)$

Equation of QG is $x = 2a + am^2$

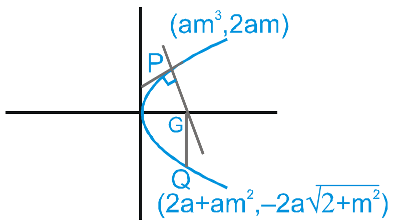
Solving with parabola we get

$$y = \pm 2a\sqrt{2+m^2}$$

$QG^2 - PG^2 =$

$4a^2(2 + m^2) - (am^2 - am^2 - 2a)^2 - (2am)^2$

$= 4a^2$ which is constant



17. Let point on parabola $y^2 = 4ax$ is $P(at^2, 2t)$

Given $at^2 = 4a \Rightarrow t = \pm 2$

taking positive $t = 2$

$P(4a, 4a)$

equation of tangent at P is $2y = x + 4a$

If intersect x-axis at T then $T(-4a, 0)$

Normal at $(4a, 4a)$ meet again parabola at

$Q(at_2^2, 2at_2)$ (using $t_2 = -t_1 - \frac{2}{t_1} = -3$)

$\therefore$ $Q(9a, -6a)$

Now $P(4a, 4a), T(-4a, 0), Q(9a, -6a)$

$$PT = \sqrt{(4a+4a)^2 + (4a)^2} = \sqrt{80a^2}$$

$$PQ = \sqrt{(4a-9a)^2 + (4a+6a)^2} = \sqrt{125a^2}$$

$$\frac{PT}{PQ} = \sqrt{\frac{80a^2}{125a^2}} = \frac{4}{5}$$

20. Let point be (h, k)

Equation of normal at $(am^2, 2am)$

$y + mx = 2am + am^3$

$k = mh - 2am - am^3$

$am^3 + m(2a - h) + k = 0$ .....**(i)**

Its slope is m_1, m_2 & m_3

$$m_1.m_2.m_3 = \frac{-k}{a}$$

$m_3 = \frac{k}{a}$ Put in (i) [Given $m_1m_2 = -1$]

$\Rightarrow$ $y^2 = a(x - 3a)$

22. Equation of normal at $(am^2, 2am)$

on $y^2 = 4ax$

$y + mx = 2am + am^3$ .....**(i)**

It cuts x – axis at y = 0 i.e. $(2a + am^2, 0)$

Let middle point (h, k)

$2h = am^2 + 2a + am^2$

$h = am^2 + a$ & $k = am$ .....**(ii)**

from (1) & (2)

$$h = a\frac{k^2}{a^2} + a$$

Locus $y^2 = a(x - a)$

vertex $(a, 0)$ L.R. = a

23. Let $P(at_1^2, 2at_1)$ & $Q(at_2^2, 2at_2)$

on $y^2 = 4ax$

co-ordinate of $T(at_1t_2, a(t_1 + t_2))$ which is point of intersection of tangent at P & Q

equation of PQ which is normal at P

$y + t_1x = 2at_1 + at_1^3$ .....**(i)**

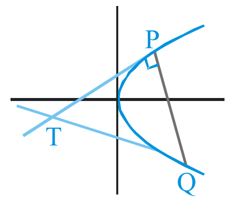
equation of PQ is

$(t_1 + t_2)y = 2x + 2at_1t_2$ .....**(ii)**

equation (i) & (ii) are same

Compare slope $\frac{2}{t_1 + t_2} = -t_1$

$\Rightarrow$ $t_1^2 + t_1t_2 = -2$



Now mid point of TP

$$x = \frac{at_1^2 + at_1t_2}{2} = \frac{a(t_1^2 + t_1t_2)}{2}$$

$$x = \frac{a(-2)}{2} = -a$$

$x = -a$ which is directrix

Hence TP bisect the directrix

24. Let the point P be (p, 0) and the equation of the chord through P be

$$\frac{x-p}{\cos\theta}=\frac{y-0}{\sin\theta}=r \qquad (r \in R) \qquad ....(i)$$

Therefore,

$(r\cos\theta + p, r\sin\theta)$ lies on the parabola $y^2 = 4ax$.

So, $r^2\sin^2\theta - 4ar\cos\theta - 4ap = 0$ ....(ii)

If AP = r_1 and BP = $-r_2$, then r_1 and r_2 are the roots of (ii).

Therefore,

$$r_1 + r_2 = \frac{4a\cos\theta}{\sin^2\theta},\ r_1r_2 = \frac{-4ap}{\sin^2\theta}$$

Now, $\frac{1}{AP^2}+\frac{1}{BP^2}=\frac{1}{r_2^2}+\frac{1}{r_2^2}$

$$=\frac{(r_1+r_2)^2-2r_1r_2}{r_1^2r_2^2}$$

$$=\frac{\cos^2\theta}{p^2}+\frac{\sin^2\theta}{2ap}$$

Since $\frac{1}{AP^2}+\frac{1}{BP^2}$

should be independent of θ, we take p = 2a. Then,

$$\frac{1}{AP^2}+\frac{1}{BP^2}=\frac{1}{4a^2}(\cos^2\theta+\sin^2\theta)=\frac{1}{4a^2}$$

Hence, $\frac{1}{AP^2}+\frac{1}{BP^2}$

is independent of θ for all the position of the chord if P ≡ (2a, 0).

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

3. It is a fundamental theorem.

4. Given parabolas are

$y^2 = 4ax$ ..... (i)

$x^2 = 4ay$ ..... (ii)

Putting the value of y from (ii) in (i), we get

$$\frac{x^2}{16a^2}=4ax \quad \Rightarrow \quad x(x^3-64a^3)=0$$

$\Rightarrow$ x = 0, 4a

from (ii), y = 0, 4a. Let A ≡ (0, 0); B ≡ (4a, 4a)

Since, given line 2bx + 3cy + 4d = 0 passes through A and B,

∴ d = 0 and 8ab + 12ac = 0

$\Rightarrow$ 2b + 3c = 0. (∵ a ≠ 0)

Obviously, $d^2 + (2b + 3c)^2 = 0$

5. $y=\frac{a^3x^2}{3}+\frac{a^2x}{2}-2a$

$$\Rightarrow y=\frac{a^3}{3}\left[x^2+\frac{x}{2}\times\frac{3}{a}\times\frac{2}{2}\right]-2a$$

$$\Rightarrow y=\frac{a^3}{3}\left[\left(x+\frac{3}{4a}\right)^2\right]-\frac{3a}{16}-2a$$

$$\Rightarrow y+\frac{35a}{16}=\frac{4a^3}{12}\left(x+\frac{3}{4a}\right)^2$$

∴ Vertices will be (α, β)

So that $\alpha=-\frac{3}{4a}$ and $\beta=-\frac{35a}{16}$

or $\alpha\beta=\left(\frac{-3}{4a}\right)\times\left(\frac{-35a}{16}\right)=\frac{105}{64}$

∴ Required locus will be $xy=\frac{105}{64}$

6. Point must be on the directrix of the parabola

Hence the point is (–2, 0)

8. Locus of point of intersection of perpendicular tangent is directrix of the parabola.

so x = –1

9. tangent of slope m of $y^2 = 4\sqrt{5}\,x$

is $y = mx + \dfrac{\sqrt{5}}{m}$

also tangent to $\dfrac{x^2}{5/2} + \dfrac{y^2}{5/2} = 1$

$\Rightarrow \dfrac{5}{m^2} = \dfrac{5}{2}m^2 + \dfrac{5}{2}$

$\Rightarrow 2 = m^4 + m^2$

$\Rightarrow m^4 + m^2 - 2 = 0$

$m = \pm 1$

which satisfy $m^4 - 3m^2 + 2 = 0$

which gives $y = x + \sqrt{5}$ as tangent

So I & II both are true.

11. $(2t^2, 4t)$ $(0, -6)$

$F(t) = 4t^2 + (4t + 6)^2$

$= 4(t^4 + 4t^2 + 9 + 12t)$

$= 4(t^4 + 4t^2 + 12t + 9)$

$F'(t) = 4(4t^3 + 8t + 12) = 0$

$\Rightarrow t^3 + 2t + 3 = 0$

$t = -1$

$x^2 + y^2 - 4x + 8y + 12 = 0$

Part # II : IIT-JEE ADVANCED

1. (a) The parabola is $y^2 = 4.\dfrac{k}{4}\left(x - \dfrac{8}{k}\right)$

Putting $y = Y$, $x - \dfrac{8}{k} = X$,

the equation $Y^2 = 4.\dfrac{k}{4}.X$

$\therefore$ The directrix is $X + \dfrac{k}{4} = 0$,

i.e. $x - \dfrac{8}{k} + \dfrac{k}{4} = 0$

But $x - 1 = 0$ is the directrix.

So, $\dfrac{8}{k} - \dfrac{k}{4} = 1 \quad \Rightarrow \quad k = -8, 4$

(b) Any normal is $y + tx = 6t + 3t^2$. It is identical

with $x + y = k$ if $\dfrac{t}{1} = \dfrac{1}{1} = \dfrac{6t + 3t^3}{k}$

$\therefore$ $t = 1$ and $1 = \dfrac{6+3}{k} \quad \Rightarrow \quad k = 9$

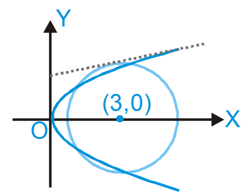
Aliter : $y = -x + k$

$\therefore$ $c = -[2am + am^3]$

$\Rightarrow$ $c = -[6(-1) + 3(-1)^3]$

$\therefore$ $c = \pm 9$

2. (a) Any tangent to $y^2 = 4x$ is $y = mx + \dfrac{1}{m}$.



It touches the circle, if $3 = \left|\dfrac{3m + \frac{1}{m}}{\sqrt{1 + m^2}}\right|$

or $9(1 + m^2) = \left(3m + \dfrac{1}{m}\right)^2$

or $\dfrac{1}{m^2} = 3$, $\quad \therefore \quad m = \pm\dfrac{1}{\sqrt{3}}$

For the common tangent to be above the x-axis, $m = \dfrac{1}{\sqrt{3}}$

$\therefore$ Common tangent is,

$y = \dfrac{1}{\sqrt{3}}x + \sqrt{3} \quad \Rightarrow \quad \sqrt{3}\,y = x + 3$

3. $\alpha = \dfrac{at^2 + a}{2}$, $\beta = \dfrac{2at + 0}{2}$

$\Rightarrow$ $2\alpha = at^2 + a$, $at = \beta$

$\therefore$ $2\alpha = a.\dfrac{\beta^2}{a^2} + a$ or $2a\alpha = \beta^2 + a^2$

$\therefore$ The locus is $y^2 = \dfrac{4a}{2}\left(x - \dfrac{a}{2}\right) = 4b(x - b)$, $\left(b = \dfrac{a}{2}\right)$

$\therefore$ Directrix is $(x - b) + b = 0$ or $x = 0$

4. The given curves are

$y^2 = 8x$ .....**(i)**

and $xy = -1$ .....**(ii)**

If m is the slope of tangent to (1), then equation of tangent is $y = mx + 2/m$.

If this tangent is also a tangent to (2), then

$$x\left(mx + \frac{2}{m}\right) = -1$$

$\Rightarrow$ $mx^2 + \frac{2}{m}x + 1 = 0$

$\therefore$ $m^2x^2 + 2x + m = 0$

We should get repeated roots for this equation (conditions of tangency)

$\Rightarrow$ $D = 0$

$\therefore$ $(2)^2 - 4m^2 . m = 0$

$\Rightarrow$ $m^3 = 1$

$\Rightarrow$ $m = 1$

Hence required tangent is $y = x + 2$.

6. Let P be the point (h, k). Then equation of normal to parabola $y^2 = 4x$ from point (h, k), if m is the slope of normal, is $y = mx - 2m - m^3 = 0$

As it passes through (h, k), therefore

$mh - k - 2m - m^3 = 0$

or $m^3 + (2 - h)m + k = 0$ .....**(i)**

which is cubic in m, giving three values of m say m_1, m_2 and m_3. Then $m_1m_2m_3 = -k$ (from equation)

but given that $m_1m_2 = \alpha$

$\therefore$ We get $m_3 = -\frac{k}{\alpha}$

But m_3 must satisfy equation **(i)**

$$\therefore \quad \frac{-k^3}{\alpha^3} + (2 - h)\left(\frac{-k}{\alpha}\right) + k = 0$$

$\Rightarrow$ $k^2 - 2\alpha^2 - h\alpha^2 - \alpha^3 = 0$

$\therefore$ Locus of P(h, k) is $y^2 = \alpha^2x + (\alpha^3 - 2\alpha^2)$

But ATQ, locus of P is a part of parabola $y^2 = 4x$, therefore comparing the two, we get $\alpha^2 = 4$ and $\alpha^3 - 2\alpha^2 = 0$

$\Rightarrow$ $\alpha = 2$

8. The given equation of parabola is

$y^2 - 2y - 4x + 5 = 0$ .....**(i)**

$\Rightarrow$ $(y - 1)^2 = 4(x - 1)$

Any parametric point on this parabola is

$P(t^2 + 1, 2t + 1)$

Differentiating **(i)** w.r.t. x, we get

$$2y\frac{dy}{dx} - 2\frac{dy}{dx} - 4 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{y-1}$$

$\therefore$ Slope of tangent to **(i)** at pt.

$P(t^2 + 1, 2t + 1)$ is $m = \frac{2}{2t} = \frac{1}{t}$

$\therefore$ Equation of tangent at $P(t^2 + 1, 2t + 1)$ is

$$y - (2t + 1) = \frac{1}{t}(x - t^2 - 1)$$

$\Rightarrow$ $yt - 2t^2 - t = x - t^2 - 1$

$\Rightarrow$ $x - yt + (t^2 + t - 1) = 0$ .....**(ii)**

Now direction of given parabola is

$(x - 1) = -1 \Rightarrow x = 0$

Tangent to (2) meets directrix at $Q\left(0, \frac{t^2 + t - 1}{t}\right)$

Let pt. R be (h, k)

ATQ R divides the line joining QP in the ratio $\frac{1}{2} : 1$ i.e. 1 : 2 externally.

$$\therefore \quad (h, k) = \left[\frac{1(1 + t^2) - 0}{-1}, \frac{t + 2t^2 - 2t^2 - 2t + 2}{-t}\right]$$

$\Rightarrow$ $h = -(1 + t^2)$ and $k = \frac{t-2}{t}$

$\Rightarrow$ $t^2 = -1 - h$ and $t = \frac{2}{1-k}$

Eliminating t we get $\left(\frac{2}{1-k}\right)^2 = -1 - h$

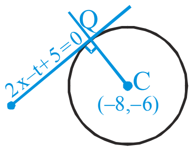
$\Rightarrow$ $4 = -(1 - k)^2(1 - h)$

$\Rightarrow$ $(h - 1)(k - 1)^2 + 4 = 0$

$\therefore$ locus of R(h, k) is, $(x - 1)(y - 1)^2 + 4 = 0$

9. The given curve is $y = x^2 + 6$

Equation of tangent at (1, 7) is



$$\frac{1}{2}(y + 7) = x.1 + 6$$

$\Rightarrow$ $2x - y + 5 = 0$ .....**(i)**

ATQ this tangent (1) touches the circle

$x^2 + y^2 + 16x + 12y + C = 0$

at Q. (centre of circle (– 8, – 6)).

Then equation of CQ which is perpendicular to **(i)** and passes through (– 8, – 6) is

$$y + 6 = -\frac{1}{2}(x + 8)$$

$\Rightarrow$ $x + 2y + 20 = 0$ .....**(ii)**

Now Q is pt. of intersection of (1) and (2) i.e. $x = -6, y = -7$

$\therefore$ Req. pt. is (– 6, – 7).

13. Without loss of generality we can assume the square ABCD with its vertices A(1, 1), B(–1, 1), C(–1,–1), D(1,–1)

P to be the point (0, 1) and Q as ($\sqrt{2}$, 0)

Then, $\dfrac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2}$

$$= \frac{1+1+5+5}{2[(\sqrt{2}-1)^2 + 1] + 2((\sqrt{2}+1)^2 + 1)} = \frac{12}{16} = 0.75$$

14. Let C' be the said circle touching C_1 and L, so that C_1 and C' are on the same side of L. Let us draw a line T parallel to L at a distance equal to the radius of circle C_1, on opposite side of L.

Then for N, centre of circle C', MN = NO

$\Rightarrow$ N is equidistant from a line and a point

$\Rightarrow$ locus of N is a parabola.

15. Since S is equidistant from A and line BD, it traces a parabola. Clearly AC is the axis, A(1, 1) is the focus and $T_1\left(\frac{1}{2}, \frac{1}{2}\right)$ is the vertex of parabola,

$AT_1 = \frac{1}{\sqrt{2}}$.

T_2T_3 = latus rectum of parabola $= 4 \times \frac{1}{\sqrt{2}} = 2\sqrt{2}$

$\therefore$ Area $(\Delta T_1\, T_2\, T_3) = \frac{1}{2} \times 2\sqrt{2} = \frac{1}{2}$ = 1 sq. units.

16. $$\frac{Ar\Delta PQS}{Ar\Delta PQR} = \frac{\frac{1}{2}QP \times ST}{\frac{1}{2}PQ \times TR} = \frac{ST}{TR} = \frac{2}{8} = \frac{1}{4}$$

17. For ΔPRS,

$Ar(\Delta PRS) = \Delta = \frac{1}{2} \times SR \times PT = \frac{1}{2} \times 10 \times 2\sqrt{2}$

$\therefore$ $\Delta = 10\sqrt{2}$, $a = PS = 2\sqrt{3}$,

$b = PR = 6\sqrt{2}$, $c = SR = 10$

$\therefore$ radius of circumference

$$= R = \frac{abc}{4\Delta} = \frac{2\sqrt{3} \times 6\sqrt{2} \times 10}{4 \times 10\sqrt{2}}\ 3\sqrt{3}$$

18. Radius of incircle

$$= \frac{\text{area of } \Delta PQR}{\text{semiperimeter of} \Delta PQR} = \frac{\Delta}{s}$$

We have $a = PR = 6\sqrt{2}$, $b = QP = PR = 6\sqrt{2}$

$c = PQ = 4\sqrt{2}$

and $\Delta = \frac{1}{2} \times PQ \times TR = 16\sqrt{2}$

$\therefore$ $s = \dfrac{6\sqrt{2} + 6\sqrt{2} + 4\sqrt{2}}{2} = 8\sqrt{2}$

$\therefore$ $r = \dfrac{16\sqrt{2}}{8\sqrt{2}} = 2$

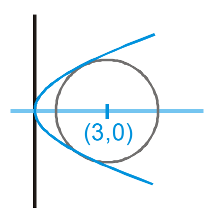
20. $C_1 : y^2 = 4x$

$C_2 : x^2 + y^2 - 6x + 1 = 0$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$ $\Rightarrow$ $x = 1$

$y = \pm 2$

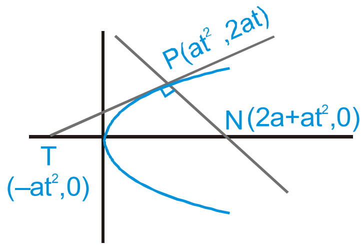


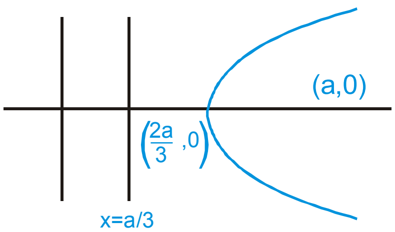
so the curves touches each other at two points (1, 2) & (1, –2)

21. $3h = 2a + at^2$

$3k = 2at$

$3h = 2a + \dfrac{a \cdot 9k^2}{4a^2}$





$y^2 = \dfrac{4a}{9}(3x - 2a)$

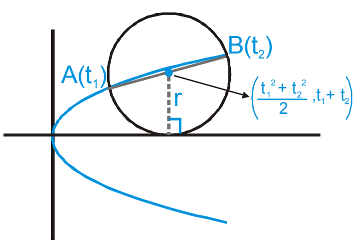
$y^2 = \dfrac{4a}{3}\left(x - \dfrac{2a}{3}\right)$

22. $t_1 + t_2 = r$

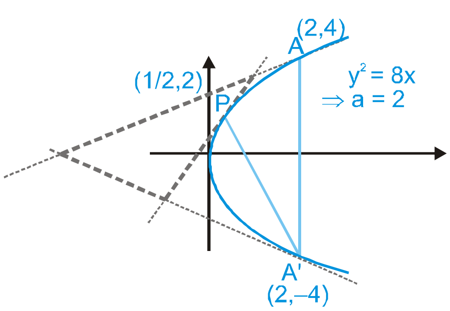
$\dfrac{2}{r} = \dfrac{2}{t_1 + t_2}$

similarly $-\dfrac{2}{r}$ is

also possible



23.



Δ_1 = area of Δ PAA' $= \dfrac{1}{2}.8.\dfrac{3}{2} = 6$

$\Delta_2 = \dfrac{1}{2}(\Delta_1)$

(Using property : Area of triangle formed by tangents is always half of original triangle)

$\Rightarrow \dfrac{\Delta_1}{\Delta_2} = 2$

24. Let P be (h, k)

on using section formula $P\left(\dfrac{x}{4}, \dfrac{y}{4}\right)$

$\therefore$ $h = \dfrac{x}{4}$ and $k = \dfrac{y}{4}$

$\Rightarrow$ x = 4h and y = 4k

$\because$ (x, y) lies on $y^2 = 4x$

$\therefore$ $16k^2 = 16h$ $\Rightarrow$ $k^2 = h$

Locus of point P is $y^2 = x$.

25. Equation of normal is $y = mx - 2m - m^3$

It passes through the point (9, 6) then

$6 = 9m - 2m - m^3$

$\Rightarrow$ $m^3 - 7m + 6 = 0$

$\Rightarrow$ $(m - 1)(m - 2)(m + 3) = 0$

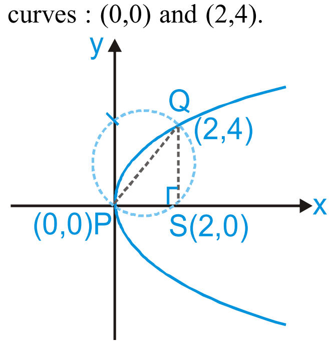
$\Rightarrow$ m = 1, 2, –3

Equations of normals are

y – x + 3 = 0, y + 3x – 33 = 0

& y – 2x + 12 = 0

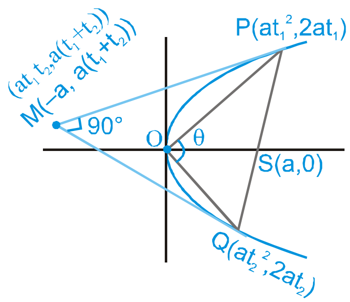
26. Focus of parabola S(2,0) points of intersection of given curves : (0,0) and (2,4).



Area (ΔPSQ) $= \dfrac{1}{2}.2.4 = 4$ sq. units

Paragraph for Question 27 and 28

27. Single tangent at the extrimities of a focal



chord will intersect on directrix.

∴ M(–a, a(t_1 + t_2))

lies on y = 2x + a

$a(t_1 + t_2) = -2a + a \Rightarrow t_1 + t_2 = -1$

& $t_1t_2 = -1$

$$\tan\theta = \left|\frac{\frac{2}{t_1} - \frac{2}{t_2}}{1 + \frac{4}{t_1t_2}}\right| = \left(\frac{2(t_2 - t_1)}{3}\right)$$

∵ $(t_2 - t_1)^2 = (t_2 + t_1)^2 - 4t_1t_2 = 5$

$t_2 - t_1 = \pm\sqrt{5}$

∴ $\tan\theta = \pm\frac{2\sqrt{5}}{3}$

but θ is obtuse because O is the interior point of the circle for which PQ is diameter.

∴ $\tan\theta = \frac{-2\sqrt{5}}{3}$

28. Length of focal chord

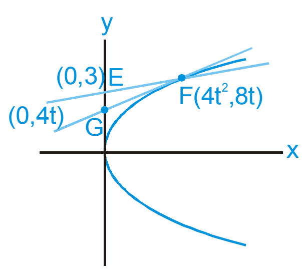
PQ $= a(t_1 - t_2)^2$

$= a[(t_1 + t_2)^2 - 4t_1t_2]$

$= a[1 + 4] = 5a$

29. Let F($4t^2$, 8t)

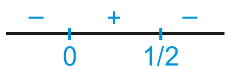
where $0 \le 8t \le 6 \Rightarrow 0 \le t \le \frac{3}{4}$



$$\Delta EFG = \frac{1}{2}(3 - 4t)4t^2$$

$$\Delta = (6t^2 - 8t^3)$$

$$\frac{d\Delta}{dt} = 12t - 24t^2 = 0 \quad \begin{cases} t = 0\,(\text{minima}) \\ t = \frac{1}{2}\,(\text{maxima}) \end{cases}$$



$$\Rightarrow m = \frac{8t - 3}{4t^2 - 0} = \frac{4 - 3}{1} = 1$$

$$(\Delta EFG)_{max} = \frac{6}{4} - 1 = \frac{1}{2}$$

$y_0 = 8t = 4$ & $y_1 = 4t = 2$

35. $x^2 + y^2 = 3$

$x^2 = 2y$

Intersection point is $P \equiv \left(\sqrt{2}, 1\right)$

Equation of tangents is $\sqrt{2}x + y = 3$

$\tan(\theta) = -\sqrt{2}$

$\tan(\alpha) = \tan(\theta - 90) = -\cot\theta = \frac{1}{\sqrt{2}}$

$\sin(\alpha) = \frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{Q_3T}$

$\Rightarrow Q_3T = 6$

∴ $Q_2Q_3 = 2Q_3T = 12$

$\tan(\alpha) = \frac{1}{\sqrt{2}} = \frac{2\sqrt{3}}{R_3T} \Rightarrow R_3T = 2\sqrt{6}$

∴ $R_2R_3 = 2R_3T = 4\sqrt{6}$

⊥ distance of o from R_2R_3 is $\left|\frac{3}{\sqrt{\left(\sqrt{2}\right)^2 + 1^2}} = \sqrt{3}\right|$

∴ Area (OR2R3) $= \frac{1}{2} \times \sqrt{3} \times 4\sqrt{6} = 6\sqrt{2}$ square units

Similarly, Area (PQ2Q3) $= \frac{1}{2} \times \sqrt{2} \times 12 = 6\sqrt{2}$ square units

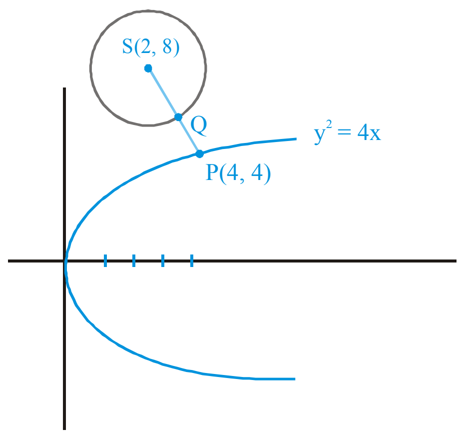
36. $x^2 + y^2 - 4x - 16y + 64 = 0$

Centre $S \equiv (2, 8)$

$r = \sqrt{4 + 64 - 64} = 2$

Normal $\quad y = mx - 2m - m^3$

As shortest distance $\Rightarrow$ common normal



$\Rightarrow$ It passes S(2, 8)

$\Rightarrow$ $8 = 2m - 2m - m^3$

$\Rightarrow$ $m = -2$

Normal at P $\; y = -2x + 12$

Point $P \equiv (am^2, -2am) \equiv (4, 4)$

$SP = \sqrt{(4-2)^2 + (8-4)^2} = 2\sqrt{5}$

$SQ : QP = 2 : \left(2\sqrt{5} - 2\right)$

Slope of tangent at Q is $= \dfrac{1}{2}$

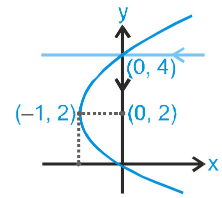
MOCK TEST

1. **(A)**

Given curve is $\quad (y-2)^2 = 4(x+1)$

focus $\quad (x + 1 = 1, y - 2 = 0) \;\Rightarrow\; (0, 2)$

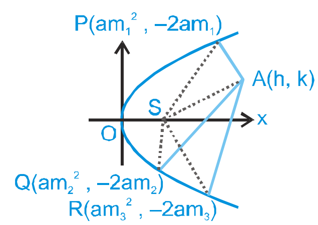
Point of intersection of the curve and



y = 4 is (0,4) from the reflection property of parabola reflected ray passes through the focus.

$\therefore \quad x = 0$ is required line

3. **(C)**



$|SP|. |SQ|. |SR|$

$= a^3 (1 + m_1^2)(1 + m_2^2)(1 + m_3^2)$

$= a^3 \left| 1 + (\Sigma m_1)^2 - 2\Sigma m_1 m_2 + (\Sigma m_1 m_2)^2 - 2m_1 m_2 m_3 \Sigma m_1 + (m_1 m_2 m_3)^2 \right|$

$$= a^3 \left| 1 + 0 + \frac{2(h-2a)}{a} + \frac{(h-2a)^2}{a^2} - 0 + \frac{k^2}{a^2} \right|$$

$$= a\left|k^2 + (h-a)^2\right| = a\,(SA)^2$$

5. **(C)**

Since (4, – 4) and (9, 6) lie on $y^2 = 4a(x - b)$

$\therefore \quad 4 = a(4 - b)$ and $9 = a(9 - b)$

$\therefore \quad a = 1$ and $b = 0$

$\therefore \quad$ the parabola is $y^2 = 4x$

let the point R be $(t^2, 2t)$

$\therefore \quad$ Area of Δ PRQ

$$= \frac{1}{2}\begin{vmatrix} 4 & -4 & 1 \\ 9 & 6 & 1 \\ t^2 & 2t & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 4 & -4 & 1 \\ 5 & 10 & 0 \\ t^2-4 & 2t+4 & 0 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 5 & 10 \\ t^2-4 & 2t+4 \end{vmatrix}$$

$$= \frac{1}{2}(10t + 20 - 10t^2 + 40)$$

$$= -5t^2 + 5t + 30 = -5(t^2 - t + \frac{1}{4}) + 30 + \frac{5}{4}$$

$$= -5\left(t - \frac{1}{2}\right)^2 + \frac{125}{4}$$

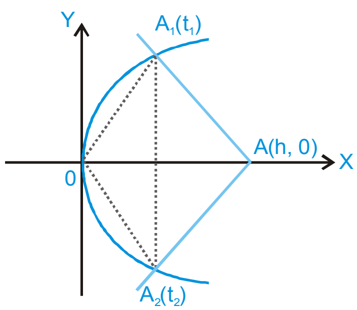
$\therefore$ Area in largest when $t = \frac{1}{2}$

7. (C)

$\because$ OA_1A_2 is equilateral triangle

$\therefore$ $\angle A_1OA = \frac{\pi}{6}$

$\Rightarrow$ Slope of $OA_1 = \frac{1}{\sqrt{3}}$.



$\Rightarrow$ $\frac{2}{t_1} = \frac{1}{\sqrt{3}}$ $\Rightarrow$ $t_1 = 2\sqrt{3}$.

$\because$ equation of normal at A_1 is $y = -t_1x + 4t_1 + 2t_1^3$

$\because$ it passes through A(h, 0)

$\Rightarrow$ $h = 4 + 2t_1^2$ $\Rightarrow$ $h = 4 + 2\times4\times3 = 28$

$\therefore$ Coordinates of the point R are $(\frac{1}{4}, 1)$

9. (D)

S_1 : Tangent at A $(at^2, 2at)$ is $y = \frac{x}{t} + at$

This intersect the x - axis at $(-at^2, 0)$ and foot of $\perp$ from A on the x - axis is $(at^2, 0)$ clearly origin is the mid point.

$\therefore$ S1 is true.

S_2 : Equation of normal at $(at^2, 2at)$ is $y = -tx + 2at + at^3$

it intersect x - axis at $(2a + at^2, 0)$

$\therefore$ subnormal = 2a

$\therefore$ S2 is False.

S_3 : Let A $(at^2, 2at)$ be a point on the parabola $y^2 = 4ax$. S (a, 0) be the focus equation of circle having AS as diameter is $(x - at^2)(x - a) + y(y - 2at) = 0$ and tangent at the vertex to the parabola is $x = 0$.

It can be easily checked that $x = 0$ touches this circle.

$\therefore$ S3 is true.(by D = 0)

S_4 : equation of such circle is

$$(x - at^2)\left(x - \frac{a}{t^2}\right) + (y - 2at)\left(y + \frac{2a}{t}\right) = 0$$

Directrix $x = -a$ which is tangent.

$\therefore$ S4 is true.

11. (A, B, C, D)

$R \equiv (1, 1)$

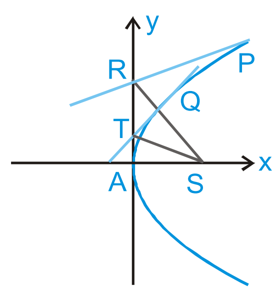
$T \equiv (2, 2)$

Equation of RS is $4x - 3y - 1 = 0$

Equation of TS is $3x - 2y - 2 = 0$

$\therefore$ focus $S \equiv (4, 5)$

length of latus rectum $= 4 \times \frac{1}{\sqrt{2}} = 2\sqrt{2}$



axis is $x + y - 9 = 0$

vertex $\equiv \left(\frac{9}{2}, \frac{9}{2}\right)$

13. (A)

$\because$ A $(r_1 \cos\theta, r_1 \sin\theta)$ lies on the parabola

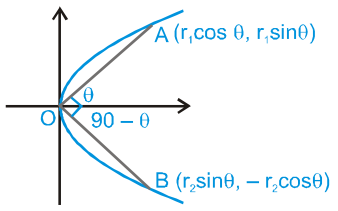
$\therefore$ $r_1^2 \sin^2\theta = 4ar_1 \cos\theta$ ......**(i)**

and $\because$ B $(r_2 \sin\theta, -r_2 \cos\theta)$ also lies on the parabola

$\therefore$ $r_2^2 \cos^2\theta = 4ar_2 \sin\theta$ ......**(ii)**

from **(i)** $r_1^2 \sin^4\theta = 16a^2 \cos^2\theta$ ......**(iii)** from **(iii)** and **(ii)**

$$\sin^3\theta = \frac{64a^3}{r_1^2 r_2} \qquad \text{......(iv)}$$



$$\text{Similarly } \cos^3\theta = \frac{64a^3}{r_1 r_2^2} \qquad \text{......(v)}$$

15. (A,B,C,D)

Obvious

16. (A)

Statement- 1 : $y^2 = 4x$

Clearly $x = 0$ is tangent to the parabola at (0,0)

$y^2 = 4(-y-1) \Rightarrow (y+2)^2 = 0$

$\therefore$ $x + y + 1 = 0$ is a tangent to the parabola

Again $y^2 = 4(y-1)$ i.e. $y^2 - 4y + 4 = 0$ i.e. $(y-2)^2 = 0$

$\therefore$ $x - y + 1 = 0$ is a tangent to the parabola

(1, 0) is the focus.

$\therefore$ Statement- 1 true

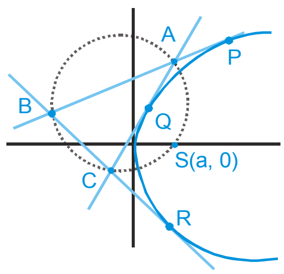
Consider a parabola $y^2 = 4ax$

Let $P(t_1)$, $Q(t_2)$ and $R(t_3)$ be these point on it.

Tangents are drawn at these points which intersect at $A \equiv (a t_1 t_2, a(t_1 + t_2))$

$B \equiv (a t_1 t_3, a(t_1 + t_3))$

$C \equiv (a t_2 t_3, a(t_2 + t_3))$



Let $\angle SAC = \alpha$ & $\angle SBC = \beta$

$$\Rightarrow \tan\alpha = \left|\frac{\frac{1}{t_2} - \frac{t_1+t_2}{t_1t_2-1}}{1+\frac{1}{t_2}\left(\frac{t_1+t_2}{t_1t_2-1}\right)}\right| = \left|\frac{1}{t_1}\right|$$

$$\text{similarly } \tan\beta = \left|\frac{1}{t_1}\right|$$

$\Rightarrow$ $\alpha = \beta$ or $\alpha + \beta = \pi$

$\Rightarrow$ A, B, C and S are concyclic.

17. Given $C : (y-1)^2 = 8(x+2)$ (which is a parabola.)

Clearly $P(-4, 1)$ lies on directrix $x = -4$.

Also $P(-4, 1)$ lies on axis of parabola i.e., $y = 1$.

So from any point on directrix of parabola, if two tangents are drawn to the parabola then these two tangents will be mutually perpendicular.

18. (C)

Statement- 2 : Area of triangle formed by these tangents and their corresponding chord of contact is

$$\frac{(y_1^2 - 4ax_1)^{\frac{3}{2}}}{2|a|}$$

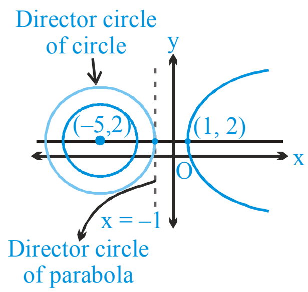
$\therefore$ Statement is false.

Statement - 1 : $x_1 = 12$, $y_1 = 8$

$$\text{Area} = \frac{\left(y_1^2 - 4x_1\right)^{3/2}}{2} = \frac{(64-48)^{3/2}}{2} = 32$$

Statement is true.

19. Equation of director circle of $(x+5)^2 + (y-2)^2 = 8$ will be $(x+5)^2 + (y-2)^2 = 16$ and of $(y-2)^2 = 8(x-1)$ is $x = -1$.



Clearly the line $x = -1$ touches $(x+5)^2 + (y-2)^2 = 16$

Hence only one such point exist.

20. (D)

STATEMENT-I is false

since here $t^2 = 4$

$\therefore$ the normal subtends a right angle at the focus (not on the vertex)

STATEMENT-II true (A standared result)

21. (A) → (q), (B) → (r), (C) → (s), (D) → (p)

22. (A) → (s), (B) → (q), (C) → (s), (D) → (r)

(A) Point (a, a) lies on $y^2 = 4x$

$\therefore$ $a^2 = 4a$ i.e. $a = 0, 4$

$\therefore$ $a = 4$

(B) The line $3x - y + 8 = 0$ passes through the focus (– 2, 2) so the tangents at the end points on the chord is $\frac{\pi}{2}$ $\Rightarrow$ $p = 8$

(C) $y^2 = k(x - 8/k)$

equation of directrix is $x - \frac{8}{k} = -\frac{k}{4} \Rightarrow x = \frac{8}{k} - \frac{k}{4}$

compare with $x = 1 \Rightarrow \frac{8}{k} - \frac{k}{4} = 1 \Rightarrow k = 4$

(D) end points of the normal chord will be (8, 8) &

$\left(2\left(\frac{-5}{2}\right)^2, 2.\,2.\left(\frac{-5}{2}\right)\right)$

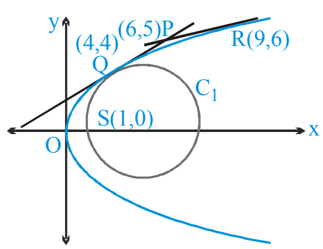
$\therefore$ length of the chord will be $= 10\sqrt{5}$

23. Equation of tangent of slope m to $y^2 = 4x$ is

$y = mx + \frac{1}{m}$ ....**(i)**

1. As (1) passes through P(6, 5), so

$5 = 6m + \frac{1}{m}$



$\Rightarrow$ $6m^2 - 5m + 1 = 0$ $\Rightarrow$ $m = \frac{1}{2}$ or $m = \frac{1}{3}$

Points of contact are $\left(\frac{1}{m_1^2}, \frac{2}{m_1}\right)$ and $\left(\frac{1}{m_2^2}, \frac{2}{m_2}\right)$

Hence P (4, 4) and Q (9, 6)

Area of $\Delta PQR = \frac{1}{2}\begin{vmatrix} 6 & 5 & 1 \\ 4 & 4 & 1 \\ 9 & 6 & 1 \end{vmatrix} = \frac{1}{2}$ $\Rightarrow$ **(A)**

2. $y = \frac{1}{2}x + 2 \Rightarrow x - 2y + 4 = 00$ ....**(ii)**

and $y = \frac{1}{3}x + 3 \Rightarrow x - 3y + 9 = 0$

Now equation of circle C_2 touching $x - 3y + 9 = 0$ at (9, 6), is

$(x-9)^2 + (y-6)^2 + \lambda(x - 3y + 9) = 0$

As above circle passes through (1, 0), so

$64 + 36 + 10\lambda = 0 \Rightarrow \lambda = -10$

Circle C_2 is $x^2 + y^2 - 28x + 18y + 27 = 0$ ....**(iii)**

Radius of C_2 is

$r_2^2 = 196 + 81 - 27 = 277 - 27 = 250$

$\Rightarrow$ $r_2 = 5\sqrt{10}$ $\Rightarrow$ **(B)**

3. Equation of C_1

$(x-4)^2 + (y-4)^2 + \lambda(x - 2y + 4) = 0$

As above circle passes through (1, 0)

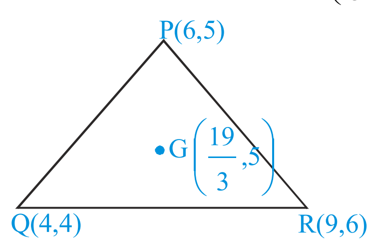
$9 + 16 + \lambda(5) = 0 \Rightarrow \lambda = -5$

Now C_1 is $x^2 + y^2 - 13x + 2y + 12 = 0$ ....**(iv)**

$\therefore$ Common chord of **(iii)** and **(v)** is

$15x - 16y - 15 = 0$ ....**(v)**

Also centroid (G) of ΔPQR is $\left(\frac{19}{3}, 5\right)$



Clearly $\left(\frac{19}{3}, 5\right)$ satisfies equation **(v)**

Hence **(C)**

24.

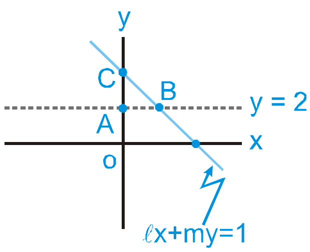
1 **(C)**

2 **(D)**

3 **(B)**

$\because\ C \equiv \left(0, \frac{1}{m}\right)$

$B \equiv \left(\frac{1-2m}{\ell}, 2\right)$, A (0, 2)



Let (h, k) be the circumcentre of ΔABC

$\therefore\ h = \frac{1-2m}{2\ell}$; $k = \frac{1+2m}{2m}$

$\therefore\ 2h = \frac{1-2m}{\ell}$; $k = 1 + \frac{1}{2m}$

$\therefore\ m = \frac{1}{2k-2}$; $\ell = \frac{k-2}{2h(k-1)}$

$\because$ (ℓ, m) lies on $y^2 = 4ax$

$\therefore$ $m^2 = 4a\ell$

$\Rightarrow \left(\frac{1}{2k-2}\right)^2 = 4a\left\{\frac{k-2}{2h(k-1)}\right\}$

$h = 8a(k^2 - 3k + 2)$

$\therefore$ locus of (h, k) is

$x = 8a(y^2 - 3y + 2)$

$\Rightarrow \left(y - \frac{3}{2}\right)^2 = \frac{1}{8a}(x + 2a)$

$\therefore$ vertex is $\left(-2a, \frac{3}{2}\right)$

$\because$ Length of smallest focal chord = length of latus rectum $= \frac{1}{8a}$

From the equation of curve C it is clear that it is symmetric about line $y = \frac{3}{2}$.

25.

1. (A)

minimum distance is along common normal

so firstly find out the common normal

Length of LR $4a = 1$ $\therefore$ $a = \frac{1}{4}$

Equation of normal for $y^2 = x - 1$ at point P

$y = mx - \frac{3m}{2} - \frac{m^3}{4}$ ....**(i)**

Equation of normal for $x^2 = y - 1$ at point Q

$y = mx + \frac{3}{2} + \frac{1}{4m^2}$ ....**(ii)**

(i) and **(ii)** are similar so compare coefficient

$1 = \dfrac{\frac{-3m}{2} - \frac{m^3}{4}}{\frac{3}{2} + \frac{1}{4m^2}} \Rightarrow \frac{3}{2} + \frac{1}{4m^2} = -\frac{3}{2}m - \frac{m^3}{4}$

$\Rightarrow$ $m^5 + 6m^3 + 6m^2 + 1 = 0$

Its one root is $m = -1$ and remaining roots are imaginary

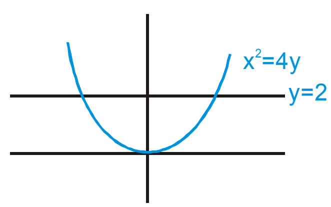
so coordinates of $P \equiv \left(1 + \frac{1}{4}, 2 \times \frac{1}{4}\right) \equiv \left(\frac{5}{4}, \frac{1}{2}\right)$

$Q \equiv \left(-\dfrac{2 \times \frac{1}{4}}{-1}, 1 + \frac{1}{4}\right) = \left(\frac{1}{2}, \frac{5}{4}\right)$

Distance $= \sqrt{\left(\frac{5}{4} - \frac{1}{2}\right)^2 + \left(\frac{5}{4} - \frac{1}{2}\right)^2}$ is $\left(\frac{5}{4} - \frac{1}{2}\right)\sqrt{2}$

$= \frac{5-2}{4}\sqrt{2}$ is $\frac{3\sqrt{2}}{4}$

2. (B)



Let any point on parabola $x^2 = 4y \Rightarrow (2t, t^2)$

slope of tangent is 't'

slope of normal is $-\frac{1}{t}$

equation of normal

$t^3 - (y-2)t - x = 0$ (roots t_1, t_2, t_3) ....**(i)**

$t_1 + t_2 + t_3 = 0$, $t_1.t_2.t_3 = x$

solving y = 2 and equation (i), then we get

$x = t^3$

t_1^3, t_2^3, t_3^3 are in A.P.

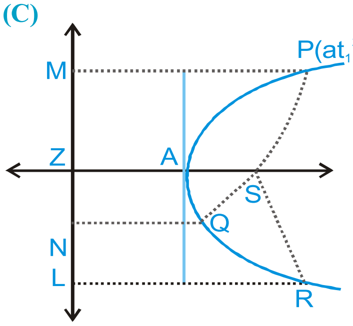
$2t_2^3 = t_1^3 + t_3^3$

$= (t_1 + t_3)^3 - 3t_1t_3(t_1 + t_3) = (-t_2)^3 - 3t_1t_3(-t_2)$

$3t_2^3 = 3t_1t_2t_3 \Rightarrow t_2^2 = t_1t_3$

t_1, t_2, t_3 are in G.P.

3. (C)



$= |SP|.|SQ|.|SR| = |PM|.|QN|.|RL|$

$= (a + at_1^2)(a + at_2^2)(a + at_3^2)$

$= a^3((1 + t_1^2)(1 + t_2^2)(1 + t_3^2)$

$= a^3[(1 + t_1^2 + t_2^2 + t_1^2t_2^2](1 + t_3^2))$

$= a^3\left[1 + \sum t_1^2 + \sum (t_1 t_2)^2 + t_1^2t_2^2t_3^2\right]$

$= a^3$

$$\left[1 + \left(\sum t_1\right)^2 - 2\left(\sum t_1 t_2\right) + \left(\sum t_1 t_2\right)^2 - 2t_1t_2t_3\left(\sum t_1\right) + (t_1t_2t_3)^2\right]$$

$$= a^3\left[1 + 0 - 2\left(\frac{2a - x}{a}\right) + \left(\frac{2a - x}{a}\right)^2 + 0 + \frac{y^2}{a^2}\right]$$

$$= a^3\left[1 - \frac{2}{a}(2a - x) + \frac{(2a - x)^2}{a^2} + \frac{y^2}{a^2}\right]$$

$= a[a^2 - 4a^2 + 2ax + 4a^2 - 4ax + x^2 + y^2] = a[(x - a)^2 + y^2]$

$= a(SO)^2$

26. for $y^2 = 4ax$ equation of normal $y = mx - 2am - am^3$ ....(i)

for $y^2 = 4(a - 1)(x - b)$

equation of normal $y = m(x - b) - 2(a - 1)m - (a - 1)m^3$ ....(ii)

for common – Normal equation (i) & (ii) are same, so compare the coeff.

$$1 = \frac{2am + am^3}{mb + 2cm + cm^2} \qquad \text{where } c = (a - 1)$$

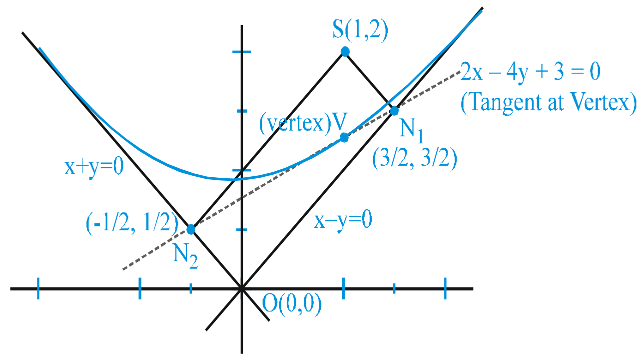
$$\Rightarrow 1 = \frac{2a + am^2}{b + 2(a - 1) + (a - 1)m^2}$$

$\Rightarrow b + 2a - 2 + am^2 - m^2 = 2a + am^2$

$\Rightarrow m^2 = b - 2$

$b - 2 > 0 \Rightarrow b > 2$

27.



Feet of the perpendicular (N_1 and N_2) from focus upon any tangent to parabola lies on the tangent line at the vertex.

Now equation of SN_1 is $x + y = \lambda$ passing through (1, 2)

$\Rightarrow \lambda = 3$

Equation of SN_1 is $x + y = 3$

Solving $x + y = 3$ and $y = x$, we get $N_1 \equiv \left(\frac{3}{2}, \frac{3}{2}\right)$

|||ly equation of SN_2 is $x - y = \lambda$ passing through (1, 2)

$\Rightarrow \lambda = -1$

Equation of SN_2 is $y - x = 1$

Solving $y - x = 1$ and $y = -x$, we get $N_2 \equiv \left(\frac{-1}{2}, \frac{1}{2}\right)$

Now equation of tangent line at vertex is,

$2x - 4y + 3 = 0$

Distance of S(1, 2) from tangent at vertex is

$$= \frac{|2 - 8 + 3|}{\sqrt{20}} = \frac{3}{2\sqrt{5}} = \frac{1}{4} \times \text{latus rectum}.$$

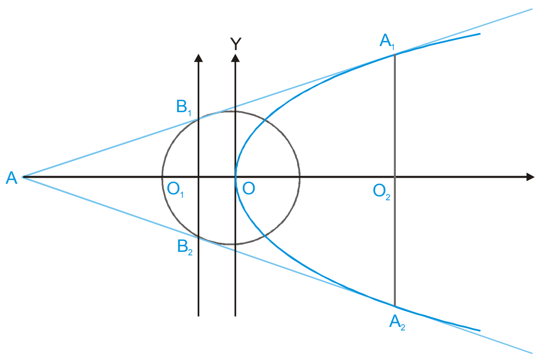
and hence length of latus rectum $= \frac{6}{\sqrt{5}} = \frac{m}{\sqrt{n}}$

Hence $m + n = 6 + 5 = 11$

29. Let a common tangent through A meet the circle at

$B_1\left(\frac{a}{\sqrt{2}}\cos\theta, \frac{a}{\sqrt{2}}\sin\theta\right)$ and the parabola at

A_1 $(at^2, 2at)$ (figure).



Equation of the tangent to the parabola at A_1 is

$t\,y = x + at^2$ ...... **(i)**

Equation of the tangent to the circle at

B_1 is $x\cos\theta + y\sin\theta = \frac{a}{\sqrt{2}}$ ..... **(ii)**

Since **(i)** and **(ii)** represent the same line.

$-\frac{1}{\cos\theta} = \frac{t}{\sin\theta} = \sqrt{2}\,t^2$ ..... **(iii)**

$\Rightarrow \frac{1}{2t^4} + \frac{1}{2t^2} = 1 \quad \Rightarrow \quad 1 + t^2 = 2t^4$

$\Rightarrow 2t^4 - t^2 - 1 = 0$

$\Rightarrow (t^2 - 1)(2t^2 + 1) = 0$

which gives two real values of t, equal to ± 1 giving two common tangents through A to the given circle and the parabola. Let the other common tangent meet the circle at B_2 and the parabola at A_2.

$\Rightarrow$ coordinate of A_1 are $(a, 2a)$ and coordinate of A_2 are $(a, -2a)$

$\Rightarrow A_1A_2 = 4a$.

From **(iii)** we get coordinate of B_1 are $\left(-\frac{a}{2}, \frac{a}{2}\right)$ and the coordinate of B_2 as $\left(-\frac{a}{2}, -\frac{a}{2}\right)$

$\Rightarrow B_1B_2 = a$.

The quadrilateral $A_1B_1B_2A_2$ formed by the common tangents and the chords of contact B_1B_2 of the circle and A_1A_2 of the parabola is a trapezium whose area.

$= \frac{1}{2}(A_1A_2 + B_1B_2) \times \left(\frac{a}{2} + a\right)$

$= \frac{1}{2} \times 5a \times \frac{3a}{2} = \frac{15a^2}{4}$.

30. Equation of any normal to the parabola $y^2 = 4ax$ is

$y = mx - 2am - am^3$ .....**(i)**

If the slope $m = \sqrt{2}$ then the equation **(i)** becomes

$y = \sqrt{2}x - 2a.\sqrt{2} - a(\sqrt{2})^3$

$\Rightarrow y = \sqrt{2}x - 2a\sqrt{2} - 2a\sqrt{2}$

$\Rightarrow y - x\sqrt{2} + 4a\sqrt{2} = 0$

So the given line is a normal to the parabola.

Length of chord $= \frac{4}{m^2}\sqrt{(1+m^2)(a-mc)}$

$= \frac{4}{2}\sqrt{(1+2)(a+8a)}$

$= 6\sqrt{3}a$