PARABOLA

INTRODUCTION

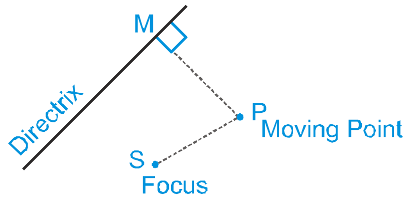
This chapter focusses on parabolic curves, which constitutes one category of various curves obtained by slicing a cone by a plane, called conic sections. A cone (not necessarily right circular) can be out in various ways by a plane, and thus different types of conic sections are obtained.

Let us start with the definition of a conic section and then we will see how are they obtained by slicing a right circular cone.

CONIC SECTIONS

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

- The fixed point is called the Focus.
- The fixed straight line is called the Directrix.
- The constant ratio is called the Eccentricity denoted by e.



$$\frac{PS}{PM} = e$$

- The line passing through the focus & perpendicular to the directrix is called the Axis.
- A point of intersection of a conic with its axis is called a Vertex.

General Equation of a Conic

If S is (p, q) & directrix is $\ell x + my + n = 0$

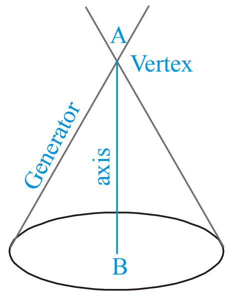
then $\quad PS = \sqrt{(x-\alpha)^2 + (y-\beta)^2}$ & $PM = \dfrac{|\ell x + my + n|}{\sqrt{\ell^2 + m^2}}$

$$\frac{PS}{PM} = e \Rightarrow (\ell^2 + m^2)\,[(x-p)^2 + (y-q)^2] = e^2\,(\ell x + my + n)^2$$

Which is of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

SECTION OF RIGHT CIRCULAR CONE BY DIFFERENT PLANES

A right circular cone is as shown in the figure – 1



Circular Base

Figure 1

(i) Section of a right circular cone by a plane passing through its vertex is a pair of straight lines passing through the vertex as shown in the figure - 2.

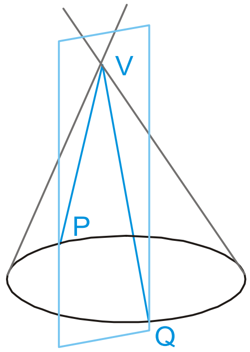


Figure –2

(ii) Section of a right circular cone by a plane parallel to its base is a circle as shown in the figure – 3.

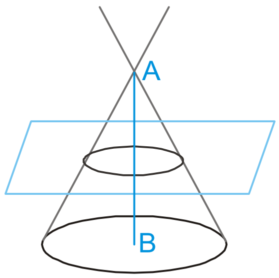


Figure- 3

(iii) Section of a right circular cone by a plane parallel to a generator of the cone is a parabola as shown in the figure–4.

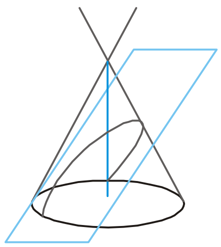


Figure-4

(iv) Section of a right circular cone by a plane neither parallel to any generator of the cone nor perpendicular or parallel to the axis of the cone is an ellipse or hyperbola as shown in the figure – 5 & 6.

Figure -5

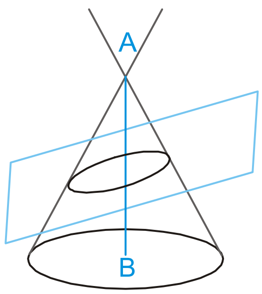
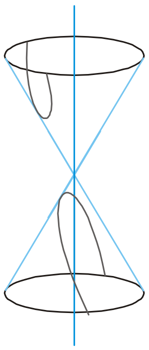
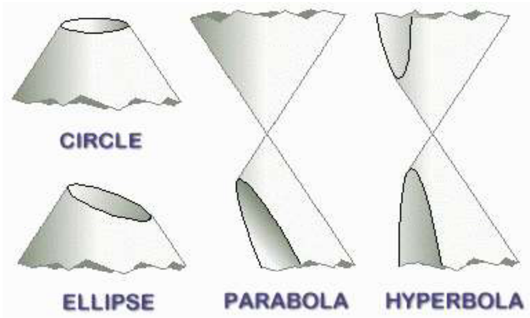


Figure -6



3D View



Distinguishing Various Conics

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (I) When The Focus Lies On The Directrix.

In this case $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines if:

$e > 1 \equiv h^2 > ab$ the lines will be real & distinct intersecting at S.

$e = 1 \equiv h^2 = ab$ the lines will coincident.

$e < 1 \equiv h^2 < ab$ the lines will be imaginary.

Case (II) When The Focus Does Not Lie On Directrix.

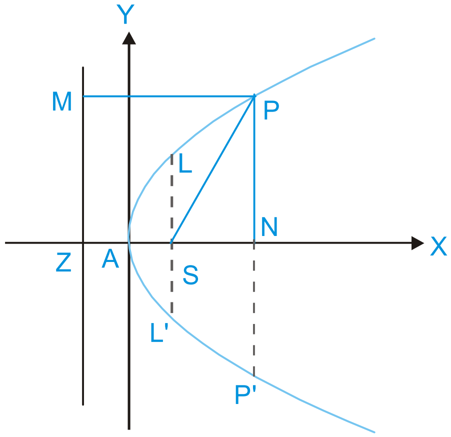
a parabola	an ellipse	a hyperbola	a rectangular hyperbola
$e = 1; \Delta \neq 0,$	$0 < e < 1; \Delta \neq 0;$	$e > 1; \Delta \neq 0;$	$e > 1; \Delta \neq 0$
$h^2 = ab$	$h^2 < ab$	$h^2 > ab$	$h^2 > ab; a + b = 0$

PARABOLA

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

Standard equation of a parabola is $y^2 = 4ax$. For this parabola :

(i) Vertex is (0, 0) **(ii)** Focus is (a, 0) **(iii)** Axis is $y = 0$ **(iv)** Directrix is $x + a = 0$



$y^2 = 4ax$

(a) **Focal distance :**

The distance of a point on the parabola from the focus is called the focal distance of the point.

(b) **Focal chord :**

A chord of the parabola, which passes through the focus is called a focal chord.

(c) **Double ordinate :**

A chord of the parabola perpendicular to the axis of the symmetry is called a double ordinate.

(d) **Latus rectum :**

A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the latus rectum. For $y^2 = 4ax$.

- ❖ Length of the latus rectum = 4a.
- ❖ Length of the semi latus rectum = 2a.
- ❖ Ends of the latus rectum are L(a, 2a) & L'(a, – 2a)

ETOOS KEY POINTS

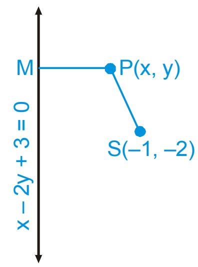
(i) Perpendicular distance from focus on directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

(iii) Two parabolas are said to be equal if they have the same latus rectum.

Ex. Find the equation of the parabola whose focus is at $(-1, -2)$ and the directrix is $x - 2y + 3 = 0$.

Sol. Let $P(x, y)$ be any point on the parabola whose focus is $S(-1, -2)$ and the directrix $x - 2y + 3 = 0$. Draw PM perpendicular to directrix $x - 2y + 3 = 0$. Then by definition,



$$SP = PM$$

$$\Rightarrow \quad SP^2 = PM^2$$

$$\Rightarrow \quad (x+1)^2 + (y+2)^2 = \left(\frac{x-2y+3}{\sqrt{1+4}}\right)^2$$

$$\Rightarrow \quad 5\left[(x+1)^2 + (y+2)^2\right] = (x-2y+3)^2$$

$$\Rightarrow \quad 5(x^2 + y^2 + 2x + 4y + 5) = (x^2 + 4y^2 + 9 - 4xy + 6x - 12y)$$

$$\Rightarrow \quad 4x^2 + y^2 + 4xy + 4x + 32y + 16 = 0$$

This is the equation of the required parabola.

Ex. Find the vertex, axis, directrix, focus, latus rectum and the tangent at vertex for the parabola $9y^2 - 16x - 12y - 57 = 0$.

Sol. The given equation can be rewritten as $\left(y - \frac{2}{3}\right)^2 = \frac{16}{9}\left(x + \frac{61}{16}\right)$ which is of the form $Y^2 = 4AX$.

Hence the vertex is $\left(-\frac{61}{16}, \frac{2}{3}\right)$

The axis is $y - \frac{2}{3} = 0 \quad \Rightarrow \quad y = \frac{2}{3}$

The directrix is $X + A = 0$

$$\Rightarrow \quad x + \frac{61}{16} + \frac{4}{9} = 0 \Rightarrow \quad x = -\frac{613}{144}$$

The focus is $X = A$ and $Y = 0$

$$\Rightarrow x + \frac{61}{16} = \frac{4}{9} \quad \text{and} \quad y - \frac{2}{3} = 0$$

$$\Rightarrow \quad \text{focus} = \left(-\frac{485}{144}, \frac{2}{3}\right)$$

Length of the latus rectum $= 4A = \frac{16}{9}$

The tangent at the vertex is $X = 0$

$$\Rightarrow \quad x = -\frac{61}{16}.$$

Ex. The length of latus rectum of a parabola, whose focus is $(2, 3)$ and directrix is the line $x - 4y + 3 = 0$ is -

Sol. The length of latus rectum $= 2 \times$ perp. from focus to the directrix

$$= 2 \times \left|\frac{2 - 4(3) + 3}{\sqrt{(1)^2 + (4)^2}}\right| = \frac{14}{\sqrt{17}}$$

PARAMETRIC REPRESENTATION

The simplest & the best form of representing the co–ordinates of a point on the parabola is $(at^2, 2at)$ i.e. the equations $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

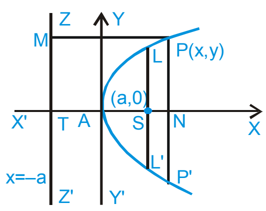
Parametric form for : $y^2 = -4ax$ $(-at^2, 2at)$

$x^2 = 4ay$ $(2at, at^2)$

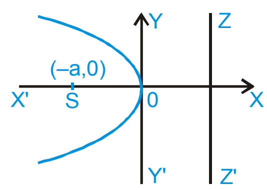
$x^2 = -4ay$ $(2at, -at^2)$

TYPE OF PARABOLA

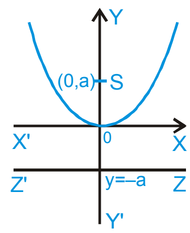
Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$



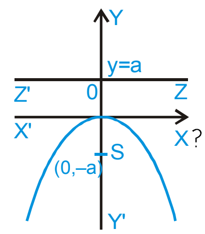
$y^2 = 4ax$



$y^2 = -4ax$



$x^2 = 4ay$



$x^2 = -4ay$

Parabola	Vertex	Focus	Axis	Directrix	Length of Latus rectum	Ends of Latus rectum	Parametric equation	Focal length
$y^2 = 4ax$	(0,0)	(a,0)	y= 0	x=–a	4a	(a, ±2a)	$(at^2,2at)$	x + a
$y^2 = -4ax$	(0,0)	(–a,0)	y = 0	x=a	4a	(–a, ±2a)	$(-at^2,2at)$	x–a
$x^2 = +4ay$	(0,0)	(0,a)	x = 0	y=–a	4a	(±2a, a)	$(2at,at^2)$	y+a
$x^2 = -4ay$	(0,0)	(0,–a)	x = 0	y=a	4a	(±2a, –a)	$(2at, -at^2)$	y–a
$(y-k)^2 = 4a(x-h)$	(h,k)	(h+a,k)	y = k	x+a– h =0	4a	(h+a, k±2a)	$(h+at^2,k+2at)$	x–h+a
$(x-p)^2 = 4b(y-q)$	(p,q)	(p, b+q)	x=p	y+b–q=0	4b	(p±2a,q+a)	$(p+2at,q+at^2)$	y–q+b

Ex. The extreme points of the latus rectum of a parabola are (7, 5) and (7, 3). Find the equation of the parabola.

Sol. Focus of the parabola is the mid-point of the latus rectum.

$\Rightarrow$ S is (7, 4). Also axis of the parabola is perpendicular to the latus rectum and passes through the focus. Its equation is

$$y - 4 = \frac{0}{5-3}(x-7) \Rightarrow y = 4$$

Length of the latus rectum = (5 – 3) = 2

Hence the vertex of the parabola is at a distance 2/4 = 0.5 from the focus. We have two parabolas, one concaveigh towards and the other concave leftwards.

The vertex of the first parabola is (6.5, 4) and its equation is $(y-4)^2 = 2(x-6.5)$ and it meets the x-axis at (14.5, 0).

The equation of the second parabola is $(y-4)^2 = -2(x-7.5)$. It meets the x-axis at (–0.5, 0).

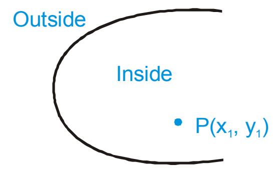
Ex. Find the parametric equation of the parabola $(x-1)^2 = -12(y-2)$

Sol. $\because$ $4a = -12$ $\Rightarrow$ $a = -3$, $y - 2 = at^2$

$x - 1 = 2at$ $\Rightarrow$ $x = 1 - 6t$, $y = 2 - 3t^2$

POSITION OF A POINT WITH RESPECT TO A PARABOLA

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.



$S_1 : y_1^2 - 4ax_1$ $\qquad$ $S_1 < 0 \to$ Inside $\qquad$ $S_1 > 0 \to$ Outside

Ex. Check whether the point (3, 4) lies inside or outside the paabola $y^2 = 4x$.

Sol. $y^2 - 4x = 0$

$\because$ $\quad S_1 \equiv y_1^2 - 4x_1 = 16 - 12 = 4 > 0$

$\therefore$ $\quad$ (3, 4) lies outside the parabola.

Ex. Find the value of α for which the point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$.

Sol. $\because$ $\quad$ Point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$

$\therefore$ $\quad y_1^2 - 4ax_1 < 0$

$\Rightarrow$ $\quad \alpha^2 - 4(\alpha - 1) < 0$

$\Rightarrow$ $\quad \alpha^2 - 4\alpha + 4 < 0$

$\quad (\alpha - 2)^2 < 0$

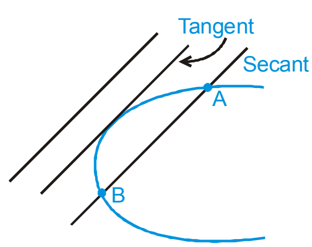
$\Rightarrow$ $\quad \alpha \in \phi$

LINE & A PARABOLA

The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a \gtreqless cm$

$\Rightarrow$ $\quad$ condition of tangency is, $c = a/m$.

Length of the chord intercepted by the parabola

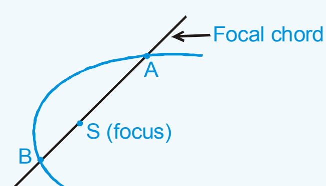


on the line $y = mx + c$ is :

$$\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}\,.$$

1. The equation of a chord joining t_1 & t_2 is $2x - (t_1 + t_2)\, y + 2\, at_1t_2 = 0$.

2. If t_1 & t_2 are the ends of a focal chord of the parabola $y^2 = 4ax$ then $t_1t_2 = -1$. Hence the co–ordinates at the extremities of a focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$



3. Length of the focal chord making an angle α with the x– axis is $4a\,\text{cosec}^2\,\alpha$.

Ex. Discuss the position of line $y = x + 1$ with respect to parabola $y^2 = 4x$.

Sol. Solving we get $(x + 1)^2 = 4x \Rightarrow (x - 1)^2 = 0$

so $y = x + 1$ is tangent to the parabola.

Ex. If the line $y = 3x + \lambda$ intersect the parabola $y^2 = 4x$ at two distinct points then set of values of λ is -

Sol. Putting value of y from the line in the parabola -

$(3x+\lambda)^2 = 4x$

$\Rightarrow \quad 9x^2 + (6\lambda - 4)x + \lambda^2 = 0$

$\because$ line cuts the parabola at two distinct points

$\therefore \quad D > 0$

$\Rightarrow \quad 4(3\lambda - 2)^2 - 4.9\lambda^2 > 0$

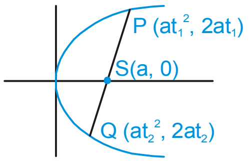
$\Rightarrow \quad 9\lambda^2 - 12\lambda + 4 - 9\lambda^2 > 0$

$\Rightarrow \quad \lambda < 1/3$

Hence, $\lambda \in (-\infty, 1/3)$

Ex. If t_1, t_2 are end points of a focal chord then show that $t_1 t_2 = -1$.

Sol. Let parabola is $y^2 = 4ax$



since P, S & Q are collinear

$\therefore \quad m_{PQ} = m_{PS}$

$$\Rightarrow \quad \frac{2}{t_1 + t_2} = \frac{2t_1}{t_1^2 - 1}$$

$\Rightarrow \quad t_1^2 - 1 = t_1^2 + t_1 t_2$

$\Rightarrow \quad t_1 t_2 = -1$

TANGENT TO THE PARABOLA $y^2 = 4ax$

(a) Point form

Equation of tangent to the given parabola at its point (x_1, y_1) is

$\mathbf{yy_1 = 2a(x + x_1)}$

(b) Slope form

Equation of tangent to the given parabola whose slope is 'm', is

$$y = mx + \frac{a}{m}, \; (m \neq 0)$$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(c) Parametric form

Equation of tangent to the given parabola at its point P(t), is $ty = x + at^2$

❖ Point of intersection of the tangents at the point t_1 & t_2 is $[\,at_1t_2, a(t_1 + t_2)]$.

Equation of tangent at a point on the parabola can be obtained by replacement method or using derivatives.

In replacement method, following changes are made to the second degree equation to obtain T.

$x^2 \to x x_1,\ y^2 \to y y_1,\ 2xy \to xy_1 + x_1y,\ 2x \to x + x_1,\ 2y \to y + y_1$

So, it follows that the targents are :

(i) $y y_1 = 2a(x + x_1)$ at the point (x_1, y_1) ;

(ii) $y = mx + \dfrac{a}{m}$ $(m \neq 0)$ at $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$

(iii) $ty = x + at^2$ at $(at^2, 2at)$.

(iv) Point of intersection of the tangents at the point t_1 & t_2 is $\{ at_1t_2,\ a(t_1 + t_2) \}$.

Ex. Find the equation to the tangents to the parabola $y^2 = 9x$ which goes through the point (4, 10).

Sol. Equation of tangent to parabola $y^2 = 9x$ is $y = mx + \dfrac{9}{4m}$

Since it passes through (4, 10)

$\therefore$ $10 = 4m + \dfrac{9}{4m}$

$\Rightarrow$ $16m^2 - 40m + 9 = 0$

$m = \dfrac{1}{4}, \dfrac{9}{4}$

$\therefore$ equation of tangent's are $y = \dfrac{x}{4} + 9$

& $y = \dfrac{9}{4}x + 1.$

Ex. A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Find its equation and its point of contact.

Sol. Let the slope of the tangent be m

$\therefore$ $\tan 45° = \left|\dfrac{3 - m}{1 + 3m}\right|$

$\Rightarrow$ $1 + 3m = \pm(3 - m)$

$\therefore$ $m = -2$ or $\dfrac{1}{2}$

As we know that equation of tangent of slope m to the parabola $y^2 = 4ax$ is $y = mx + \dfrac{a}{m}$ and point of contact is $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$

for $m = -2$, equation of tangent is $y = -2x - 1$ and point of contact is $\left(\dfrac{1}{2}, -2\right)$

for $m = \dfrac{1}{2}$, equation of tangent is $y = \dfrac{1}{2}x + 4$ and point of contact is (8, 8)

Ex. Prove that the straight line $y = mx + c$ touches the parabola $y^2 = 4a(x + a)$ if $c = ma + \frac{a}{m}$

Sol. Equation of tangent of slope 'm' to the parabola $y^2 = 4a(x + a)$ is

$y = m(x + a) + \frac{a}{m}$ $\Rightarrow$ $y = mx + a\left(m + \frac{1}{m}\right)$

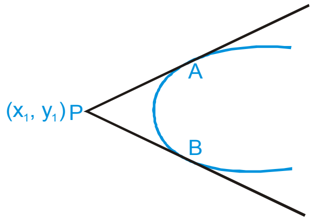
but the given tangent is $y = mx + c$

$\therefore$ $c = am + \frac{a}{m}$

Pair of Tangents

The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the parabola $y^2 = 4ax$ is given by: $SS_1 = T^2$ where :

$S \equiv y^2 - 4ax$; $S_1 = y_1^{\,2} - 4ax_1$; $T \equiv y y_1 - 2a(x + x_1)$.



Ex. Write the equation of pair of tangents to the parabola $y^2 = 4x$ drawn from a point $P(-1, 2)$

Sol. We know the equation of pair of tangents are given by $SS_1 = T^2$

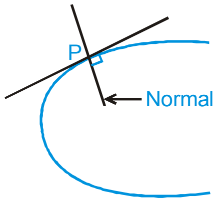
$\therefore$ $(y^2 - 4x)(4 + 4) = (2y - 2(x - 1))^2$

$\Rightarrow$ $8y^2 - 32x = 4y^2 + 4x^2 + 4 - 8xy + 8y - 8x$

$\Rightarrow$ $y^2 - x^2 + 2xy - 6x - 2y = 1$

NORMALS TO THE PARABOLA $y^2 = 4ax$

Normal is obtained using the slope of tangent.



Slope of tangent at $(x_1, y_1) = \frac{2a}{y_1}$

$\Rightarrow$ Slope of normal $= -\frac{y_1}{2a}$

(a) Point form

Equation of normal to the given parabola at its point (x_1, y_1) is

$$\mathbf{y - y_1 = -\frac{y_1}{2a}(x - x_1)}$$

(b) Slope form

Equation of normal to the given parabola whose slope is 'm', is

$y = mx - 2am - am^3$

foot of the normal is $(am^2, -2am)$

(c) Parametric form

Equation of normal to the given parabola at its point P(t), is

$y + tx = 2at + at^3$

(i) Point of intersection of normals at t_1 & t_2 is $(a(t_1^2 + t_2^2 + t_1t_2 + 2), - at_1t_2 (t_1 + t_2))$.

(ii) If the normal to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.

(iv) If normal drawn to a parabola passes through a point P(h,k) then $k = mh - 2am - am^3$, i.e. $am^3 + m(2a - h) + k = 0$.

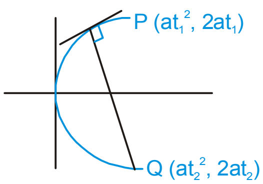
This gives $m_1 + m_2 + m_3 = 0$; $m_1m_2+m_2m_3+m_3m_1 = \frac{2a-h}{a}$; $m_1m_2m_3 = \frac{-k}{a}$

where m_1, m_2, & m_3 are the slopes of the three concurrent normals :

- Algebraic sum of slopes of the three concurrent normals is zero.
- Algebraic sum of ordinates of the three co-normal points on the parabola is zero.
- Centroid of the Δ formed by three co-normal points lies on the axis of parabola (x-axis).

Ex. If the normal at point 't_1' intersects the parabola again at 't_2' then show that $t_2 = -t_1 - \frac{2}{t_1}$

Sol. Slope of normal at P $= -t_1$ and slope of chord PQ $= \frac{2}{t_1 + t_2}$



$\therefore \quad -t_1 = \frac{2}{t_1 + t_2}$

$t_1 + t_2 = -\frac{2}{t_1} \quad \Rightarrow \quad t_2 = -t_1 - \frac{2}{t_1}$.

Ex. If two normals drawn from any point to the parabola $y^2 = 4ax$ make angle α and β with the axis such that $\tan\alpha \, . \tan\beta = 2$, then find the locus of this point.

Sol. Let the point is (h, k). The equation of any normal to the parabola $y^2 = 4ax$ is

$$y = mx - 2am - am^3$$

passes through (h, k)

$$k = mh - 2am - am^3$$

$$am^3 + m(2a - h) + k = 0 \qquad ....(i)$$

m_1, m_2, m_3 are roots of the equation, then $m_1 . m_2 . m_3 = -\frac{k}{a}$

but $\quad m_1m_2 = 2, m_3 = -\frac{k}{2a}$

m_3 is root of **(i)**

$\therefore \quad a\left(-\frac{k}{2a}\right)^3 - \frac{k}{2a}(2a - h) + k = 0 \Rightarrow k^2 = 4ah$

Thus locus is $y^2 = 4ax$.

Ex. If the normals at points t_1, t_2 meet at the point t_3 on the parabola then prove that

(i) $t_1\, t_2 = 2$ **(ii)** $t_1 + t_2 + t_3 = 0$

Sol. Since normal at t_1 & t_2 meet the curve at t_3

$$\therefore \quad t_3 = -t_1 - \frac{2}{t_1} \quad .....(i)$$

$$t_3 = -t_2 - \frac{2}{t_2} \quad .....(ii)$$

$$\Rightarrow \quad (t_1^2 + 2)\, t_2 = t_1\, (t_2^2 + 2)$$

$$t_1 t_2\, (t_1 - t_2) + 2\, (t_2 - t_1) = 0$$

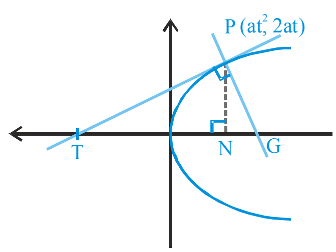
$$\because \quad t_1 \neq t_2\,, \quad t_1 t_2 = 2 \quad ......(iii)$$

Hence **(i)** $t_1\, t_2 = 2$

from equation **(i)** & **(iii)**, we get $t_3 = -t_1 - t_2$

Hence **(ii)** $t_1 + t_2 + t_3 = 0$

LENGTH OF SUBTANGENT & SUBNORMAL



PT and PG are the tangent and normal respectively at the point P to the parabola $y^2 = 4ax$. Then

TN = length of subtangent = twice the abscissa of the point P (Subtangent is always bisected by the vertex)

NG = length of subnormal which is constant for all points on the parabola & equal to its semilatus rectum (2a).

DIRECTOR CIRCLE

Locus of the point of intersection of the perpendicular tangents to the parabola $y^2 = 4ax$ is called the **director circle.** It's equation is $x + a = 0$ which is parabola's own directrix.

Ex. The circle drawn with variable chord $x + ay - 5 = 0$ (a being a parameter) of the parabola $y^2 = 20x$ as diameter will always touch the line -

Sol. Clearly $x + ay - 5 = 0$ will always pass through the focus of $y^2 = 20x$ i.e. (5, 0). Thus the drawn circle will always touch the directrix of the parabola i.e. the line $x + 5 = 0$.

CHORD JOINING TWO POINTS

The equation of a chord of the parabola $y^2 = 4ax$ joining its two points $P(t_1)$ and $Q(t_2)$ is $y(t_1 + t_2) = 2x + 2at_1t_2$

❖ If PQ is focal chord then $t_1t_2 = -1$.

❖ Extremities of focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

Ex. Through the vertex O of a parabola $y^2 = 4x$ chords OP and OQ are drawn at right angles to one another. Show that for all position of P, PQ cuts the axis of the parabola at a fixed point.

Sol. The given parabola is $y^2 = 4x$ ...... **(i)**

Let $P \equiv (t_1^2, 2t_1)$, $Q \equiv (t_2^2, 2t_2)$

Slope of OP $= \dfrac{2t_1}{t_1^2} = \dfrac{2}{t_1}$ and slope of OQ $= \dfrac{2}{t_2}$

Since OP $\perp$ OQ, $\dfrac{4}{t_1 t_2} = -1$ or $t_1 t_2 = -4$ ...... **(ii)**

The equation of PQ is $y(t_1 + t_2) = 2(x + t_1 t_2)$

$\Rightarrow \quad y\left(t_1 - \dfrac{4}{t_1}\right) = 2(x-4)$ [from **(ii)**]

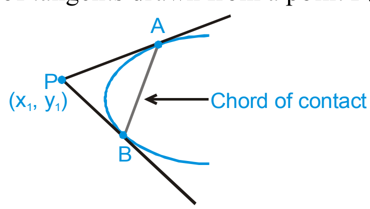
$\Rightarrow \quad 2(x-4) - y\left(t_1 - \dfrac{4}{t_1}\right) = 0 \quad \Rightarrow \quad L_1 + \lambda L_2 = 0$

$\therefore$ variable line PQ passes through a fixed point which is point of intersection of $L_1 = 0$ & $L_2 = 0$

i.e. (4, 0)

CHORD OF CONTACT

Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$.



The area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is

$$\frac{1}{2a}(y_1^2 - 4ax_1)^{3/2} = \frac{(S_1)^{3/2}}{2a}$$

Ex. Find the length of chord of contact of the tangents drawn from point (x_1, y_1) to the parabola $y^2 = 4ax$.

Sol. Let tangent at $P(t_1)$ & $Q(t_2)$ meet at (x_1, y_1)

$\therefore \quad at_1t_2 = x_1 \quad \& \quad a(t_1 + t_2) = y_1$

$\because \quad PQ = \sqrt{(at_1^2 - at_2^2)^2 + (2a(t_1 - t_2))^2}$

$$= a\sqrt{((t_1+t_2)^2 - 4t_1t_2)((t_1+t_2)^2 + 4)} = \sqrt{\frac{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}{a^2}}$$

Ex. If the line $x - y - 1 = 0$ intersect the parabola $y^2 = 8x$ at P & Q, then find the point of intersection of tangents at P & Q.

Sol. Let (h, k) be point of intersection of tangents then chord of contact is

$$yk = 4(x + h)$$

$$4x - yk + 4h = 0 \qquad \text{....... (i)}$$

But given line is

$$x - y - 1 = 0 \qquad \text{....... (ii)}$$

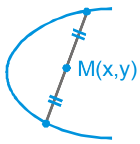
Comparing **(i)** and **(ii)**

$$\therefore \quad \frac{4}{1} = \frac{-k}{-1} = \frac{4h}{-1} \quad \Rightarrow \quad h = -1, k = 4$$

$$\therefore \quad \text{point} \equiv (-1, 4)$$

CHORD WITH A GIVEN MIDDLE POINT

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is $y - y_1 = \frac{2a}{y_1}\left(x - x_1\right)$.



This reduced to $T = S_1$, where $T \equiv yy_1 - 2a(x + x_1)$ & $S_1 \equiv y_1^2 - 4ax_1$.

Ex. Find the locus of middle point of the chord of the parabola $y^2 = 4ax$ which pass through a given point (p, q).

Sol. Let P(h, k) be the mid point of chord of parabola $y^2 = 4ax$,

so equation of chord is $yk - 2a(x + h) = k^2 - 4ah$.

Since it passes through (p, q)

$$\therefore \quad qk - 2a(p + h) = k^2 - 4ah$$

$\therefore$ Required locus is

$$y^2 - 2ax - qy + 2ap = 0.$$

Ex. Find the locus of middle point of the chord of the parabola $y^2 = 4ax$ whose slope is 'm'.

Sol. Let P(h, k) be the mid point of chord of parabola $y^2 = 4ax$,

so equation of chord is $yk - 2a(x + h) = k^2 - 4ah$.

but slope $= \frac{2a}{k} = m$

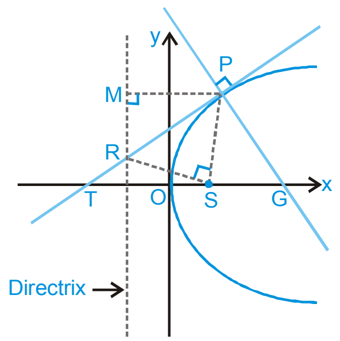
$$\therefore \quad \text{locus is } y = \frac{2a}{m}$$

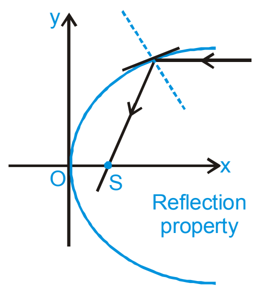
DIAMETER

The locus of the middle points of a system of parallel chords of a Parabola is called a DIAMETER. Equation to the diameter of a parabola is $y = 2a/m$, where m = slope of parallel chords.

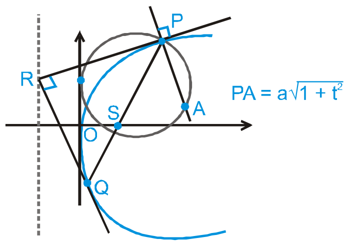
IMPORTANT CONCEPT

(i) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.



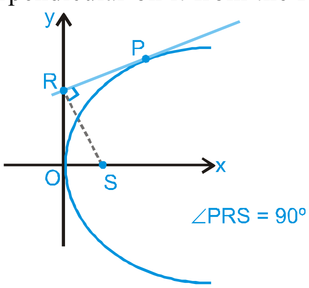


(ii) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the focus. See figure above.



(iii) The tangents at the extremities of a focal chord intersect at right angles on the directrix, and hence a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P $(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.

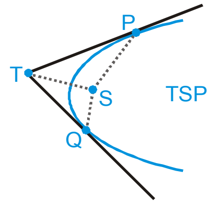
(iv) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.



(v) If the tangents at P and Q meet in T, then:

⇒ TP and TQ subtend equal angles at the focus S.

⇒ $ST^2 = SP. SQ$ & ⇒ The triangles SPT and STQ are similar.



(vi) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola is ; $2a = \frac{2bc}{b+c}$ i.e. $\frac{1}{b} + \frac{1}{c} = \frac{1}{a}$.

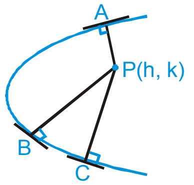
(vii) The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

(viii) If normal are drawn from a point P(h, k) to the parabola $y^2 = 4ax$ then

$k = mh - 2am - am^3$ i.e. $am^3 + m(2a - h) + k = 0$.

$m_1 + m_2 + m_3 = 0$; $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$; $m_1m_2m_3 = -\frac{k}{a}$.

Where m_1, m_2 & m_3 are the slopes of the three concurrent normals. Note that



A, B, C $\rightarrow$ Conormal points

$\Rightarrow$ algebraic sum of the slopes of the three concurrent normals is zero.

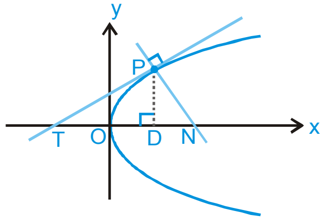
$\Rightarrow$ algebraic sum of the ordinates of the three conormal points on the parabola is zero

$\Rightarrow$ Centroid of the Δ formed by three co–normal points lies on the x–axis.

$\Rightarrow$ Condition for three real and distinct normals to be drawn froma point P (h, k) is

$h > 2a$ & $k^2 < \frac{4}{27A}(h - 2a)^3$.

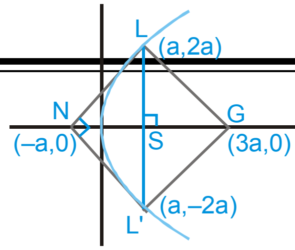
(ix) Length of subtangent at any point P(x, y) on the parabola $y^2 = 4ax$ equals twice the abscissa of the point P. Note that the subtangent is bisected at the vertex.



$TD = 2(OD)$, $DN = 2a$

(x) Length of subnormal is constant for all points on the parabola & is equal to the semi latus rectum. See figure above.

(xi) Tangents and Normals at the extremities of the latus rectum of a parabola



$y^2 = 4ax$ constitute a square, their points of intersection being (–a, 0) & (3a, 0).

(xii) The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

(xiii) If a family of straight lines can be represented by an equation $\lambda^2 P + \lambda Q + R = 0$ where λ is a parameter and P, Q, R are linear functions of x and y then the family of lines will be tangent to the curve $Q^2 = 4PR$.

(a) The two tangents at the extremities of focal chord meet on the foot of the directrix.

(b) Figure LNL'G is square of side $2\sqrt{2}a$

TIPS & FORMULAS

1. Conic Section

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

(a) The fixed point is called the FOCUS.

(b) The fix straight line is called the DIRECTRIX.

(c) The constant ratio is called the ECCENTRICITY denoted by e.

(d) The line passing through the focus & perpendicular to the directrix is called the AXIS.

(e) A point of intersection of a conic with its axis called a VERTEX.

2. General Equation of a Conic : Focal Direction Property

The general equation of a conic section with focus(p, q) & directrix $lx + my + n = 0$ is

$(l^2 + m^2)\,[(x - p)^2 + (y\text{-}q^2)] = e^2\,(lx + my + n)^2 \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

3. Distinguishing Between the Conic

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (i) When the Focus Lies on the Directrix :

In this case $D \equiv abc + 2\,fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines and if :

$e > 1$, $h^2 > ab$ the lines will be real & distinct intersecting at S.

$e = 1$, $h^2 = ab$ the lines will coincident.

$e < 1$, $h^2 < ab$ the lines will be imaginary.

Case (ii) When the Focus does not Lie on the Directrix :

The conic represents :

a parabola	an ellipse	a hyperbola	a rectangular hyperbola
$e = 1;\ D \neq 0$	$0 < e < 1\ ;\ D \neq 0$	$D \neq 0\ ;\ e > 1$	$e > 1\ ;\ D \neq 0$
$h^2 = ab$	$h^2 < ab$	$h^2 > ab$	$h^2 > ab\ ;\ a + b = 0$

4. Parabola

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

Standard equation of a parabola is $y^2 = 4\,ax$. For this parabola :

(i) Vertex is (0,0) **(ii)** Focus is (a, 0) **(iii)** Axis is $y = 0$ **(iv)** Directrix is $x + a = 0$

(a) Focal Distance

The distance of a point on the parabola from the focus is called the FOCAL DISTANCE OF THE POINT.

(b) Focal Chord :

A chord of the parabola, which passes through the focus is called a FOCAL CHORD.

(c) Double Ordinate :

A chord of the parabola perpendicular to the axis of the symmetry is called a DOUBLE ORDINATE with respect to axis as diameter.

(d) Latus Rectum :

A focal chord perpendicular to the axis of parabola is called the LATUS RECTUM. For $y^2 = 4ax$

(i) Length of the latus rectum = 4a

(ii) Length of the semi latus rectum = 2a

(iii) Ends of the latus rectum are L(a, 2a) & L'(a, –2a)

Note that

(i) Perpendicular distance from focus on directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

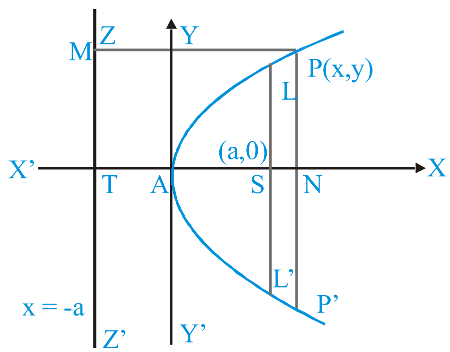
(iii) Two parabolas are said to be equal if they have latus rectum of same length.

5. Parametric Representation

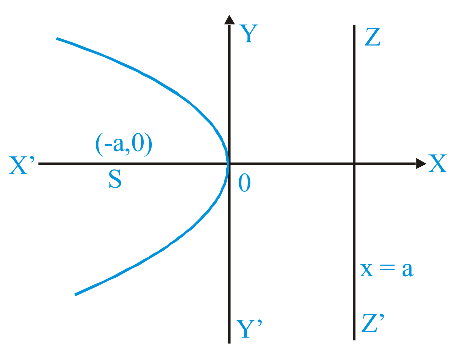
The simplest & the best form of representing the co-ordinates of a point on the parabola $y^2 = 4ax$ is $(at^2, 2at)$. The equation $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

6. Type of Parabola

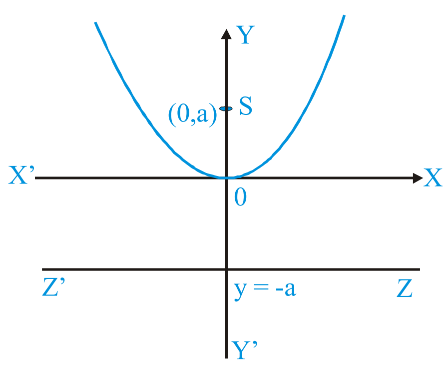
Four standards forms of the parabola are $y^2 = 4ax$; $y^2 = –4ax$; $x^2 = 4ay$; $x^2 = –4ay$



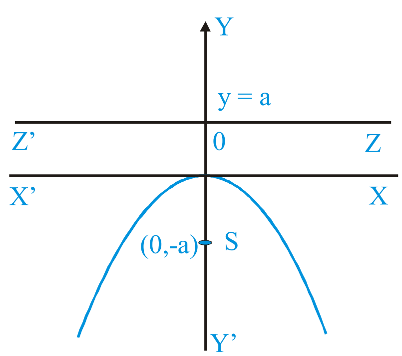
$y^2 = 4ax$



$y^2 = -4ax$



$x^2 = 4ay$



$x^2 = -4ay$

Parabola	Vertex	Focus	Axis	Directrix	Length of Latus rectum	Ends of Latus rectum	Parametric equation	Focal length
$y^2 = 4ax$	(0,0)	(a,0)	y=0	x = -a	4a	(a,± 2a)	$(at^2, 2at)$	x+a
$y^2 = -4ax$	(0,0)	(-a,0)	y=0	x = a	4a	(-a,± 2a)	$(-at^2, 2at)$	x-a
$x^2 = +4ay$	(0,0)	(0,a)	x=0	y = -a	4a	(± 2a,a)	$(2at, at^2)$	y+a
$x^2 = -4ay$	(0,0)	(0,-a)	x=0	y = a	4a	(± 2a, -a)	$(2at, -at^2)$	y-a
$(y-k)^2 = 4a(x-h)$	(h,k)	(h+a,k)	y=k	x+a-h = 0	4a	(h+a, k ± 2a)	$(h+at^2, k+2at)$	x-h+a
$(x-p)^2 = 4b(y-q)$	(p,q)	(p, b+q)	x = p	y+b-q = 0	4b	(p ± 2a, q+a)	$(q+2at, q+at^2)$	y-q+b

7. Position of a Point Relative to a Parabola

The point $(x_1 ; y_1)$ lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 = -4ax_1$ is positive, zero ot negative.

8. Chord Joining two Points

The equation of a chord of the parabola $y^2 = 4ax$ joining its two points $P(t_1)$ and $Q(t_2)$ is $y(t_1 + t_2) = 2x + 2at_1t_2$

Note

(i) If PQ is focal chord then $t_1t_2 = -1$.

(ii) Extremities of focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

(iii) If $t_1t_2 = k$ then chord always passes a fixed point (-ka, 0).

9. Line & a Parabola

(a) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as a >=<cm

$\Rightarrow$ condition of tangency is, $c = \frac{a}{m}$.

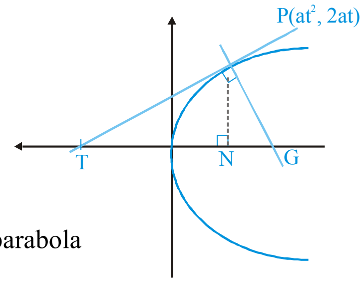
Note : Line $y = mx + c$ will be tangent to parabola

$x^2 = 4ay$ if $c = -am^2$.

(b) Length of the chord intercepted by the parabola$y^2 = 4ax$ on the line $y = mx + c$ is : $\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$.

Note : Length of the focal chord making an angle α with the x-axis is $4a\,\mathrm{cosec}^2\,\alpha$.

10. Length of Subtangentubnormal



PT and PG are the tangent and normal respectively at the point P to the parabola $y^2 = 4ax$. Then

TN = length of subtangent = twice the abscisaa of the point P

(Subtangent is always bisected by the vertex)

NG = length of the subnormal which is constant for all points on the parabola & equal to its semilatus rectum (2a).

11. Tangent to the Parabola $y^2 = 4ax$

(a) Point Form :

Equation of tangent to the given parabola at its point (x_1, y_1) is $yy_1 = 2a(x + x_1)$

(b) Slope Form :

Equation of tangent to the given parabola whose slope is 'm', is

$$y = mx + \frac{a}{m}, (m \neq 0)$$

point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(c) Parametric Form :

Equation of tangent to the given parabola at its point P(t), is - $ty = x + at^2$

Note : Point of intersection of the tangents at the point t_1 & t_2 is $[at_1 t_2, a(t_1 + t_2)]$. [i.e. G.M. and A.M. of abscissae and ordinates of the points]

12. Normal to the Parabola $y^2 = 4ax$

(a) Point Form :

Equation of normal to the given parabola at its point (x_1, y_1) is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$

(b) Slope Form :

Equation of normal to the given parabola whose slope is 'm', is $y = mx - 2am - am^3$ foot of the normal is $(am^2, -2am)$

(c) Parametric Form :

Equation of normal to the given parabola at its point P(t), is $y + tx = 2at + at^2$

Note :

If the normal to the parabola $y^2 = 4ax$ at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

13. Pair of Tangents

The equation of the pair of tangents which can be drawn from any point $P(x_1, y_1)$ outside the parabola $y^2 = 4ax$ is given by : $SS = T^2$, where :

$S = y^2 - 4ax$; $S_1 = y_1^2 - 4ax_1$; $T = yy_1 - 2a(x + x_1)$.