

• ROTATIONAL MOTION •

INTRODUCTION

Till so far we have learnt kinematics and kinetics of translation motion in which all the particles of a body undergo identical motions i.e. at any instant of time all of them have equal velocities and equal accelerations and in any interval of time they all follow identical trajectories. Therefore kinematics of any particle of a body or of its mass center in translation motion is representative of kinematics of the whole body. But when a body is in rotation motion, all of its particles and the mass center do not undergo identical motions. Newton's laws of motion, which are the main guiding laws of mechanics, are applicable to a point particle and if applied to a rigid body or system of particles, they predict motion of the mass center. Therefore, it becomes necessary to investigate how mass center and different particles of a rigid body move when the body rotates. In kinematics of rotation motion we investigate relations existing between time, positions, velocities and accelerations of different particles and mass center of a rigid body in rotation motion.

RIGID BODY

A rigid body is an assemblage of a large number of material particles, which do not change their mutual distances under any circumstance or in other words, they are not deformed under any circumstance.

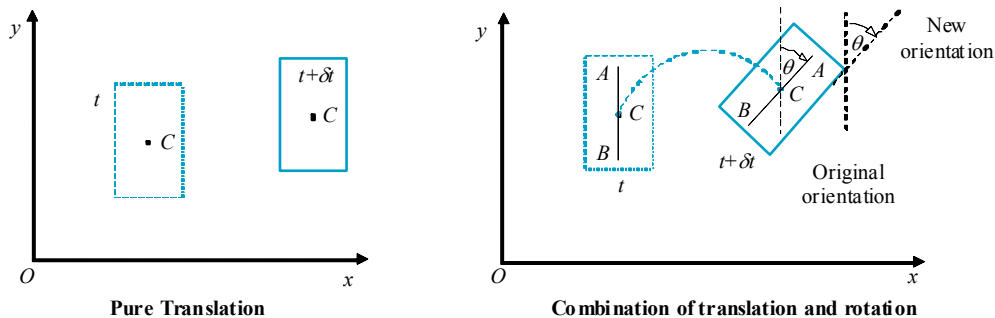
Actual material bodies are never perfectly rigid and are deformed under action of external forces. When these deformations are small enough to be considered during their course of motion, the body is assumed a rigid body. Hence, all solid objects such as stone, ball, vehicles etc. are considered as rigid bodies while analyzing their translation as well as rotation motion.

To analyze rotation of a body relative motion between its particles cannot be neglected and size of the body becomes a considerable factor. This is why study of rotation motion is also known as *mechanics of rigid bodies*.

ROTATION MOTION OF A RIGID BODY

Any kind of motion of a body is identified by change in position or change in orientation or change in both. If a body changes its orientation during its motion it said to be in rotation motion.

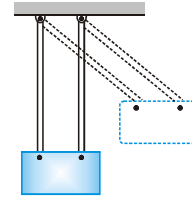
In the following figures, a rectangular plate is shown moving in the x - y plane. The point C is its mass center. In the first case it does not changes orientation, therefore is in pure translation motion. In the second case it changes its orientation by during its motion. It is a combination of translation and rotation motion.



Rotation i.e. change in orientation is identified by the angle through which a linear dimension or a straight line drawn on the body turns. In the figure this angle is shown by θ .

Ex. Identify Translation and rotation motion

A rectangular plate is suspended from the ceiling by two parallel rods each pivoted at one end on the plate and at the other end on the ceiling. The plate is given a side-push to oscillate in the vertical plane containing the plate. Identify motion of the plate and the rods.



Sol.

Neither of the linear dimensions of the plate turns during the motion. Therefore, the plate does not change its orientation. Here edges of the body easily fulfill our purpose to measure orientation; therefore, no line is drawn on it.

The plate is in *curvilinear translation motion* and the rods are in rotation motion.

Types of Motions involving Rotation

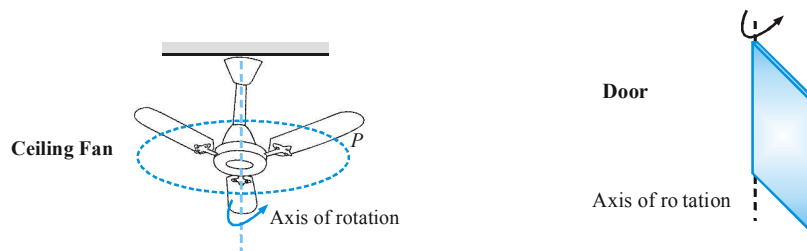
Motion of body involving rotation can be classified into following three categories.

- I Rotation about a fixed axis.
- II Rotation about an axis in translation.
- III Rotation about an axis in rotation

Rotation about a fixed axis

Rotation of ceiling fan, potter’s wheel, opening and closing of doors and needles of a wall clock etc. come into this category.

When a ceiling fan it rotates, the vertical rod supporting it remains stationary and all the particles on the fan move on circular paths. Circular path of a particle *P* on one of its blades is shown by dotted circle. Centers of circular paths followed by every particle are on the central line through the rod. This central line is known as *axis of rotation* and is shown by a dashed line. All the particles on the axis of rotation are at rest, therefore the axis is stationary and the fan is in rotation about this fixed axis.

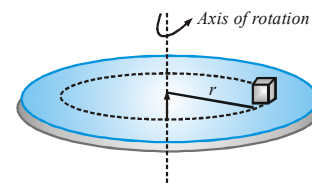


A door rotates about a vertical line that passes through its hinges. This vertical line is the axis of rotation. In the figure, the axis of rotation is shown by dashed line.

Axis of rotation

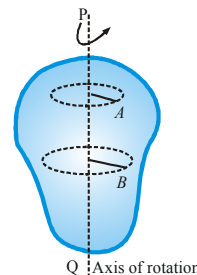
An imaginary line perpendicular to plane of circular paths of particles of a rigid body in rotation and containing the centers of all these circular paths is known as *axis of rotation*.

It is not necessary that the axis of rotation pass through the body. Consider system shown in the figure, where a block is fixed on a rotating disk. The axis of rotation passes through the center of the disk but not through the block.



Important observations

Let us consider a rigid body of arbitrary shape rotating about a fixed axis PQ passing through the body. Two of its particles A and B shown are moving on their circular paths.

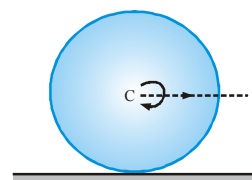


- (i) All of its particles, not on the axis of rotation, move on circular paths with centers on the axis of rotation. All these circular paths are in parallel planes that are perpendicular to the axis of rotation.
- (ii) All the particles of the body cover same angular displacement in the same time interval, therefore all of them move with the same angular velocity and angular acceleration.
- (iii) Particles moving on circular paths of different radii move with different speeds and different magnitudes of linear acceleration. Furthermore, no two particles in the same plane perpendicular to the axis of rotation have same velocity and acceleration vectors.
- (iv) All the particles on a line parallel to the axis of rotation move circular paths of the same radius therefore have same velocity and acceleration vectors.
- (v) Consider two particles in a plane perpendicular to the rotational axis. Every such particle on a rigid body in rotation motion moves on circular path relative to another one. Radius of the circular path equals to the distance between the particles. In addition, angular velocity and angular acceleration equals to that of rotation motion of the body.

Rotation about an axis in translation

Rotation about an axis in translation includes a broad category of motions. Rolling is an example of this kind of motion. A rod lying on table when pushed from its one end in its perpendicular direction also executes this kind of motion. To understand more let us discuss few examples.

Consider rolling of wheels of a vehicle, moving on straight level road. Relative to a reference frame, moving with the vehicle wheel appears rotating about its stationary axis. The rotation of the wheel from this frame is rotation about fixed axis. Relative to a reference frame fixed with the ground, the wheel appears rotating about the moving axis, therefore, rolling of a wheel is superposition of two simultaneous but distinct motions – rotation about the axis fixed with the vehicle and translation of the axis together with the vehicle.



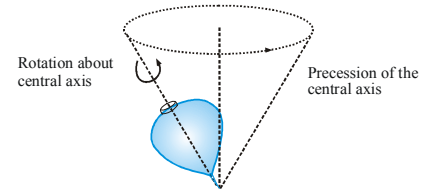
Important observations

- (i) Every particle of the body always remains in a plane perpendicular to the rotational axis. Therefore, this kind of motion is also known as *general plane motion*.
- (ii) Relative to every particle another particle in a plane perpendicular to axis of rotation moves on circular path. Radius of the circular path equals to the distance between the particles and angular velocity and angular acceleration equals to that of rotation motion of the body.
- (iii) Rotation about axis in translation is superposition of pure rotation about the axis and simultaneous translation motion of the axis.

Rotation about an axis in rotation.

In this kind of motion, the body rotates about an axis that also rotates about some other axis. Analysis of rotation about rotating axes is not in the scope of JEE, therefore we will discuss it to have an elementary idea only.

As an example consider a rotating top. The top rotates about its central axis of symmetry and this axis sweeps a cone about a vertical axis. The central axis continuously changes its orientation, therefore is in rotation motion. This type of rotation in which the axis of rotation also rotates and sweeps out a cone is known as *precession*.

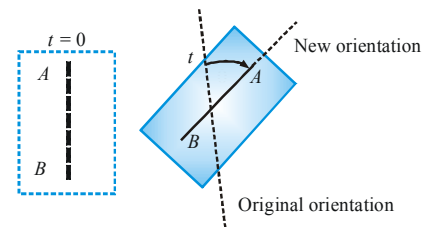


Another example of rotation about axis in rotation is a table-fan swinging while rotating. Table-fan rotates about its horizontal shaft along which axis of rotation passes. When the rotating table-fan swings, its shaft rotates about a vertical axis.

Angular Displacement, Angular Velocity and Angular Acceleration

Rotation motion is the change in orientation of a rigid body with time. It is measured by turning of a linear dimension or a straight line drawn on the body.

In the figure is shown at two different instants $t = 0$ and t a rectangular plate moving in its own plane. Change in orientation during time t equals to the angle θ through which all the linear dimensions of the plate or a line AB turns.



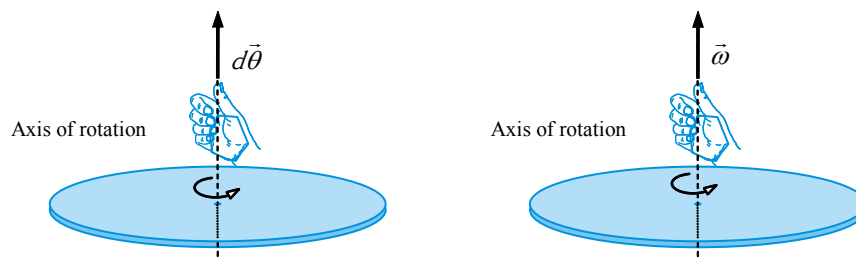
If the angle θ continuously changes with time t , instantaneous angular velocity ω and angular acceleration α for rotation of the body are defined by the following equations.

$$\omega = \frac{d\theta}{dt} \quad \dots\dots\dots(1)$$

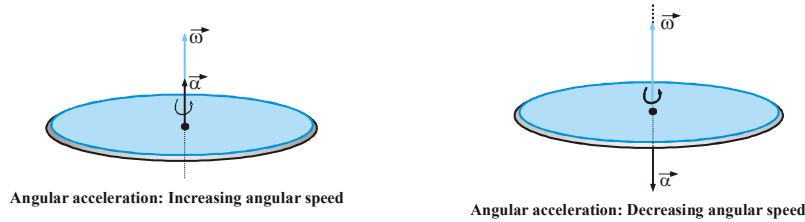
$$\alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \omega \frac{d\omega}{d\theta} \quad \dots\dots\dots(2)$$

Direction of angular motion quantities

Angular displacement, angular velocity and angular acceleration are known as angular motion quantities. Infinitesimally small angular displacement, instantaneous angular velocity and angular acceleration are vector quantities. Direction of infinitesimally small angular displacement and instantaneous angular velocity is given by the right hand rule. For a disk rotating as shown in the figure, the angular velocity points upwards along the axis of rotation.



The direction of angular acceleration depends on whether angular velocity increases or decreases with time. For increasing angular velocity, the angular acceleration vector points in the direction of angular velocity vector and for decreasing angular velocity, the angular acceleration vector points opposite to the angular velocity vector.



In rotation about fixed axis and rotation about axis in translation, the axis of rotation does not rotate and angular velocity and acceleration always point along the axis of rotation. Therefore, in dealing these kinds of motions, the angular motion quantities can be used in scalar notations by assigning them positive sign for one direction and negative sign for the opposite direction.

These quantities have similar mathematical relations as position coordinate, velocity, acceleration and time have in rectilinear motion.

- (i) A body rotating with constant angular velocity ω and hence zero angular acceleration is said to be uniform rotation. Angular position θ is given by equation

$$\theta = \theta_o + \omega t \quad \dots\dots\dots(3)$$

- (ii) Thus for a body rotating with uniform angular acceleration α , the angular position θ and angular velocity ω can be expressed by the following equation.

$$\omega = \omega_o + \alpha t \quad \dots\dots\dots(4)$$

$$\theta = \theta_o + \omega_o t + \frac{1}{2} \alpha t^2 = \theta_o + \frac{1}{2} (\omega_o + \omega) t \quad \dots\dots\dots(5)$$

$$\omega^2 = \omega_o^2 + 2\alpha(\theta - \theta_o) \quad \dots\dots\dots(6)$$

Angular motion quantities in rotation and assumption of axis of rotation

Rotation is identified by change in orientation, which is measured by turning of a linear dimension of the body or a line drawn on the body. It remains unchanged relative to all inertial frames.

Therefore, if we assume axis of rotation anywhere but parallel to the original one, angular displacement, angular velocity and angular acceleration of rotation motion remain the same.

Ex. A wheel is rotating with angular velocity 2 rad/s. It is subjected to uniform angular acceleration 2.0 rad/s².

- (a) How much angular velocity does the wheel acquire after 10 s?
- (b) How many complete revolution it makes in this time interval?

Sol. The wheel is in uniform angular acceleration, therefore from eq. [4]

$$\omega = \omega_o + \alpha t \rightarrow \text{Substituting values of } \omega_o, \alpha \text{ and } t, \text{ we have} \quad \omega = 2 + 2 \times 10 = 22 \text{ rad/s}$$

From eq.[5], we have

$$\theta = \theta_o + \frac{1}{2} (\omega_o + \omega) t \rightarrow \text{Substituting } \theta_o = 0 \text{ for initial position, and } \omega_o \text{ from above equation, we have}$$

$$\theta = 0 + \frac{1}{2} (2 + 22) 10 = 120 \text{ rad.}$$

In one revolution, the wheel rotates through 2π radians. Therefore, number of complete revolutions n is

$$n = \frac{\theta}{2\pi} = \frac{120}{2\pi} \approx 19$$

Ex. A disk rotates about a fixed axis. Its angular velocity ω varies with time according to equation $\omega = at + b$. At the instant $t = 0$ its angular velocity is 1.0 rad/s and at angular position is 2 rad and at the instant $t = 2$ s, angular velocity is 5.0 rad/s. Determine angular position θ and angular acceleration α when $t = 4$ s.

Sol. The given equation $\omega = at + b$ has form similar to eq.[4], therefore motion is rotation with uniform angular acceleration.

Initial angular velocity = $\omega_o = b = 1.0$ rad/s

Angular acceleration $\alpha = a$

Substituting these values in eq.[5], we get

$$\theta = \frac{1}{2}at^2 + \omega_o t + c$$

Since at $t = 0$, $\omega = 1.0$ rad/s, we obtain the constant c .

Initial angular position = $\theta_o = c = 2.0$ rad

Since at $t = 2.0$ s angular velocity is 5.0 rad/s, from given expression of angular velocity, we have

$\omega = at + b \rightarrow$ Substituting $b = 1.0$ rad/s, $t = 2.0$ s and $\omega = 5.0$ rad/s, we have

$$a = 2.0 \text{ rad/s}^2$$

Now we can write expressions for angular position, angular velocity and angular acceleration.

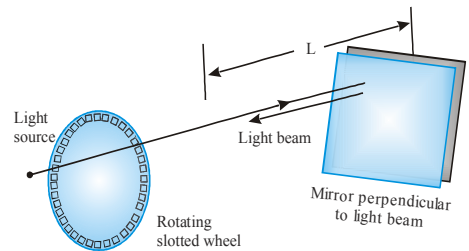
$$\theta = t^2 + t + 2.0 \quad \dots\dots\dots(1)$$

$$\omega = 2.0t + 1.0 \quad \dots\dots\dots(2)$$

From the above equations, we can calculate angular position, angular velocity and angular acceleration at $t = 4.0$ s

$$\theta_4 = 22 \text{ rad}, \omega_4 = 9.0 \text{ rad/s}, \alpha = 2.0 \text{ rad/s}^2$$

Ex. An early method of measuring the speed of light makes use of a rotating slotted wheel. A beam of light passes through slot at the outside edge of the wheel, as shown in figure below, travels to a distant mirror, and returns to the wheel just in time to pass through the next slot in the wheel. One such slotted wheel has a radius of 5.0 cm and 500 slots at its edge. Measurements taken when the mirror is $L = 500$ m from the wheel indicate a speed of light of 3.0×10^5 km/s.



- (a) What is the (constant) angular speed of the wheel ?
- (b) What is the linear speed of a point on the edge of the wheel?

Sol. (a) During the time light goes from the wheel to the mirror and comes back again, the wheel turns through

an angle of $\theta = \frac{2\pi}{500} = 1.26 \times 10^{-2}$ rad. That time is $t = \frac{2\ell}{c} = \frac{2(500\text{m})}{2.998 \times 10^8 \text{ m/s}} = 3.34 \times 10^{-4}$ s

So the angular speed of the wheel is $\omega = \frac{\theta}{t} = \frac{1.26 \times 10^{-2} \text{ rad}}{3.34 \times 10^{-6} \text{ s}} = 3.8 \times 10^3$ rad/s

- (b) Linear speed of a point on the edge of a wheel $v = \omega r = 3.8 \times 10^3 \times 0.05 = 1.9 \times 10^2$ m/s

PHYSICS FOR JEE MAINS & ADVANCED

Ex. A pulsar is rapidly rotating neutron star that emits a radio beam like a lighthouse emits a light beam. We receive a radio pulse for each rotation of the star. The period T of rotation is found by measuring the time between pulses. The pulsar in the Crab nebula has a period of rotation of $T = 0.033$ s that is increasing at the rate of 1.26×10^{-5} second/year.

- (a) What is the pulsar's angular acceleration?
 (b) If its angular acceleration is constant, how many years from now will the pulsar stop rotating?
 (c) The pulsar originated in a supernova explosion seen in the year 1054. What was the initial T for the pulsar? (Assume constant angular acceleration since the pulsar originated.)

Sol. (a) The angular velocity in rad/s $\omega = 2\pi/T$.

$$\text{The angular acceleration} = \alpha = \frac{d\omega}{dt} = -\frac{2\pi}{T^2} \frac{dT}{dt}$$

$$\text{For the pulsar described } \frac{dT}{dt} = \frac{1.26 \times 10^{-5} \text{ s/y}}{3.16 \times 10^7 \text{ s/y}} = 4.00 \times 10^{-13}$$

$$\text{So } \alpha = -\left[\frac{2\pi}{(0.033\text{s})^2}\right] (4.00 \times 10^{-13}) = -2.3 \times 10^{-3} \text{ rad/s}^2$$

The negative sign indicates that the angular acceleration is opposite the angular velocity and the pulsar is slowing down.

- (b) $\omega = \omega_0 + \alpha t$ for the time t when $\omega=0$.

$$t = -\frac{\omega}{\alpha} = \frac{2\pi}{aT} = -\frac{2\pi}{(-2.3 \times 10^{-9} \text{ rad/s}^2)(0.033\text{s})} = 8.3 \times 10^{10} \text{ s. This is about 2600 years.}$$

- (c) The pulsar was born $1992 - 1054 = 938$ years ago.

This is equivalent to $(938 \text{ y})(3.16 \times 10^7 \text{ s/y}) = 2.96 \times 10^{10} \text{ s}$. Its angular velocity was then

$$\omega = \omega_0 + \alpha t = \frac{2\pi}{T} + \alpha t = \frac{2\pi}{0.033\text{s}} + (-2.3 \times 10^{-9} \text{ rad/s}^2)(-2.96 \times 10^{10}\text{s}) = 258 \text{ rad/s.}$$

$$\text{Its period was } T = \frac{2\pi}{\omega} = 2.4 \times 10^{-2} \text{ s.}$$

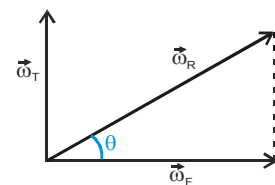
Ex. A turn table is rotating in a horizontal plane about the vertical axis passing through its centre with an angular velocity 20 rad/s. It carries upon it a flywheel rotating with an angular velocity 40 rad/s about a horizontal axle mounted in bearings. Find the angular velocity of the wheel as seen by an observer in the room.

Sol. As the axis of the turn table is vertical its angular velocity ω_T is directed vertical. The axis of flywheel is horizontal therefore its angular velocity ω_F is directed horizontal, hence the resultant angular velocity is $\vec{\omega}_R = \vec{\omega}_F + \vec{\omega}_T$

$$|\vec{\omega}_R| = \sqrt{\omega_F^2 + \omega_T^2} = \sqrt{40^2 + 20^2} = 20\sqrt{5} \text{ rad/s}$$

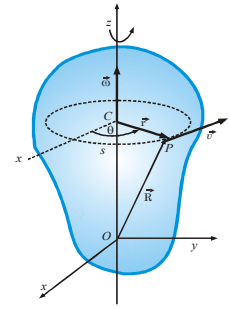
$\vec{\omega}_R$ lies in a plane which makes an angle θ with the horizontal

$$\text{plane, given by } \theta = \tan^{-1}\left(\frac{\omega_T}{\omega_F}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$



Kinematics of rotation about fixed axis

In figure is shown a rigid body of arbitrary shape rotating about the z -axis. In the selected frame (here the coordinate system) all the three axes are at rest, therefore the z -axis that is the axis of rotation is at rest and the body is in fixed axis rotation. All of its particles other than those on the z -axis move on circular paths with their centers on the z -axis. All these circular paths are parallel to the x - y plane. In the figure, one of its particles P is shown moving with velocity \vec{v} on a circular path of radius r and center C . Its position vector is \vec{R} . It were at the line Cx at $t = 0$ and at the position shown at the instant t . During time interval t , it covers the circular arc of length s and its radius vector turns through angle θ .



In an infinitesimally small time interval dt let, the particle covers infinitesimally small distance ds along its circular path.

$$d\vec{s} = d\vec{\theta} \times \vec{r} = d\vec{\theta} \times \vec{R} \quad \dots\dots\dots(7) \qquad \vec{v} = \frac{d\vec{s}}{dt} = \frac{d\vec{\theta}}{dt} \times \vec{r} = \frac{d\vec{\theta}}{dt} \times \vec{R} \quad \dots\dots\dots(8)$$

From eq. [7] and [8] we have

$$\vec{v} = \vec{\omega} \times \vec{r} = \vec{\omega} \times \vec{R} \quad \dots\dots\dots(9)$$

The above equation tells us the relation between the linear and angular velocity. Now we explore relation between the linear and angular accelerations. For the purpose, differentiate the above equation with respect to time.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v} \quad \dots\dots\dots(10)$$

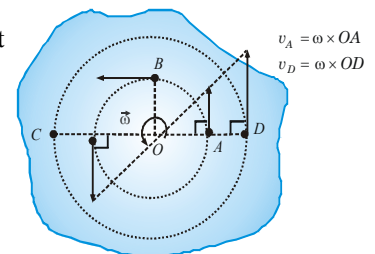
The first term on the RHS points along the tangent in the direction of the velocity vector and it is known as tangential acceleration \vec{a}_t same as we have in circular motion. In addition, the second term point towards the center C . It is known as centripetal acceleration or normal component \vec{a}_n of acceleration same as in circular motion. Now we have

Tangential acceleration $\vec{a}_t = \vec{\alpha} \times \vec{r} \quad \dots\dots\dots(11)$

Normal acceleration $\vec{a}_n = \vec{\omega} \times \vec{v} = -\omega^2 \vec{r} \quad \dots\dots\dots(12)$

How to Locate Axis of Rotation

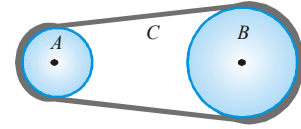
Every particle in a plane perpendicular to the axis of rotation move with different velocities and accelerations, moreover, they all have the same angular velocity and angular acceleration. Such a section of a body in rotation is shown here. The particles A, B and C at equal distance from the axis of rotation move with equal speeds v_A and the particle D moves with speed v_D on concentric circular paths. The location of rotational axis can be determined by any of the two graphical techniques.



- (i) Lines perpendicular to velocity vectors and passing through the particles, whose velocity vectors are neither parallel nor antiparallel intersect at the axis of rotation. See pairs of particles A and B, B and C and B and D .
- (ii) Lines perpendicular to velocity vectors and passing through the particles, whose velocity vectors are either parallel or antiparallel, coincide and intersect the line joining tips of their velocity vectors at the axis of rotation. Refer pairs of particles A and C, A and D and C and D .

PHYSICS FOR JEE MAINS & ADVANCED

Ex. A belt moves over two pulleys A and B as shown in the figure. The pulleys are mounted on two fixed horizontal axils. Radii of the pulleys A and B are 50 cm and 80 cm respectively. Pulley A is driven at constant angular acceleration 0.8 rad/s^2 until the pulley B acquires an angular velocity of 10 rad/s . The belt does not slide on either of the pulleys.



- (a) Find acceleration of a point C on the belt and angular acceleration of the pulley B .
- (b) How long after the pulley B achieve angular velocity of 10 rad/s .

Sol. Since the belt does not slide on the pulleys, magnitude of velocity and acceleration of any point on the belt are same as velocity and tangential acceleration of any point on periphery of either of the pulleys.

Using the above fact with eq.[11], we have

$$\vec{a}_T = \vec{\alpha} \times \vec{r} \rightarrow a_C = \alpha_A r_A = \alpha_B r_B$$

Substituting $r_A = 0.5 \text{ m}$, $r_B = 0.8 \text{ m}$ and $\alpha = 0.8 \text{ rad/s}^2$, we have

$$a_C = 0.4 \text{ m/s}^2 \text{ and } \alpha_B = \frac{a_C}{r_B} = \frac{\alpha_A r_A}{r_B} = 0.5 \text{ rad/s}^2$$

From eq. [4], we have

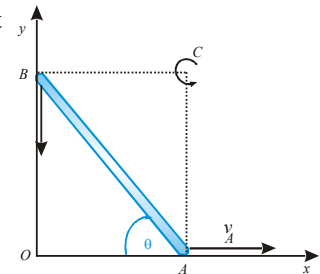
$$\omega = \omega_o + \alpha t \rightarrow t = \frac{\omega_B - \omega_{Bo}}{\alpha_B}$$

Substituting $\omega_{Bo} = 0$, $\omega_B = 10 \text{ rad/s}$ and $\alpha_B = 0.5 \text{ rad/s}^2$, we have $t = 20 \text{ s}$

KINEMATICS OF ROTATION ABOUT AXIS IN TRANSLATION

In this kind of motion, the body rotates about an axis and the axis moves without rotation. Rolling is a very common example of this kind of motion.

As an example consider a rod whose ends A and B are sliding on the x and y -axis as shown in the figure. Change in its orientation measured by change in angle θ indicates that the rod is in rotation. Perpendiculars drawn to velocity vector of its end points intersect at the axis of rotation, which is continuously changing its position.

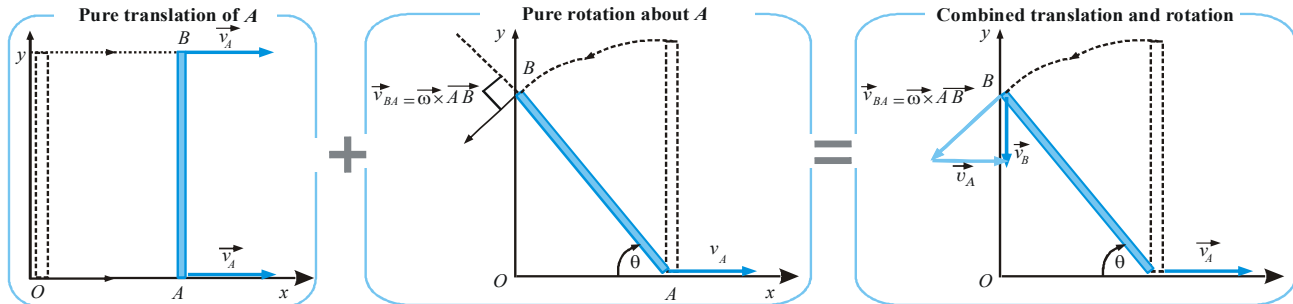


Instantaneous Axis of Rotation (IAR)

It is a mathematical line about that a body in combined translation and rotation can be conceived in pure rotation at an instant. It continuously changes its location.

Now we explore how the combined translation and rotational motion of the rod is superposition of translation motion of any of its particle and pure rotation about an axis through that particle.

Consider motion of the rod from beginning when it was parallel to the y -axis. In the following figure translation motion of point A is superimposed with pure rotation about A .



The motion of the rod can be conceived as superposition of translation of point A and simultaneous rotation about an axis through A .

The same experiment can be repeated to demonstrate that motion of the rod can be conceived as superposition of translation of any of its particle and simultaneous rotation about an axis through that particle.

Considering translation of A and rotation about A this fact can be expressed by the following equation.

Combined Motion = Translation of point A + Pure rotation about point A

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} \quad \dots\dots(13)$$

Since point B moves relative A moving on circular path its velocity relative to A is given by the equation

$$\vec{v}_{BA} = \vec{\omega} \times \overline{AB}.$$

Now we have $\vec{v}_B = \vec{v}_A + \vec{\omega} \times \overline{AB} \quad \dots\dots(14)$

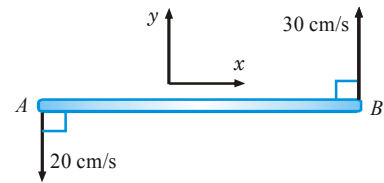
The above fact is true for any rigid body in combined translation and rotation motion.

Rotation about an axis in translation of a rigid body can be conceived as well as analyzed as superposition of translation motion of any of its particle and simultaneous rotation about an axis passing through that particle provided that the axis is parallel to the actual one.

Similar to eq.[13], we can write equation for acceleration.

$$\left. \begin{aligned} \vec{a}_A &= \vec{a}_B + \vec{a}_{BA} \\ \vec{a}_A &= \vec{a}_B + \overbrace{\vec{a}_{BA_T} + \vec{a}_{BA_n}} \\ \vec{a}_A &= \vec{a}_B + \vec{\alpha} \times \overline{AB} + -\omega^2 \overline{AB} \end{aligned} \right\} \quad \dots\dots(26)$$

Ex. A 100 cm rod is moving on a horizontal surface. At an instant, when it is parallel to the x -axis its ends A and B have velocities 30 cm/s and 20 cm/s as shown in the figure.



- (a) Find its angular velocity and velocity of its center.
- (b) Locate its instantaneous axis of rotation.

Sol. Let the rod is rotating anticlockwise, therefore its angular velocity is given by $\vec{\omega} = \omega \hat{k}$. Velocity vectors of all the points on the rod and its angular velocity must satisfy the relative motion eq.[14].

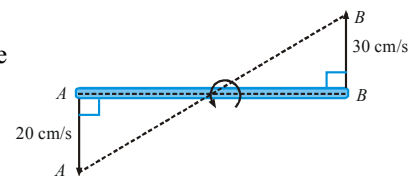
(a) Substituting velocities $\vec{v}_A = -20\vec{j}$ cm/s and $\vec{v}_B = 30\vec{j}$ cm/s and angular velocity $\vec{\omega}$ in eq.[14], we have

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \overline{AB} \rightarrow \omega = 0.5 \text{ rad/s}$$

Velocity vector of the center C of the rod also satisfy the following equation.

$$\vec{v}_C = \vec{v}_A + \vec{\omega} \times \overline{AC} \rightarrow \vec{v}_C = -20\vec{j} + 0.5\vec{k} \times 50\vec{i} = 5.0\vec{j} \text{ cm/s}$$

(b) Here velocity vectors of the particles A and B are antiparallel, therefore the instantaneous axis of rotation passes through intersection of the common perpendicular to their velocity vectors and a line joining tips of the velocity vectors. The required geometrical construction is shown in the following figure.



Since triangles $AA'P$ and $BB'P$ are similar and $AB = 100$ cm, we have $AP = 40$ cm.

The instantaneous axis of rotation passes through the point P , which is 40 cm from A .

Analytical Approach.

The instantaneous center of rotation is at instantaneous rest. Using this fact in eq.[14], we have

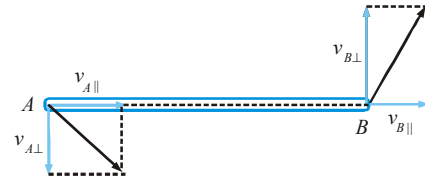
$$\vec{v}_P = \vec{v}_A + \vec{\omega} \times \overline{AP} \rightarrow \vec{0} = -20\vec{j} + 0.5\vec{k} \times (AP)\vec{j} \Rightarrow AP = 40 \text{ cm}$$

PHYSICS FOR JEE MAINS & ADVANCED

Ex. Can you suggest a quick way to find angular velocity of a rod, if velocities of two of its points are known?

Sol. The eq.[14] suggest a quick way to determine angular velocity, when distance between two points and their velocity components perpendicular to the lining joining them are known.

Angular velocity of the rod
$$\omega = \frac{|\vec{v}_{BA}|}{AB} = \frac{|\vec{v}_{B\perp} - \vec{v}_{A\perp}|}{AB}$$

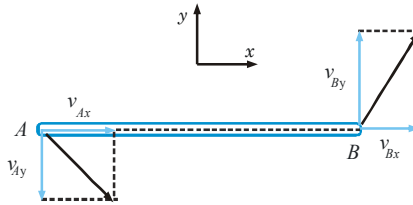


Ex. A 50 cm long rod AB is in combined translation and rotation motion on a table. At an instant velocity component of point A perpendicular to the rod is 10 cm/s, velocity component of point B parallel to the rod is 6.0 cm/s and angular velocity of the rod is 0.4 rad/s in anticlockwise sense as shown in the given figure.



- (a) Find velocity vectors of point A and B .
- (b) Locate the instantaneous axis of rotation.

Sol. Let x - y plane of a coordinate system coincides with the tabletop and the rod is parallel to the x -axis at the instant considered. The rod is shown in this coordinate frame in the following figure.



(a) Since distance between any two points remains unchanged, the velocity components of any two points parallel to the line joining them must be equal. Therefore, we have

$$v_{Ax} = v_{Bx} = 6.0 \text{ cm/s} \quad \dots\dots(1)$$

Velocities of points A and B must satisfy eq.[14]. Substituting angular velocity $\vec{\omega} = 0.4\hat{k} \text{ rad/s}$, in this equation we have

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{AB} \rightarrow 6.0\vec{i} + v_{By}\vec{j} = 6.0\vec{i} - 10\vec{j} + 0.4\hat{k} \times 50\vec{i}$$

Equating y -components of both the sides, we have

$$v_{By} = 10 \text{ cm/s} \quad \dots\dots(2)$$

From eq. (1), (2) and the given information, we can express the velocity vectors of the points A and B .

$$\vec{v}_A = 6.0\vec{i} - 10\vec{j} \text{ cm/s, and } \vec{v}_B = 6.0\vec{i} + 10\vec{j} \text{ cm/s}$$

(b) The instantaneous axis of rotation is at instantaneous rest. Let the end A of the rod is at the origin and coordinates of the point P in the x - y plane through which the IAR passes is (x, y) . Now from eq.[14], we have

$$\vec{v}_P = \vec{v}_A + \vec{\omega} \times \vec{AP} \rightarrow \vec{0} = 6.0\vec{i} - 10\vec{j} + 0.4\hat{k} \times (x\vec{i} + y\vec{j})$$

Equating coefficients of x and y -components of both the sides, we have

$$x = 25 \text{ cm and } y = 15 \text{ cm}$$

Therefore, coordinates of the point P through which IAR passes the x - y plane are (25, 15).

ROLLING AS ROTATION ABOUT AN AXIS IN TRANSLATION

Wheels of a moving vehicle roll on road. A ball rolls on ground when pushed. In fact, a body of round section can roll smoothly under favorable conditions. On the other hand, objects with corners, such as dice, roll by successive rotations about the edge or corner that is in contact with the ground. This type of motion is usually known as toppling.

If the point of contact of the of the rolling body does not slide it is known as rolling without slipping or pure rolling or simply rolling and if the point of contact slides it is known as rolling with slipping.

All kind of rolling motion is examples of rotation about an axis in translation.

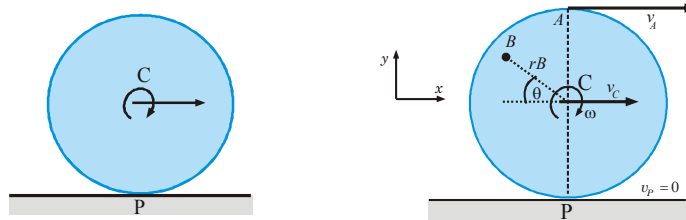
Rolling without slipping on stationary surface.

We first discuss velocity relations and thereafter accelerations relations of two points of a body of round section rolling on a stationary surface. For the purpose, we can use any of the following methods.

- I** Analytical Method: By using relative motion equations.
- II** Superposition Method: By superimposing translation of a point and pure rotation about that point.
- III** Use of ICR.

Velocity relations by Analytical Method

Its point of contact P does not slide on the surface, therefore velocity of the point of contact relative to the surface is zero. In the next figure, velocity vectors of its center C and top point A are shown.



Velocity of the center C can be obtained with the help of relative motion equation.

$$\vec{v}_C = \vec{v}_P + \vec{\omega} \times \overline{PC} \rightarrow \vec{v}_C = \vec{0} + (-\omega \vec{k}) \times R\vec{j}$$

$$\vec{v}_C = \omega R\vec{i} \quad \dots\dots(16)$$

The above equation is used as condition of rolling without slipping on stationary surface.

Velocity of the top point A can be obtained by relative motion equation.

$$\vec{v}_A = \vec{v}_P + \vec{\omega} \times \overline{PA} \rightarrow \vec{v}_A = \vec{0} + (-\omega \vec{k}) \times (R\vec{j})$$

$$\vec{v}_A = 2\omega R\vec{i} = 2\vec{v}_C \quad \dots\dots(17)$$

Once velocity of the center is obtained, we can use relative motion between A and C as well.

$$\vec{v}_A = \vec{v}_C + \vec{\omega} \times \overline{CA} \rightarrow \vec{v}_A = \omega R\vec{i} + (-\omega \vec{k}) \times (2R\vec{j})$$

$$\vec{v}_A = 2\omega R\vec{i} = 2\vec{v}_C \quad \dots\dots(18)$$

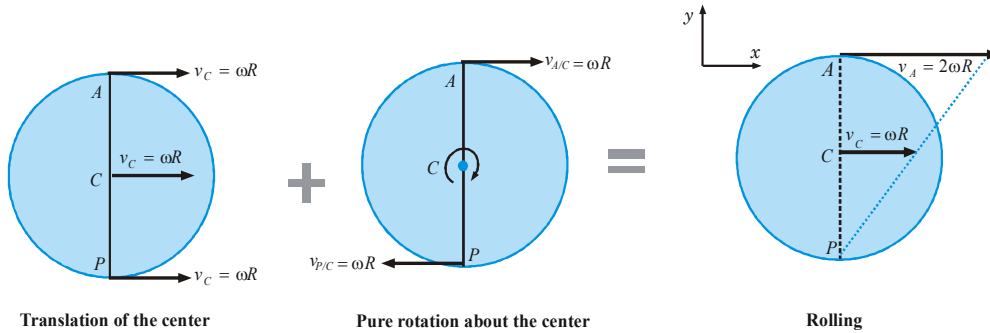
In similar fashion, velocity vector of an arbitrarily chosen point B .

$$\vec{v}_B = \vec{v}_C + \vec{\omega} \times \overline{CB} \rightarrow \vec{v}_B = v_C\vec{i} + (-\omega \vec{k}) \times (-r \cos \theta \vec{i} + r \sin \theta \vec{j})$$

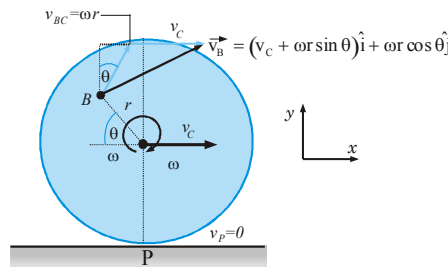
$$\vec{v}_B = (v_C + \omega r \sin \theta)\vec{i} + \omega r \cos \theta \vec{j} \quad \dots\dots(19)$$

Velocity relations by Superposition Method

Now we will see that the above velocity relation can also be obtained by assuming rolling of the wheel as superposition of translation of its center and simultaneous rotation about the center.



Velocity of an arbitrary point *B* as superposition of translation of the center and rotation about the center.



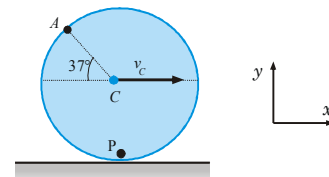
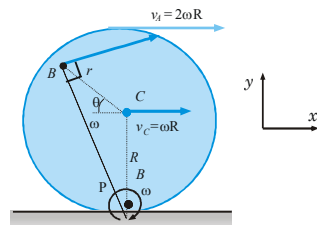
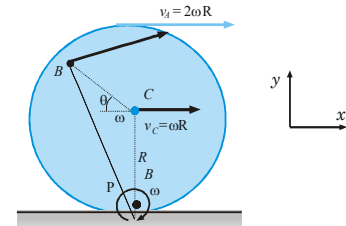
Velocity relations by Use of ICR

In rolling without slipping on stationary surface the point of contact is at instantaneous rest, therefore the ICR and the IAR passes through it. We will see how velocity of the center *C*, the top point *A* and an arbitrarily chosen point *B* can be calculated by assuming the body in state of pure rotation about the ICR.

Velocity of center *C* $\vec{v}_C = \vec{\omega} \times \overline{PC} = \omega R \hat{i}$

Velocity of the top point *A* $\vec{v}_A = \vec{\omega} \times \overline{PA} = 2\omega R \hat{i}$

Velocity of the point *B* $\vec{v}_B = \vec{\omega} \times \overline{PB} = (v_C + \omega r \sin \theta) \hat{i} + \omega r \cos \theta \hat{j}$



Ex. A cylinder of radius 5 m rolls on a horizontal surface. Velocity of its center is 25 m/s. Find its angular velocity and velocity of the point *A*.

Sol. In rolling the angular velocity $\vec{\omega}$ and velocity of the center of a round section body satisfy condition described in the relative motion eq.[14]. So we have

$$\vec{v}_C = \vec{\omega} \times \vec{r}_{C/P} \rightarrow 25\hat{i} = \omega\hat{k} \times 5\hat{j} \Rightarrow \vec{\omega} = -5\hat{k} \text{ rad/s}$$

Angular velocity vector points in the negative *z*-axis so the cylinder rotates in clockwise sense.

Velocity of the point *A* can be calculated by either analytical method, superposition method or by using method of ICR.

Analytical Method

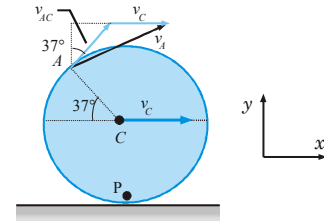
$$\vec{v}_A = \vec{v}_C + \vec{\omega} \times \vec{CA} \rightarrow \vec{v}_A = 25\vec{i} + (-5\vec{k}) \times (-5 \cos 37^\circ \vec{i} + 5 \sin 37^\circ \vec{j})$$

$$\vec{v}_A = (40\vec{i} + 20\vec{j}) \text{ m/s}$$

Superposition Method

In rolling $v_C = v_{AC} = \omega R = 25 \text{ m/s}$. The superposition i.e. vector addition of the terms of equation $\vec{v}_A = \vec{v}_C + \vec{v}_{AC}$ are shown in the following figure. Resolving $v_{AC} = \omega R = 25 \text{ m/s}$ into its Cartesian components and adding to \vec{v}_C , we obtain

$$\vec{v}_A = \vec{v}_C + \vec{v}_{A/C} \rightarrow \vec{v}_A = 25\vec{i} + 15\vec{i} + 20\vec{j} = (40\vec{i} + 20\vec{j}) \text{ m/s}$$

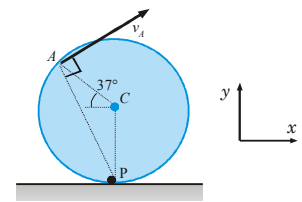


Use of ICR

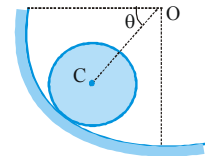
The contact point P is the ICR in rolling. The cylinder is in pure rotation about the ICR at the instant under consideration, so from the relative motion equation, we have

$$\vec{v}_A = \vec{\omega} \times \vec{r}_{AP} \rightarrow \vec{v}_A = \vec{\omega} \times (\vec{PC} + \vec{CA}) = (-5\vec{k}) \times \{(5\vec{j}) + (-4\vec{i} + 3\vec{j})\}$$

$$\vec{v}_A = (40\vec{i} + 20\vec{j}) \text{ m/s}$$



Ex. A disk of radius r is rolling down a circular track of radius R . There is no slipping between the disk and the track. When line OC is at angle θ down the horizontal, center of the disk has velocity v_C . Assume center O of the tracks origin of reference frame, find angular velocity of translation motion of the center of the disk and angular velocity of rotation motion.



Sol. The angular velocity of translation motion of the center of the disk equals to the rate of change is θ . Let us denote it by ω_o .

$$\omega_o = \frac{d\theta}{dt}$$

The center of the disk moves on circular path of radius $R - r$. Relation between velocity v_C of the center of the disk, radius R of the circular track and radius of the disk is

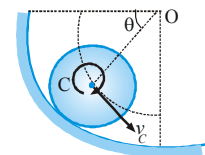
$$v_C = \omega_o (R - r)$$

Therefore, angular velocity of translation motion of center of the disk is $\omega_o = \frac{v_C}{R - r}$

Since the disk is rolling without slipping on the circular track, its angular velocity of rotation ω is given by the following equation.

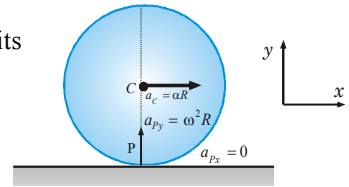
$$v_C = \omega r$$

Therefore, angular velocity of rotation of the disk is $\omega = \frac{v_C}{r}$



1. Acceleration relations by Analytical Method

The point of contact P does not slide on the surface, therefore component of its acceleration parallel to the surface must be zero. However, it has an acceleration component towards the center. The center always moves parallel to the horizontal surface and does not changes direction of its velocity; therefore, its acceleration can only be parallel to the surface.



Relation between acceleration of acceleration vector of the center C and point of contact P can be obtained with the help of relative motion [15] equation together with the above fact.

$$\vec{a}_c = \vec{a}_p + \vec{\alpha} \times \vec{PC} - \omega^2 \vec{PC} \rightarrow a_c \vec{i} = a_{py} \vec{j} + (-\alpha \vec{k}) \times R \vec{j} - \omega^2 R \vec{j} = a_{py} \vec{j} + \alpha R \vec{i} - \omega^2 R \vec{j}$$

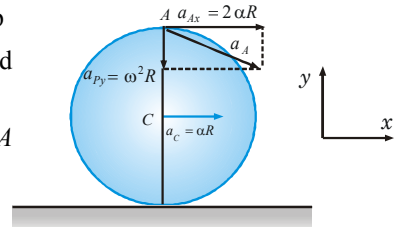
Equating coefficients of x and y -components on both the sides of the above equation, we have

$$\vec{a}_c = \alpha R \vec{i} \quad \dots\dots\dots(20)$$

$$\vec{a}_p = \omega^2 R \vec{j} \quad \dots\dots\dots(21)$$

The eq. (20) is used as condition for rolling without slipping together with eq. [16]

In the given figure, acceleration vectors the point of contact; center and the top point are shown. Now we will see how these accelerations can be calculated by using relative motion equation.



Once velocity of the center is obtained, we can use relative motion between A and C as well. Now we calculate acceleration of the top point A .

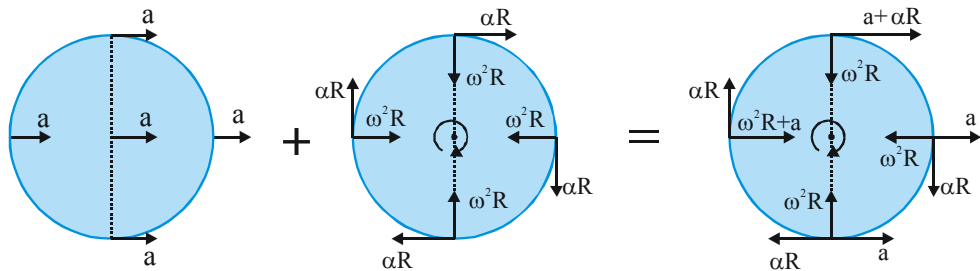
$$\vec{a}_A = \vec{a}_c + \vec{\alpha} \times \vec{CA} - \omega^2 \vec{CA} \rightarrow \vec{a}_A = \alpha R \vec{i} + (-\alpha \vec{k}) \times R \vec{j} - \omega^2 R \vec{j}$$

$$\vec{a}_A = 2\alpha R \vec{i} - \omega^2 R \vec{j} \quad \dots\dots\dots(22)$$

Acceleration vector of point A and its components are shown in the given figure.

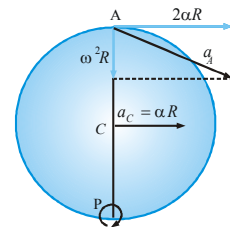
2. Acceleration relations by Superposition Method

Now we see how acceleration relations are expressed for a rolling wheel by assuming its rolling as superposition of its translation with the acceleration of center and simultaneous rotation about the centre.



3. Acceleration relations by use of ICR

Acceleration relations can also be obtained by assuming the body in pure rotation about the ICR. Here we will use relative motion equation[15]. Always keep in mind that the acceleration of the ICR is not zero, it has value $\omega^2 R$ and points towards the center of the body. Now we will see how acceleration of the center C , the top point A and an arbitrarily chosen point B can be calculated by assuming the body in state of pure rotation about the ICR.



$$\text{Acceleration of center } C \quad \vec{a}_C = \vec{a}_p + \vec{\alpha} \times \overline{PC} - \omega^2 \overline{PC} = \alpha R \vec{i}$$

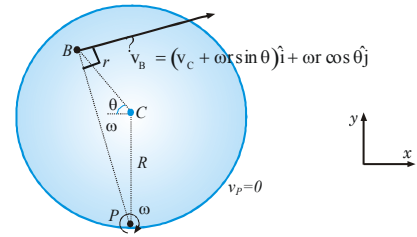
$$\vec{a}_C = \alpha R \vec{i}$$

$$\text{Acceleration of the top point } A \quad \vec{a}_A = \vec{a}_p + \vec{\alpha} \times \overline{PA} - \omega^2 \overline{PA} \vec{j}$$

$$\vec{a}_A = 2\alpha R \vec{i} - \omega^2 R \vec{j}$$

$$\text{Acceleration of the point } B \quad \vec{a}_B = \vec{a}_p + \vec{\alpha} \times \overline{PB} - \omega^2 \overline{PB}$$

$$\vec{a}_B = \left\{ \alpha(R + r \sin \theta) + \omega^2 \right\} \vec{i} + \left\{ \alpha r \cos \theta - \omega^2 r \sin \theta \right\} \vec{j}$$



Ex. A body of round section of radius 10 cm starts rolling on a horizontal stationary surface with uniform angular acceleration 2 rad/s².

(a) Find initial acceleration of the center C and top point A.

(b) Find expression for acceleration of the top point A as function of time.

Sol. Initially when the body starts, it has no angular velocity; therefore, the last term in relative motion equation [15] for acceleration vanishes and for a pair of two points A and B the equation reduces to

$$\vec{a}_A = \vec{a}_B + \vec{\alpha} \times \overline{BA}$$

The angular acceleration vector is $\vec{\alpha} = -2\vec{k}$ rad/s².

(a) Acceleration of the center C is obtained by using condition for rolling without slipping.

$$\vec{a}_C = \vec{\alpha} \times \overline{PC} \rightarrow \vec{a}_C = -2\vec{k} \times 10\vec{j} = 20\vec{i} \text{ cm/s}^2$$

Acceleration of the point A can be obtained either by analytical method, superposition method or by use of ICR. These methods for calculation of acceleration of the top point are already described; therefore, we use the result.

$$\vec{a}_A = 2\alpha R \vec{i} \rightarrow \vec{a}_A = 40\vec{i} \text{ cm/s}^2$$

(b) Initially at the instant $t = 0$, when the body starts, its angular velocity is zero. At latter time it acquires angular velocity $\vec{\omega}$, therefore acceleration of any point on the body, other than its center, has an additional component of acceleration. This additional component is accounted for by the last term in the relative motion equation [15].

Angular velocity acquired by the body at time t is obtained by eq.[4] used for a body rotating with constant angular acceleration.

$$\vec{\omega} = \vec{\omega}_o + \vec{\alpha}t \rightarrow \text{Substituting } \omega_o = 0, \text{ we have } \vec{\omega} = -2t\vec{k}$$

Analytical Method

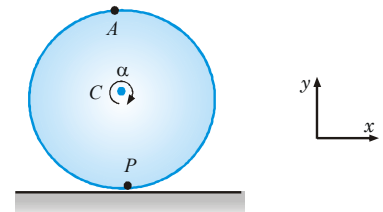
Using the relative motion equation for the pair of points C and A, we have

$$\vec{a}_A = \vec{a}_C + \vec{\alpha} \times \overline{CA} - \omega^2 \overline{CA} \rightarrow \vec{a}_A = \alpha R \vec{i} + (-\alpha \vec{k}) \times R \vec{j} - \omega^2 R \vec{j} = 2\alpha R \vec{i} - \omega^2 R \vec{j}$$

Substituting the known values

$$\vec{\alpha} = -2\vec{k} \text{ rad/s}^2, \vec{\omega} = -2t\vec{k} \text{ rad/s and } R = 10 \text{ cm}$$

$$\text{we have } \vec{a}_A = 40\vec{i} - 40t^2 \vec{j} \text{ cm/s}^2$$



Superposition Method

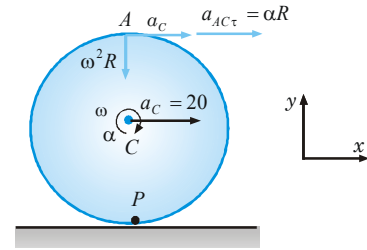
We superimpose translation motion of the center and rotation motion about the center. In fact it is vector addition of terms of above equation used in analytical method.

From the above figure, we have

$$\vec{a}_A = (a_c + \alpha R)\vec{i} - \omega^2 R\vec{j}$$

Substituting known values $\vec{\alpha} = -2\vec{k} \text{ rad/s}^2$, $\vec{\omega} = -2t\vec{k} \text{ rad/s}$ and $R=10 \text{ cm}$,

we have $\vec{a}_A = 40\vec{i} - 40t^2\vec{j} \text{ cm/s}^2$



Use of ICR

The point of contact P is the ICR, because the body is rolling without slipping. We use relative motion equation for pairs of points P and A.

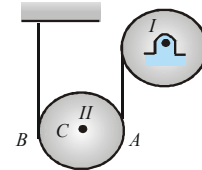
$$\vec{a}_A = \vec{a}_P + \vec{\alpha} \times \vec{PA} - \omega^2 \vec{PA} \rightarrow \vec{a}_A = \omega^2 R\vec{j} + 2\alpha R\vec{i} - 2\omega^2 R\vec{j} = 2\alpha R\vec{i} - \omega^2 R\vec{j}$$

Substituting known values

$$\alpha = 2 \text{ rad/s}^2, \omega = 4t \text{ rad/s and } R = 10 \text{ cm,}$$

we have $\vec{a}_A = 40\vec{i} - 40\vec{j} \text{ cm/s}^2$

Ex. Two identical disks, each of radius r , are connected by a cord as shown in the figure. The disk I rotates with constant angular acceleration α in anticlockwise direction. Find acceleration of the center of disk II and its angular acceleration.



Sol. As the disk I rotates, the thread unwraps it. Acceleration of a point on the portion of the thread between the two disks equals to tangential acceleration of any point on the periphery of the disk I. The extreme left point A of the disk II must also descend with the same acceleration.

Downward acceleration a_B of point B = Tangential acceleration of a point on the periphery of disk I.

$$a_B = \alpha_1 r \rightarrow a_B = \alpha r$$

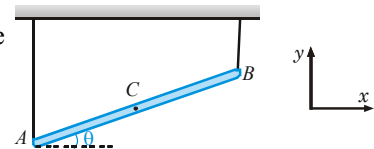
The point B on the thread is at rest relative to the ground; therefore, it can be assumed that the second disk is in a motion similar to rolling without slipping on a vertical surface.

Now applying conditions of rolling without slipping we have

$$\text{Acceleration of the center of the disk II } a_C = \frac{1}{2} a_B = \frac{1}{2} \alpha r$$

$$\text{Angular acceleration of the disk II } \alpha_{II} = \frac{a_C}{r} = \frac{1}{2} \alpha$$

Ex. A uniform rod AB of length l is supported with the help of two light inextensible threads as shown in the figure. The thread supporting the end B is cut. If magnitude of acceleration of the center C of rod is a_C immediately after the thread is cut, find angular acceleration of the rod and acceleration of its end A.



Sol. Immediately after the thread is cut, all the forces acting on the rod are in vertical direction; therefore, acceleration of its mass center is vertically downwards. The mass center of a uniform rod is at its center; therefore, acceleration of the center C immediately after the thread is cut is in vertically downward direction.

The end A can move on circular path of radius equal to length of thread supporting the end A. Therefore, acceleration a_A of the end A is in horizontal direction immediately after the thread supporting the end B is cut.

Analytical Method

If we assume angular acceleration α of the rod in clockwise direction, we can write relative motion equation for the pair of points A and C .

$\vec{a}_C = \vec{a}_A + \vec{\alpha} \times \overline{AC} - \omega^2 \overline{AC} \rightarrow$ The rod cannot acquire any angular velocity immediately after the thread is cut due to inertia therefore the last term in the relative motion equation vanishes.

Substituting $\vec{a}_C = -a_C \vec{j}$, $\vec{\alpha} = -\alpha \vec{k}$, $\vec{a}_A = a_A \vec{i}$ and $\overline{AC} = \frac{1}{2}l(\cos \theta \vec{i} + \sin \theta \vec{j})$,

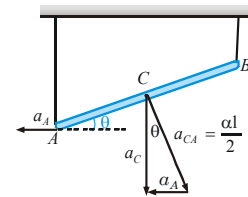
we have $-a_C \vec{j} = a_A \vec{i} - \frac{1}{2} \alpha l \cos \theta \vec{j} + \frac{1}{2} \alpha l \sin \theta \vec{i}$

Equating coefficients of x and y -components, we have

$$\alpha = \frac{2a_C}{l \cos \theta} \curvearrowright \text{ and } a_A = a_C \tan \theta$$

Superposition Method

We superimpose rod's translation with velocity of A and rotation about A . The rod cannot acquire any angular velocity immediately after the string is cut due to its inertia; therefore, point C cannot have any term involving radial acceleration. The acceleration vectors to be added are shown in the following figure.



From the figure, we have $a_C = \frac{\alpha l}{2} \cos \theta \Rightarrow \alpha = \frac{2a_C}{l \cos \theta} \curvearrowright$

$$a_A = a_C \tan \theta$$

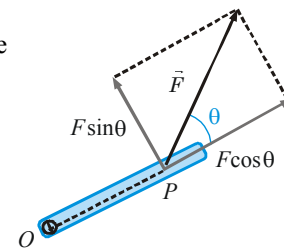
• KINETIC OF RIGID BODY •

In this section we deal with equilibrium of rigid bodies and kinetics of rigid bodies. Equilibrium of rigid bodies includes both the translational equilibrium and rotational equilibrium. In kinetics we account for causes affecting rotational motion.

TORQUE: MOMENT OF A FORCE

Torque is rotational analogue of force and expresses tendency of a force applied to an object to cause the object to rotate about a given point.

To investigate further let us discuss an experiment. Consider a rod pivoted at the point O . A force \vec{F} is applied on it at the point P . The component $F \cos \theta$ of the force along the rod is counterbalanced by the reaction force of the pivot and cannot contribute in rotating the rod. It is the component $F \sin \theta$ of the force perpendicular to the rod, which is responsible for rotation of the rod. Moreover, farther is the point P from O , where the force is applied easier is to rotate the rod. This is why handle on a door is attached as far away as possible from the hinges.



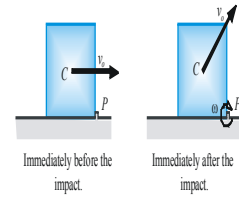
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Magnitude of torque of a force is proportional to the product of distance of point of application of the force from the pivot and magnitude of the perpendicular component $F \sin \theta$ of the force. Denoting torque by symbol τ , the distance of point of application of force from the pivot by r , we can write

$$\tau_o \propto rF \sin \theta$$

Since rotation has sense of direction, torque should also be a vector. Its direction is given by right hand rule. Now we can express torque by the cross product of \vec{r} and \vec{F} .

$$\vec{\tau}_o = \vec{r} \times \vec{F} \quad \dots\dots\dots(1)$$



Here constant of proportionality has been assumed a dimensionless number unity because a unit of torque has been chosen as product of unit of force and unit of length.

The geometrical construction shown in figure suggests a simple way to calculate torque. The line $OQ (= r \sin \theta)$ known as *moment arm*, is the length of perpendicular drawn from O on the line of action of the force. The magnitude of the torque equals to the product of OQ and magnitude of the force \vec{F} .

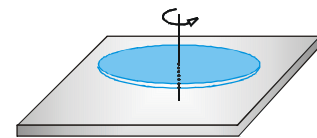
Torque about a Point and Torque about an Axes

We have defined *torque of a force about a point* as the moment of the force about that point. In dealing with rotation about a fixed axis we need to know torque about the axis rotation.

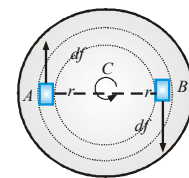
When a body is in plane motion the net torque of all the forces including the forces necessary to restrain rotation of the axis is along the axis of rotation. It is known as torque about the axis. Torque of a force about an axis of rotation equals to the moment of force about the point where plane of motion of the point of application of the force intersects the axis.

In analyzing plane motion we always consider torque about an axis under consideration and in rest of the book by the term torque of force we mean torque about an axis.

Ex. A uniform disk of mass M and radius R rotating about a vertical axis passing through its center and perpendicular to its plane is placed gently on a rough horizontal ground, where coefficient of friction is μ . Calculate torque of the frictional forces.



Sol. When the disk rotates on the ground, kinetic friction acts at every contact point. Since the gravity acts uniformly everywhere and the disk is also uniform, the normal reaction from the ground is uniformly distributed over the entire contact area. Consider two diametrically opposite identical portions A and B of the disk each of mass dm at distance r from the center as shown in the adjacent figure. The normal reaction from the ground on each of these portions equals to their weights and hence frictional forces are $df = \mu dm g$



Friction forces on these two and on all other diametrically opposite portions of the disk are equal and opposite, therefore net resultant friction force on the disk is zero. But torque of friction force on every portion is in same direction and all these torques add to contribute a net counterclockwise torque about the axis.

Consider a ring of radius r and width dr shown by dashed lines. Net torque $d\tau_c$ of friction force on this ring can easily be expressed by the following equation.

$$d\tau_c = r\mu(\text{mass of the ring})g = r\mu\left(\frac{\text{Mass of the disk}}{\text{Area of the disk}} \times \text{Area of the ring}\right)g = r\mu\left(\frac{2Mrdr}{R^2}\right)g$$

Integrating both sides of the above equation, we have

$$\tau_c = \frac{2\mu Mg}{R^2} \int_{r=0}^R r^2 dr = \frac{2}{3} \mu MgR$$

Rotational Equilibrium

A rigid body is said to be in state of rotational equilibrium if its angular acceleration is zero. Therefore a body in rotational equilibrium must either be in rest or rotation with constant angular velocity.

Since scope of JEE syllabus is confined only to rotation about a fixed axis or rotation about an axis in translation motion, the discussion regarding rotational equilibrium is limited here to situations involving only coplanar forces. Under these circumstances the necessary and sufficient condition for rotational equilibrium is

If a rigid body is in rotational equilibrium under the action of several coplanar forces, the resultant torque of all the forces about any axis perpendicular to the plane containing the forces must be zero.

In the figure a body is shown under the action of several external coplanar forces F_1, F_2, \dots, F_p and F_n .

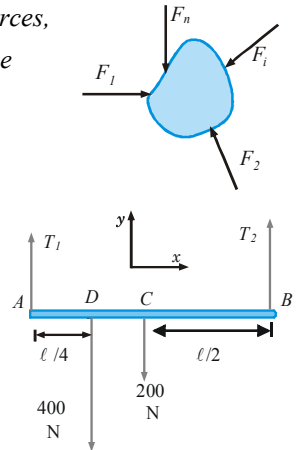
$$\sum \bar{\tau}_P = 0$$

Here P is a point in the plane of the forces about which we calculate torque of all the external forces acting on the body. The flexibility available in selection of the point P provides us with advantages that we can select such a point about which torques of several unknown forces will become zero or we can make as many number of equations as desired by selecting several different

points. The first situation yields to a simpler equation to be solved and second situation though does not give independent equation, which can be used to determine additional unknowns yet may be used to check the solution.

The above condition reveals that a body cannot be in rotational equilibrium under the action of a single force unless the line of action passes through the mass center of the body.

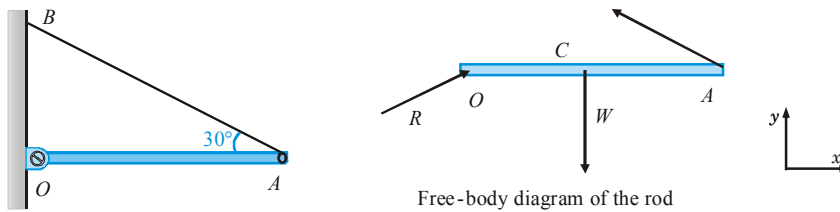
A case of particular interest arises where only three coplanar forces are involved and the body is in rotational equilibrium. It can be shown that *if a body is in rotational equilibrium under the action of three forces, the lines of action of the three forces must be either concurrent or parallel*. This condition provides us with a graphical technique to analyze rotational equilibrium.



Equilibrium of Rigid Bodies

A rigid body is said to be in equilibrium, if it is in translational as well as rotational equilibrium both. To analyze such problems conditions for both the equilibriums must be applied.

Ex. A 10 kg uniform rod OA is pivoted at O on a vertical wall with the help of a cable AB . Find the tension in the cable and reaction force applied by the pivot.



Sol. The rod is in translational and rotational equilibrium under the action of three forces that are weight (W) of the rod, the tension (T) in the cable, and the reaction (R) of the pivot. These forces are shown in the free-body diagram of the rod.

Translational equilibrium

$$\Sigma F_x = 0 \rightarrow R_x = T_x = T \cos 30^\circ \quad \dots\dots\dots(1)$$

$$\Sigma F_y = 0 \rightarrow R_y + T_y = W \Rightarrow R_y + T \sin 30^\circ = W \quad \dots\dots\dots(2)$$

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Rotational equilibrium: Let us apply the condition about O , because torque of the reaction R will become zero.

$$\Sigma \vec{\tau}_O = \vec{0} \rightarrow \quad Wl/2 = (T \sin 30^\circ)l$$

$$T = W = 100 \text{ N}$$

Now from equations (1) and (2) we have $R = 100 \text{ N}$

Ex. A uniform rod of 20 kg is hanging in horizontal position with the help of two threads. It also supports a 40 kg mass as shown in the figure. Find the tension developed in each thread.

Sol. Free body diagram of the rod is shown in the figure.

Translational equilibrium

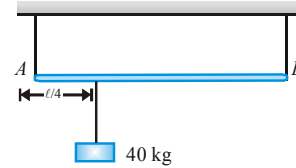
$$\Sigma F_y = 0 \rightarrow \quad T_1 + T_2 = 400 + 200 = 600 \text{ N} \quad (1)$$

Rotational equilibrium: Applying the condition about A , we get T_2 .

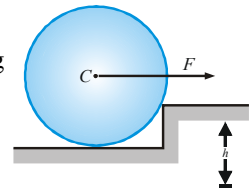
$$\Sigma \vec{\tau}_A = \vec{0} \rightarrow \quad 400(l/4) + 200(l/2) - T_2 l = 0 \Rightarrow T_2 = 200 \text{ N}$$

Similarly writing torque equation about B , we have

$$\Sigma \vec{\tau}_B = \vec{0} \rightarrow \quad T_1 = 400 \text{ N.}$$

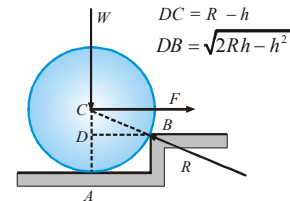


Ex. A cylinder of radius R and weight W is to be raised against a step of height h by applying a horizontal force at its center as shown in the figure. Find the required minimum magnitude of this force. Assume sufficient friction between the cylinder and the corner of the step to prevent slipping.



Sol. The forces acting on the sphere are its weight W , the horizontal pull F , reaction

R from the corner and the normal reaction from the ground. The reaction from the corner includes the normal reaction and friction. We need not to worry about this force because its torque about the corner vanishes. The moment it starts rising the normal reaction from the ground also vanishes. The requirement that the force F should be of minimum magnitude will cause the cylinder to rotate about B at very small angular velocity and with negligible angular acceleration. Therefore the cylinder can be assumed in the state of rotational equilibrium as well as translational equilibrium.



The weight W , the pull F and the reaction R from the corner are shown in the free body diagram of the cylinder.

Rotational equilibrium: The cylinder is in rotational equilibrium under the action of three coplanar forces therefore these forces must be concurrent.

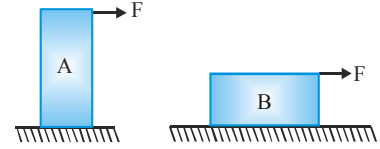
Torques equation of all the forces about the corner B to zero, we have

$$\Sigma \vec{\tau}_B = 0 \rightarrow \quad F(CD) = W(DB)$$

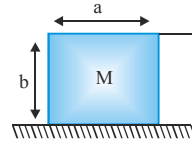
By solving above equation we have
$$F = W \frac{\sqrt{2Rh - h^2}}{R - h}$$

Toppling :

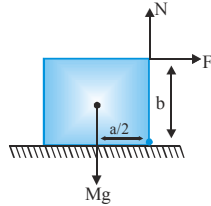
For shown situation (A) & (B), more chances of toppling in (A). In case of toppling, normal reaction must pass through end points.



Ex. Find the minimum value of F to topple about an edge.



Sol. In case of toppling



Taking torque about O

$$F(b) = Mg \left(\frac{a}{2} \right) \Rightarrow F_{\min} = \frac{Mga}{2b}$$

Ex. A uniform cylinder of height h and radius r is placed with its circular face on a rough inclined plane and the inclination of the plane to the horizontal is gradually increased. If μ is the coefficient of friction, then under what conditions the cylinder will (a) slide before toppling (b) topple before sliding.

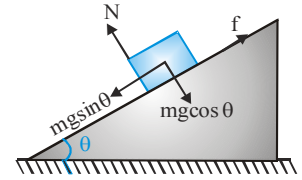
Sol. (a) The cylinder will slide if $mg \sin \theta > \mu mg \cos \theta \Rightarrow \tan \theta > \mu$ (i)

The cylinder will topple if $(mg \sin \theta) \frac{h}{2} > (mg \cos \theta)r \Rightarrow \tan \theta > \frac{2r}{h}$... (ii)

Thus, the condition of sliding is $\tan \theta > \mu$ & condition of toppling is $\tan \theta > \frac{2r}{h}$.

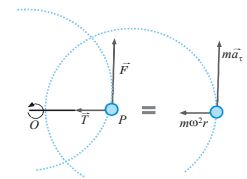
Hence, the cylinder will slide before toppling if $\mu < \frac{2r}{h}$

(b) The cylinder will topple before sliding if $\mu > \frac{2r}{h}$



Concept of Rotational Inertia

Every particle of a rigid body in rotation moves on circular paths about the axis of rotation; therefore a rigid body in rotation can be thought as a group of large number of particle moving on circular paths.



Let a particle of mass m constrained by a string to move on a circular path of radius r about a fixed point O in free space. To provide the particle tangential acceleration α there is a force \vec{F} and to provide the particle necessary centripetal acceleration there is string tension \vec{T} as shown in the figure.

$$\vec{F} = m\vec{a}_t \text{ and } \vec{T} = -m\omega^2\vec{r}_{P/O}$$

Taking moments about the center O of all the forces acting on the particle and then summing up them, the above equations yield

$$\vec{r}_{P/O} \times \vec{F} + \vec{r}_{P/O} \times \vec{T} = \vec{r}_{P/O} \times m\vec{a}_t + \vec{r}_{P/O} \times (-m\omega^2\vec{r}_{P/O})$$

The left hand side equals to resultant torque about the center O of all the external force acting on the particle. The vector product $\vec{r}_{P/O} \times \vec{a}_t$ in the first term of the right hand side become $r^2\vec{\alpha}$. The second term on both the side vanishes.

$$\Sigma \vec{\tau}_o = mr^2\vec{\alpha} \text{(2)}$$

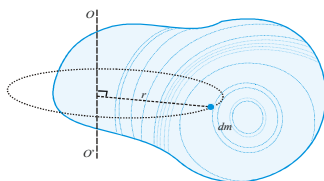
The above equation is similar to Newton’s second law of motion. It suggest that torque, angular acceleration and the term mr^2 play similar roles in rotation motion as the net force, the acceleration and mass plays in translation motion. Total mass of a body in translation motion is the measure of its inertia to translation motion, therefore the sum of all terms mr^2 arising due to all particle of the rigid body provides suitable measure of its inertia to rotation motion. *The inertial to rotation motion is known as rotational inertia or more commonly moment of inertia.*

Moment of inertia of a rigid body

A rigid body is continuous distribution of mass and can be assumed consisting of infinitely large number of point particles. If one of the point particle of infinitely small mass dm is at a distance r from the axis of rotation OO' , the moment of inertia of this point particle is given by

$$dI_o = r^2 dm$$

The moment of inertia of the whole body about the axis OO' can now be obtained by integrating term of the above equation over the limits to cover whole of the body.

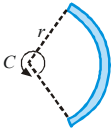
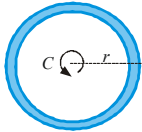
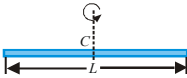
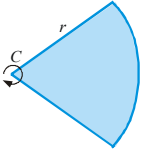
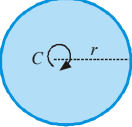

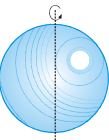
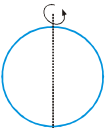


$$I_o = \int dI_o = \int r^2 dm \quad \dots\dots\dots(3)$$

Expression for moment of inertia contains product of two terms. One of them is the mass of the body and the other is a characteristic dimension, which depends on the manner how mass of the body is distributed relative to the axis of rotation. Therefore moment of inertia of a rigid body depends on the mass of the body and distribution of the mass relative to the axis of rotation. Obviously for uniform bodies expression of moment of inertia depends on their shape and location and orientation of the axis of rotation. Based on these facts we can conclude

1. If mass distribution is similar for two bodies about an axis, expressions of their moment of inertia must be of the same form about that axis.
2. If the whole body or any of its portions is shifted parallel to the axis of rotation, moment of inertia remains unchanged.

Moment of Inertia for some commonly used bodies

Body	Axis	Moment of Inertia
Uniform thin rod bent into shape of an arc of mass m 	Passing through center and perpendicular to the plane containing the arc	$I_C = mr^2$
Uniform ring of mass m 	Passing through center and perpendicular to the plane containing the arc or the centroidal axis.	$I_C = mr^2$
Straight uniform rod 	Passing through center and perpendicular to the rod or the centroidal axis.	$I_C = \frac{mL^2}{12}$
Sector of a uniform disk of mass m 	Passing through center and perpendicular to the plane containing the sector.	$I_C = \frac{mr^2}{2}$
Uniform disk of mass m 	Passing through center and perpendicular to the plane containing the disk or the centroidal axis.	$I_C = \frac{mr^2}{2}$
Homogeneous cylinder of mass m 	Axis of the cylinder or the centroidal axis.	$I_C = \frac{mr^2}{2}$
Homogeneous sphere of mass m 	Diameter or the centroidal axis	$I_C = \frac{2}{5} mR^2$
Spherical shell of mass m 	Diameter or the centroidal axis	$I = \frac{2}{3} mR^2$

Theorems on Moment of Inertia

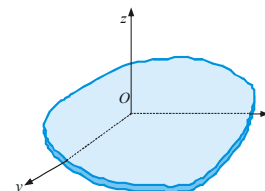
Moment of inertias of a rigid body about different axes may be different. There are two theorems known as *theorem of perpendicular axes* and *theorem of parallel axes*, which greatly simplify calculation of moment of inertia about an axis if moment of inertia of a body about another suitable axis is known.

Theorem of Perpendicular Axes

This theorem is applicable for a rigid body that lies entirely within a plane i.e. a lamina or a rod bent into shape of a plane curve. The moment of inertia I_x, I_y and I_z of the body about the x, y and z -axis can be expressed by the following equations.

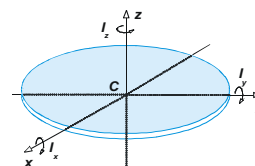
$$I_z = I_y + I_x$$

For a planar body, the moment of inertia about an axis perpendicular to the plane of the body is the sum of the moment of inertias about two perpendicular axes in the plane of the object provided that all the three axes are concurrent.



Ex. Find moment of inertia of a uniform disk of mass m and radius r about one of its diameter.

Sol. In the adjoining figure a disk is shown with two of its diameter perpendicular to each other. These diameters are along the x and the y -axis of a coordinate system. The x -axis is perpendicular to the plane of the disk and passes through its center is also shown.



Since the disk is symmetric about both the diameters, moment of inertias about both the diameters must be equal. Thus substituting this in the theorem of perpendicular axes, we have

$$I_z = I_y + I_x \rightarrow I_z = 2I_x = 2I_y$$

Moment of inertia of the disk about the z -axis is $I_z = \frac{1}{2}mr^2$. Substituting it in the above equation, we have

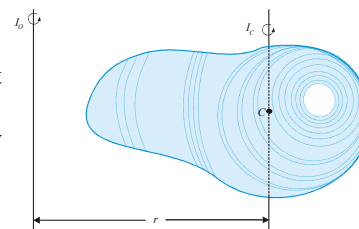
$$I_x = I_y = \frac{1}{2}I_z = \frac{1}{4}mr^2 \quad \text{Ans.}$$

Theorem of Parallel Axes

This theorem also known as Steiner's theorem can be used to determine the moment of inertia of a rigid body about any axis, if the moment of inertia of the body about a parallel axis passing through mass center of the body and perpendicular distance between both the axes is known.

Consider a body of arbitrary shape and mass m shown in the figure. Its moment of inertia I_o and I_c are defined about two parallel axes. The axis about which moment of inertia I_c is defined passes through the mass center C . Separation between the axes is r . These two moment of inertias are related by the following equation.

$$I_o = I_c + Mx_C^2$$



The above equation is known as the theorem of parallel axes or Steiner's theorem.

- (i) The moment of inertia about any axis parallel to an axis through the mass center is given by sum of moment of inertia about the axis through the mass center and product term of mass of the body and square of the distance between the axes.
- (ii) Among all the parallel axes the moment of inertia of a rigid body about the axis through the mass center is the minimum moment of inertia.

The second term added to the moment of inertia I_c about the centroidal axis in the above equation can be recognized as the moment of inertia of a particle of mass equal to that of the body and located at its mass center. It again reveals that *the plane motion of a rigid body is superposition of pure rotation about the mass center or centroidal rotation and translation of its mass center.*

Ex. Find moment of inertia of a uniform ring, uniform disk, uniform cylinder and uniform sphere each of mass m and radius r about their instantaneous axis of rotation in rolling.

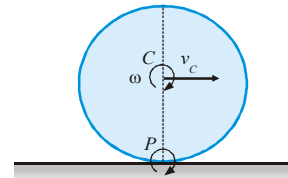
Sol. In rolling the instantaneous axis of rotation passes through the point of contact P with the surface on which the body rolls. Each of these bodies has round section of radius r and can be represented in the adjoining figure. Denoting the moment of inertia about the instantaneous axis of rotation by I_p and through parallel centroidal axis by I_c , we have from the theorem of parallel axes.

Ring $I_p = I_c + Mr^2 = Mr^2 + Mr^2 = 2Mr^2$

Disk $I_p = I_c + Mr^2 = \frac{1}{2}Mr^2 + Mr^2 = \frac{3}{2}Mr^2$

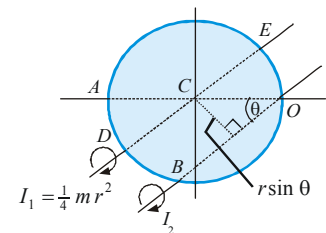
Cylinder $I_p = I_c + Mr^2 = \frac{1}{2}Mr^2 + Mr^2 = \frac{3}{2}Mr^2$

Sphere $I_p = I_c + Mr^2 = \frac{2}{5}Mr^2 + Mr^2 = \frac{7}{5}Mr^2$



Ex. Find expression for moment of inertia of a uniform disk of mass m , radius r about one of its secant making an angle θ with one of its diameter.

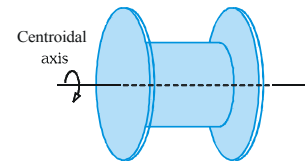
Sol. A disk, the secant OB and diameter OA are shown in the adjoining figure. The secant OB is parallel to another diameter DE . Moment of inertia of the disk about one of its diameter is $\frac{1}{4}mr^2$ and hence moment of inertia I_1 about the diameter DE . Distance between the secant OB and the parallel diameter DE is $r \sin \theta$.



Substituting above information in the theorem of parallel axes, we have

$$I_2 = I_1 + m(r \sin \theta)^2 \rightarrow I_2 = mr^2 \left(\frac{1}{4} + \sin^2 \theta \right)$$

Ex. Find moment of inertia about centroidal axis of a bobbin, which is constructed by joining coaxially two identical disks each of mass m and radius $2r$ to a cylinder of mass m and radius r as shown in the figure.

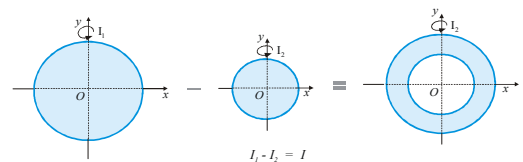


Sol. The bobbin is a composite body made by joining two identical disks coaxially to cylinder. The moment of inertia I of the bobbin equals to the sum of moment of inertias of the two disks and moment of inertia of the cylinder about their centroidal axes. Using expressions for the moment of inertia for disk and cylinder, we have

$$I = 2I_{\text{disk}} + I_{\text{cylinder}} \rightarrow I = 2 \left\{ \frac{1}{2} m (2r)^2 \right\} + \frac{1}{2} mr^2 = \frac{9}{2} mr^2$$

Ex. Find moment of inertia about one of diameter of a hollow sphere of mass m , inner radius r and outer radius R .

Sol. The hollow sphere is assumed as if a concentric smaller sphere of radius r is removed from a larger sphere of radius R . Thus the moment inertia of the hollow sphere about any axes can be obtained by subtracting moment of inertia of the smaller sphere from that of the larger sphere. As shown in the following figure.



Let the mass of the hollow sphere is m .

Density of the material used is

$$\rho = \frac{3m}{4\pi(R^3 - r^3)}$$

Masses m_1 and m_2 of the smaller spheres are

$$m_1 = \rho\left(\frac{4}{3}\pi R^3\right) = \frac{mR^3}{R^3 - r^3} \quad \text{and} \quad m_2 = \rho\left(\frac{4}{3}\pi r^3\right) = \frac{mr^3}{R^3 - r^3}$$

Subtracting I_2 from I_1 we have I .

$$I = I_1 - I_2 \rightarrow I = \frac{2m_1R^2}{5} - \frac{2m_2r^2}{5} = \frac{2m(R^5 - r^5)}{5(R^3 - r^3)}$$

Radius of Gyration

It is the radial distance from a rotation axis at which the mass of an object could be concentrated without altering the moment of inertia of the body about that axis.

If the mass m of the body were actually concentrated at a distance k from the axis, the moment of inertia about that axis would be mk^2 .

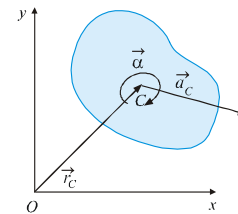
$$k = \sqrt{\frac{I}{m}}$$

The radius of gyration has dimensions of length and is measured in appropriate units of length such as meters.

Force and Torque equations in General Plane Motion

A rigid body is a system of particles in which separation between the particles remains unchanged under all circumstances. For a system of particles sum of all the external forces equals to product of mass of the whole system and acceleration of mass center. This fact we can express by the following equation and call it as force equation.

$$\Sigma \vec{F}_i = M\vec{a}_c$$



To make use of the above idea we conceive general plane motion as superposition of translation of the mass center and simultaneous centroidal rotation. In the figure is shown a body in general plane motion with acceleration of the mass center and angular acceleration about the centroidal axis. Therefore we can write torque of all the external forces about an axis parallel to the original one and passing through the origin of an inertial frame as sum of moment of effective force ($M\vec{a}_c$) on mass center and effective torque ($I_c\vec{\alpha}$) of centroidal rotation. Here I_c is the moment of inertia of the body about the centroidal axis parallel to the original one.

$$\Sigma \vec{\tau}_o = \vec{r}_c \times M\vec{a}_c + I_c\vec{\alpha}$$

If we write torque of all the external forces about the centroidal axis the first term on the right hand side vanishes and we obtain torque equation describing the centroidal rotation.

$$\Sigma \vec{\tau}_c = I_c\vec{\alpha}$$

If instantaneous axis of rotation is known we can write the torque equation about it. For the purpose, we make use of parallel axis theorem of moment of inertia.

$$\Sigma \vec{\tau}_{IAR} = I_{IAR}\vec{\alpha}$$

Dynamics of Rigid Body as a System of Particles

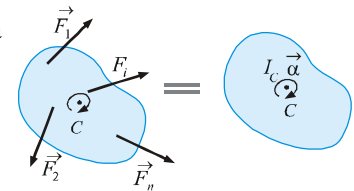
Motion of a rigid body either pure rotation or rotation about axis in translation can be thought and analyzed as superposition of translation of any of its particle and simultaneous rotation about an axis passing through that particle provided that the axis remain parallel to the original one. As far as kinematics is concerned this particle may or may not be the mass center. Whereas in dealing with kinetics, general plane motion is conceived as superposition of translation motion of the mass center and simultaneous centroidal rotation.

To make use the above idea and equations developed in the previous section we classify pure rotation i.e. rotation about fixed axis into two categories and deal with general plane motion as the third category.

Pure centroidal rotation: Rotation about fixed axis through mass centre

In this kind of rotation motion the axis of rotation passes through the mass center and remain fixed in space. Rotation of ceiling fan is a common example of this category. It is a subcategory of pure rotation. The axis of rotation passes through the mass center and remains fixed. In this kind of rotation the mass center of the body does not move.

In the figure, free body diagram and kinetic diagram of a body rotating about a fixed axis passing through its mass center C is shown. The mass center of the body does not accelerate; therefore we only need to write the torque equation.

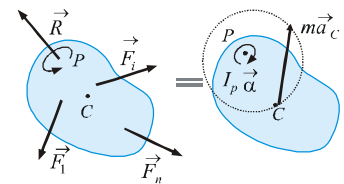


$$\Sigma \vec{\tau}_C = I_C \vec{\alpha}$$

Rotation about fixed axis not passing through mass center

In this kind of rotation the axis of rotation remains fixed and does not pass through the mass center. Rotation of door is a common example of this category. Doors are hinged about their edges; therefore their axis of rotation does not pass through the mass center. In this kind of rotation motion the mass center executes circular motion about the axis of rotation.

In the figure, free body diagram and kinetic diagram of a body rotating about a fixed axis through point P is shown. It is easy to conceive that as the body rotates its mass center moves on a circular path of radius $\vec{r}_{P/C}$. The mass center of the body is in translation motion with acceleration \vec{a}_C on circular path of radius $r_{P/C}$. To deal with this kind of motion, we have to make use of both the force and the torque equations.



Translation of mass center $\Sigma \vec{F}_i = M \vec{a}_C = M \vec{\alpha} \times \vec{r}_{C/P} - M \omega^2 \vec{r}_{C/P}$

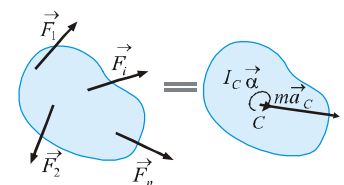
Centroidal Rotation $\Sigma \vec{\tau}_C = I_C \vec{\alpha}$

Making use of parallel axis theorem ($I_P = M r_{P/C}^2 + I_C$) and $\vec{a}_{C/P} = \vec{\alpha} \times \vec{r}_{C/P} - \omega^2 \vec{r}_{C/P}$ we can write the following equation also.

Pure Rotation about P $\Sigma \vec{\tau}_P = I_P \vec{\alpha}$

General Plane Motion: Rotation about axis in translation motion

Rotation of bodies about an axis in translation motion can be dealt with either as superposition of translation of mass center and centroidal rotation or assuming pure rotation about the instantaneous axis of rotation. In the figure is shown the free body diagram and kinetic diagram of a body in general plane motion.



Translation of mass center $\sum_{i=0}^n \vec{F}_i = M \vec{a}_C$

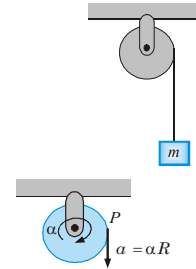
Centroidal Rotation $\sum_{i=1}^n \vec{\tau}_C = I_C \vec{\alpha}$

This kind of situation can also be dealt with considering it rotation about IAR. It gives sometimes quick solutions, especially when IAR is known and forces if acting at the IAR are not required to be found.

PHYSICS FOR JEE MAINS & ADVANCED

Ex. A block of mass m is suspended with the help of a light cord wrapped over a cylindrical pulley of mass M and radius R as shown in the figure. The system is released from rest. Find the angular acceleration of the pulley and the acceleration of the block.

Sol. After the system is released, the block is in translation motion and the pulley in rotation about an axis passing through its mass center i.e. in pure rotation.



Let the block moves vertically down with acceleration a pulling the cord down and causing the pulley to rotate clockwise. Since the cord is inextensible every point on its vertical portion and point of contact P of the pulley move down with acceleration a as shown in the adjacent figure. It is the tangential acceleration of point P so the angular acceleration α of the pulley rotating in clockwise sense is given by

$$a = \alpha R \quad \dots\dots\dots(1)$$

The forces acting on the pulley and on the block are shown in their free-body diagrams along with the effective torque $I_C \alpha$ of the pulley and effective force ma of the block. Here T is the tension in the string, R is the reaction by the axil of the pulley, Mg is weight of the pulley and mg is weight of the block.

The pulley is in rotation about fixed axis through its mass center so we use eq. .

$$\sum \vec{\tau}_c = I_c \vec{\alpha} \rightarrow TR = I_c \alpha$$

After substituting $I_c = \frac{1}{2} MR^2$ and α from eq. (1),

we have

$$T = \frac{1}{2} Ma \quad \dots\dots\dots(2)$$

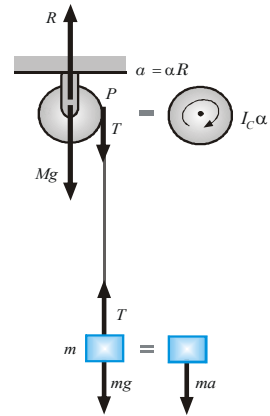
The block is in translation motion, so we use Newton's second law

$$\sum \vec{F} = m\vec{a} \rightarrow mg - T = ma \quad \dots\dots\dots(3)$$

From equation (2) and (3), we have

Acceleration of the block
$$a = \frac{2mg}{M + 2m}$$

From eq. (1) and the above, we have
$$\alpha = \frac{2mg}{R(M + 2m)}$$



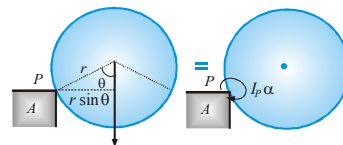
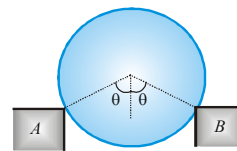
Ex. A cylinder of radius r and mass m rests on two horizontal parallel corners of two platforms. Both the platforms are of the same height. Platform B is suddenly removed. Assume friction between the corner of the platform A and cylinder to be sufficient enough to prevent sliding. Determine angular acceleration of the cylinder immediately after the removal of the platform B .

Sol. Since the cylinder does not slide at the point of contact with the corner of platform A , it rotates about fixed axis through the point of contact in subsequent motion. Torque equation should be used.

Forces acting on the cylinder and the effective torque are shown in the adjacent figure. Since forces acting at the point of contact does not contribute any torque about it, the normal reaction from the corner and the friction force are not shown in the free body diagram.

Applying the torque equation about the fixed axis through P , we have

$$\sum \vec{\tau}_p = I_p \vec{\alpha} \rightarrow mgr \sin \theta = I_p \alpha \quad \dots\dots\dots(1)$$



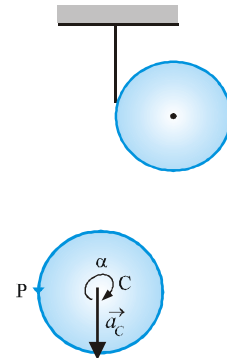
Applying the theorem of parallel axes and expression for moment of inertia about centroidal axes, we obtain moment of inertia I_p about an axis through the point P .

$$I_p = I_C + mr^2 \rightarrow I_p = \frac{1}{2}mr^2 + mr^2 = \frac{3}{2}mr^2 \quad \dots\dots\dots(2)$$

Substituting I_p from eq. (2) in (1), we have

$$\text{Angular acceleration of the cylinder } \alpha = \frac{2g \sin \theta}{3r}$$

Ex. A thread is wrapped around a uniform disk of radius r and mass m . One end of the thread is attached to a fixed support on the ceiling and the disk is held stationary in vertical plane below the fixed support as shown in the figure. When the disk is set free, it rolls down due to gravity. Find the acceleration of the center of the disk and tension in the thread.



Sol. The point P , where the thread leaves the disk is always at instantaneous rest; therefore the disk can be assumed rolling without slipping with ICR at point P . Acceleration of the mass center \vec{a}_C and angular acceleration of the disk are shown in the adjacent figure. Applying condition for rolling on stationary surface, we have

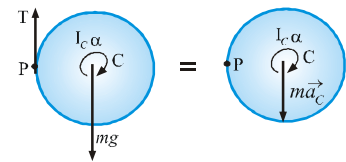
$$\vec{a}_C = \vec{\alpha} \times \vec{r}_{C/P} \rightarrow a_C = \alpha r \quad \dots\dots\dots(1)$$

The disk rolls down on the vertical stationary thread. Its motion can either be analyzed as superposition of translation of the mass center and simultaneous centroidal rotation or a pure rotation about ICR. Since tension, which acts at the ICR is asked; we prefer superposition of translation of the mass center and simultaneous centroidal rotation.

Forces acting on the disk are tension T applied by the thread at point P and weight of the disk. These forces and the effective force ma_C and effective torque $I_C \alpha$ are shown in the adjacent figure.

Applying Newton's second law for translation of mass center, we have

$$\sum \vec{F}_i = M\vec{a}_C \rightarrow mg - T = ma_C \quad \dots\dots\dots(2)$$



Applying torque equation for centroidal rotation, we have

$$\sum \vec{\tau}_C = I_C \vec{\alpha} \rightarrow Tr = I_C \alpha$$

Substituting $\frac{1}{2}mr^2$ for I_C and α from eq. (1), we have

$$T = \frac{1}{2}ma_C \quad \dots\dots\dots(3)$$

From eq. (2) and (3), we have

$$\text{Acceleration of the mass center } a_C = \frac{2}{3}g$$

$$\text{Tension in the string } T = \frac{1}{3}mg$$

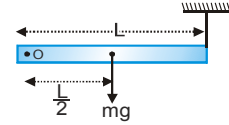
PHYSICS FOR JEE MAINS & ADVANCED

Ex. A rod is pivoted at its one edge about point O. Other edge of rod is suspended from the ceiling through rope as shown. If the rope is suddenly cut then find the angular acceleration of rod.

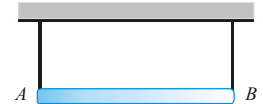


Sol. When the rope is cut, weight of rod due to force of gravity will produce torque about point O. $\tau = I\alpha$, consider force mg acting on shown in figure at CM of rod (i.e. middle point of the rod)

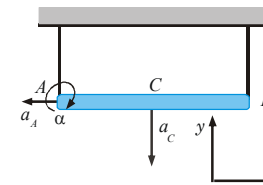
$$mg \left[\frac{L}{2} \right] = \frac{mL}{3} \alpha \quad \Rightarrow \quad \frac{g}{2} = \frac{L\alpha}{3} \quad \Rightarrow \quad \alpha = \frac{3g}{2L}$$



Ex. A uniform rod AB of mass m and length ℓ is suspended in horizontal position with the help of two strings as shown in the figure. The string supporting the end B is cut. Find acceleration of the mass center and end A immediately after the string is cut.



Sol. After the string is cut forces acting on the rod are tension in the string A and weight of the rod. Both of these forces are in vertical direction so acceleration of the mass center C must be vertically downwards. The string is inextensible so the point A can have acceleration only in horizontal direction. Let acceleration of the mass center C is denoted by a_c downwards and acceleration a_A of the point A towards the left as shown in the adjacent figure.



Applying the relative motion equation, we have

$$\vec{a}_C = \vec{a}_A + \vec{a}_{C/A} \rightarrow \vec{a}_C = \vec{a}_A + \vec{\alpha} \times \overline{AC} - \omega^2 \overline{AC}$$

Immediately after the string is cut the rod cannot acquire any angular velocity. So the last term in the above equation vanishes. Now we have

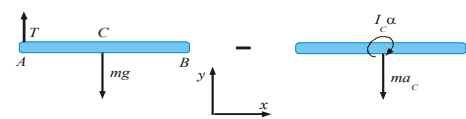
$$-a_c \vec{j} = -a_A \vec{i} + (-\alpha \vec{k}) \times \left(\frac{1}{2} \ell \vec{i} \right)$$

Comparing x and y -components on both the sides we have

$$a_c = \frac{1}{2} \alpha \ell \quad \dots\dots\dots(1)$$

$$a_A = 0$$

The rod is in plane motion, which can be analyzed as superposition of translation of the mass center and simultaneous centroidal rotation. The forces acting on the rod immediately after the string at B is cut, the effective force ma_c and the effective torque $I_c \alpha$ are shown in the adjacent figure.



Applying Newton's second law for translation of mass center, we have

$$\sum \vec{F}_i = M\vec{a}_C \rightarrow mg - T = ma_c \quad \dots\dots\dots(2)$$

Applying torque equation for centroidal rotation, we have

$$\sum \vec{\tau}_C = I_C \vec{\alpha} \rightarrow T \left(\frac{1}{2} \ell \right) = I_C \alpha$$

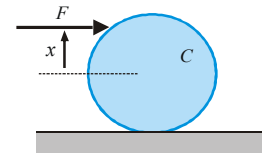
Substituting $\frac{1}{12} m \ell^2$ for I_C and α from eq. (1), we have

$$T = \frac{1}{3} ma_c \quad \dots\dots\dots(3)$$

From eq. (2) and (3), we have

$$\text{Acceleration of the mass center} \quad a_c = \frac{3}{4} g$$

Ex. A uniform rigid body of mass m and round section of radius r rests on rough horizontal surface. The radius of gyration of the body about its centroidal axis is k . It is pushed by a constant horizontal force F . The height x of the point where the force is applied can be adjusted.

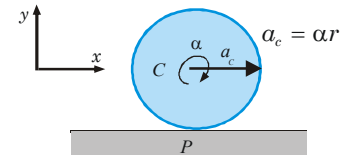


(a) Deduce suitable expression for magnitude and direction of the friction force necessary to ensure rolling.

(b) How direction of the friction force depends on x .

Sol. (a) The problem requires solution of the force and the torque equations consistent with the condition of rolling, so it is not necessary to decide the direction of friction as priory. To start with let the static friction f_s acts in the forward direction.

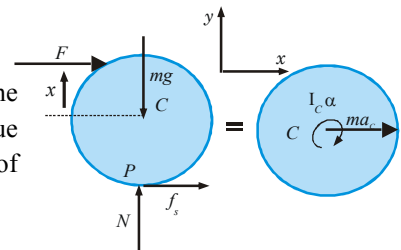
Let the mass center of the body moves towards right with acceleration a_c and angular acceleration α of the body is in clockwise sense as shown in the adjacent figure.



Necessary condition for rolling in terms of acceleration a_c of mass center and angular acceleration α of the body is

$$\vec{a}_c = \vec{\alpha} \times \vec{r}_{C/P} \rightarrow \vec{a}_c = (-\alpha \vec{k}) \times r \vec{j} \Rightarrow a_c = \alpha r \quad \dots\dots(1)$$

The forces acting on the body are its weight mg , the normal reaction N from the ground and the force of static friction f_s . These forces are shown in the adjacent figure together with the effective force ma_c and the effective torque $I_c \alpha$ about the mass center. We analyze the problem as superposition of translation of the mass center and simultaneous centroidal rotation.



Applying Newton's second law for translation of mass center, we have

$$\sum F_x = Ma_{Cx} \rightarrow F + f_s = ma_c \quad \dots\dots(2)$$

Applying torque equation for centroidal rotation, we have

$$\sum \vec{r}_C = I_C \vec{\alpha} \rightarrow Fx - f_s r = I_C \alpha$$

Substituting mk^2 for I_C , and value of α from eq. (1), we have

$$F \frac{x}{r} - f_s = \frac{mk^2 a_c}{r^2} \quad \dots\dots(3)$$

From eq. (2) and (3), we have

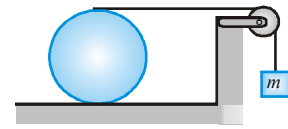
Force of static friction
$$f_s = F \left(\frac{x - k^2/r}{r + k^2/r} \right)$$

(b) The above expression shows that to ensure rolling

- For $x > k^2/r$, the friction is in direction of the applied force F .
- For $x = k^2/r$, no friction is required to ensure rolling.
- For $x < k^2/r$, the friction must be in opposite to the applied force F .

PHYSICS FOR JEE MAINS & ADVANCED

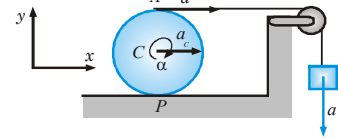
Ex. A block of mass m is attached at one end of a thin light cord, which passes over an ideal pulley. At the other end, it is wrapped around a cylinder of mass M , which can roll without slipping over a horizontal plane.



- (a) What is the acceleration of the block?
- (b) What is the friction force on the cylinder?

Sol. The problem requires solution of the force and the torque equations consistent with the condition of rolling, so it is not necessary to decide the direction of friction as priory. To start with let the static friction f_s acts in the forward direction.

- (a) Let the block descend with acceleration a . Since the cord is inextensible the top point A of the cylinder also moves with the same acceleration. Applying relative motion equation with the condition required for rolling that the particle of the cylinder at the point of contact has no acceleration parallel to the horizontal plane.

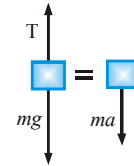


$$\vec{a}_A = \vec{a} \times \vec{r}_{A/P} \rightarrow a_A \vec{i} = (-\alpha \vec{k}) \times (2r \vec{j}) \Rightarrow \alpha = \frac{a}{2r} \dots\dots\dots(1)$$

From eq. (1) and relative motion equation for P and the center C , we have

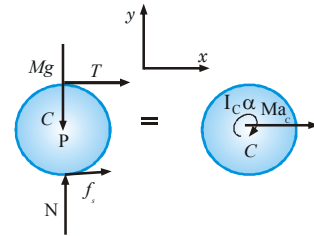
$$\vec{a}_C = \vec{a} \times \vec{r}_{C/P} \rightarrow a_C \vec{i} = (-\alpha \vec{k}) \times (r \vec{j}) \Rightarrow a_C = \frac{1}{2}a \dots\dots\dots(2)$$

The block is in translation motion under the action of its weight mg and tension T in the string. These forces and the effective force ma are shown in the adjacent figure. Applying Newton's second law for translation of mass center, we have



$$\sum \vec{F}_i = M\vec{a}_C \rightarrow mg - T = ma \dots\dots\dots(3)$$

The cylinder is in rolling under the action of its weight Mg , normal reaction N from the ground; tension T in the cord and force of static friction f_s . These forces, the effective force Ma_C and the effective torque $I_C \alpha$ are shown in the adjacent figure.



Applying Newton's second law for translation of mass center, we have

$$\sum F_x = Ma_{Cx} \rightarrow T + f_s = Ma_C$$

Substituting a_C from eq. (2), we have

$$T + f_s = \frac{1}{2}Ma \dots\dots\dots(4)$$

Applying torque equation for centroidal rotation, we have

$$\sum \vec{\tau}_C = I_C \vec{\alpha} \rightarrow Tr - f_s r = I_C \alpha$$

Substituting $\frac{1}{2}mr^2$ for I_C , and value of α from eq. (1), we have

$$T - f_s = \frac{1}{4}Ma \dots\dots\dots(5)$$

From eq. (3), (4) and (5), we have

Acceleration of the block $a = \frac{8mg}{3M + 8m}$

- (b) From eq. (4), (5) and above value of acceleration a , we have

Force of static friction $f_s = \frac{Mmg}{3M + 8m}$

ENERGY METHODS

Newton’s laws of motion tell us what is happening at an instant, while method of work and energy equips us to analyze what happens when a body moves from one place to other or a system changes its configuration. In this section, we introduce how to use methods of work and energy to analyze motion of rigid bodies.

Concept of Work in rotation motion

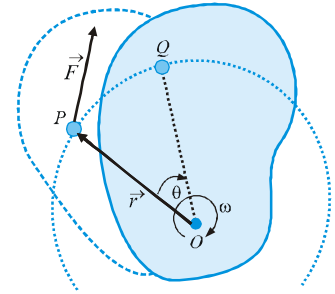
Work of a force is defined as the scalar product of the force vector and displacement vector of the point of application of the force. If during the action of a force \vec{F} its point of application moves from position \vec{r}_1 to \vec{r}_2 , the work $W_{1 \rightarrow 2}$ done by the force is expressed by the following equation.

$$W_{1 \rightarrow 2} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$$

Either we can use of this idea to calculate work of a force or its modified version in terms of torque and angular displacement.

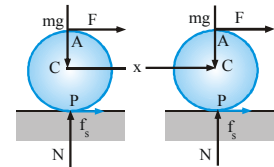
The work done by a torque during a finite rotation of the rigid body from initial value θ_i of the angle θ to final value θ_f can be obtained by integrating both the sides of the equation given

$$W_{i \rightarrow f} = \int_{\theta_i}^{\theta_f} \vec{\tau}_o \cdot d\vec{\theta}$$



Ex. A thin light cord is wound around a uniform cylinder placed on a rough horizontal ground. When free end of the cord is pulled by a constant force F the cylinder rolls. Denote radius of the cylinder by r and obtain expression for work done by each of the forces acting on the cylinder when center of the cylinder shifts by distance x .

Sol. Forces acting on the cylinder are its weight W , the normal reaction from the ground N , the tension T in the cord and the force of static friction f_s . The tension in the cord equals to the applied force F . These forces are shown in the adjacent figure.



In rolling point of contact P is at instantaneous rest, the center C moves with velocity $v_C = \omega r$ and the top point moves with velocity $v_A = 2v_C = 2\omega r$ both parallel to the surface on which body rolls. Since the cord is inextensible displacement of the top point equals to the displacement of the free end of the cord. These fact suggests that during displacement x of the center the free end of the cord shift through a distance $2x$.

Work done by the weight of the cylinder.

$W_g = 0$ The weigh is assumed to act on the center of gravity which coincides with the mass center in uniform gravitation field near the ground. The displacement x of the mass center and weight both are perpendicular to each other so the work done by gravity is zero.

Work done by the normal reaction on the cylinder

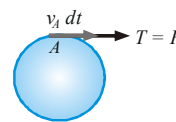
$W_N = 0$ The normal reaction acts on the particle of the body which is in contact with the ground. The particles making contact continuously change and remain at instantaneous rest during contact. Therefore normal reaction does no work.

Work done by the force of static friction.

$W_{f_s} = 0$ The force of static friction f_s acts on the particle of the body which is in contact with the ground. The particles making contact continuously change and remain at instantaneous rest during contact. Therefore force of static friction f_s does no work.

Work done by the tension in the cord.

$W_T = W_F$ The particle of the wheel on which the tension in the cord acts is at the top point. Though this particle is also continuously changing but it is not in instantaneous rest and has velocity v_A . So in every infinitesimally small time interval displacement of this particle is $v_A dt = 2v_C dt$, thus work done dW_T by the tension during a time interval dt



$$dW_T = T(2v_C dt) = 2F(v_C dt)$$

When the center shifts by a distance x the work done by the tension becomes

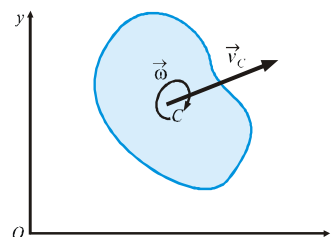
$$W_T = W_F = 2Fx$$

Potential Energy of a rigid body

Since potential energy of a system is function of its configuration and does not depend on the manner in which the system is brought into a particular configuration, hence it does not depend on motion involved whether it is translation, rotational or their combination.

Kinetic Energy of a rigid body in rotation motion

A rigid body can be represented as a system of large number of particles, which keep their mutual distances unchanged in all circumstances. Kinetic energy of the whole body must be sum of kinetic energies of all of its particles. In this section we develop expressions for kinetic energy of a rigid body.



Kinetic Energy of a rigid body in plane motion

In the figure is shown a body in plane motion. Its mass center at an instant is moving with velocity \vec{v}_C and rotating with angular velocity $\vec{\omega}$. Both these motions are shown superimposed in the given figure.

Kinetic energy too can be written as sum of kinetic energy $(\frac{1}{2}Mv_C^2)$ due to translation motion of the mass center and kinetic energy $(\frac{1}{2}I_C\omega^2)$ due to centroidal rotation.

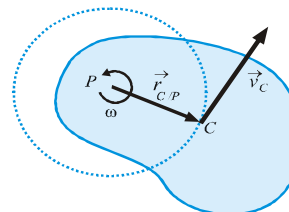
$$K = \frac{1}{2}Mv_C^2 + \frac{1}{2}I_C\omega^2$$

If location of the instantaneous axis of rotation (IAR) is known, making use of the parallel axis theorem we can write kinetic energy by the following equation also.

$$K = \frac{1}{2}I_{IAR}\omega^2$$

Kinetic Energy of a rigid body in rotation about fixed axis not passing through the mass centre

In this kind of motion the mass center is in circular motion about the axis of rotation. In the figure is shown a body rotation with angular velocity ω about a fixed axis through point P and perpendicular to plane of the paper. Mass center moves with speed $v_C = \omega r$. Kinetic energy of the body can now be expressed by the following equation.



$$K = \frac{1}{2}Mv_C^2 + \frac{1}{2}I_C\omega^2$$

Making use of the parallel axis theorem $(I_P = Mr_{P/C}^2 + I_C)$ we can write kinetic energy by the following equation also.

$$K = \frac{1}{2}I_P\omega^2$$

Kinetic Energy of a rigid body in pure centroidal rotation

In pure centroidal rotation the mass center remain at rest; therefore kinetic energy due to translation of mass center vanishes.

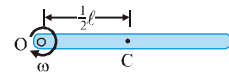
$$K = \frac{1}{2} I_C \omega^2$$

Ex. A rod of mass m and length ℓ is pivoted to a fixed support at one of its ends O . It is rotating with constant angular velocity ω . Write expression for its kinetic energy.



Sol. If the point C is the mass center of the rod, from theorem of parallel axes, the moment of inertia I_O of the rod about the fixed axis is

$$I_O = I_C + m(OC)^2 \rightarrow I_O = I_C + \frac{1}{4} m\ell^2$$



Substituting $\frac{1}{12} m\ell^2$ for I_C , we have

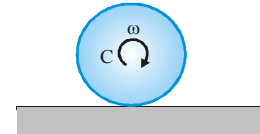
$$I_O = \frac{1}{3} m\ell^2$$

Kinetic energy of the rod equals to kinetic energy due to rotation about the fixed axis.

$$K = \frac{1}{2} I_O \omega^2 \rightarrow \text{Using above expression for } I_O, \text{ we have}$$

$$K = \frac{1}{6} m\ell^2 \omega^2$$

Ex. A uniform rigid body of mass m and round section of radius r is rolling on horizontal ground with angular velocity ω . Its radius of gyration about the centroidal axis is k .



(a) Write expression of its kinetic energy.

(b) Also express the kinetic energy as sum of kinetic energy due to translation of mass center and kinetic energy due to simultaneous centroidal rotation.

Sol. (a) The point of contact with ground of a body rolling on the ground is its ICR. Let the point P is the ICR as shown in the adjacent figure. The geometrical center C of a uniform body and the mass center coincide. Therefore moment of inertia I_P of the body about the ICR can be written by using the theorem of parallel axes.

$$I_P = I_C + m(PC)^2 \rightarrow I_P = I_C + mr^2$$

Substituting $I_C = mk^2$, we have

$$I_P = m(k^2 + r^2) \quad \dots\dots\dots(1)$$

Kinetic energy of a rigid body equals to kinetic energy due to rotation about the ICR.

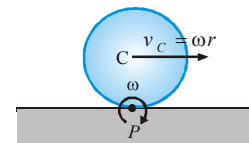
$$K = \frac{1}{2} I_P \omega^2 \rightarrow \text{Substituting } I_P \text{ from eq. (1), we have}$$

$$K = \frac{1}{2} m(k^2 + r^2) \omega^2$$

(b) Kinetic energy of the body also equals to sum of kinetic energy due to translation of its mass center and kinetic energy due to simultaneous centroidal rotation.

$$K = \frac{1}{2} mv_C^2 + \frac{1}{2} I_C \omega^2 \rightarrow \text{Substituting condition for rolling } v_C = \omega r \text{ and } I_C = mk^2, \text{ we have}$$

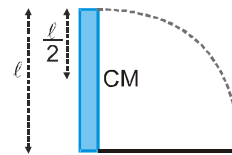
$$K = \frac{1}{2} m(\omega r)^2 + \frac{1}{2} mk^2 \omega^2 = \frac{1}{2} m(r^2 + k^2) \omega^2$$



PHYSICS FOR JEE MAINS & ADVANCED

Ex. A thin meter scale is kept vertical by placing its one end on floor, keeping the end in contact stationary, it is allowed to fall. Calculate the velocity of its upper end when it hit the floor .

Sol. Loss in PE = gain in rotational KE $\frac{mg\ell}{2} = \frac{1}{2}I\omega^2 = \frac{1}{2} \frac{m\ell^2}{3} \times \frac{v^2}{\ell^2} \Rightarrow v = \sqrt{3g\ell}$

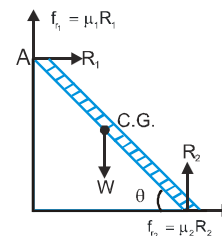


Ex. A uniform rod is made to lean between a rough vertical wall and the ground. Show that the least angle at which the rod can be leaned without slipping is given by $\theta = \tan^{-1} \left(\frac{1 - \mu_1 \mu_2}{2\mu_2} \right)$ where μ_1 is the coefficient of friction between rod and wall, μ_2 is the coefficient of friction between rod and ground.

Sol. For equilibrium of rod $\Sigma F_x = 0 \Rightarrow R_1 = \mu_2 R_2$
 $\Sigma F_y = 0 \Rightarrow \mu_1 R_1 + R_2 = W \Rightarrow (\mu_1 \mu_2 + 1) R_2 = W$

Taking torque about point A : $W \left(\frac{\ell}{2} \cos \theta \right) + \mu_2 R_2 (\ell \sin \theta) = R_2 (\ell \cos \theta) \Rightarrow W = 2R_2 (1 - \mu_2 \tan \theta)$

$$\therefore \mu_1 \mu_2 + 1 = 2 - 2\mu_2 \tan \theta \Rightarrow \tan \theta = \left(\frac{1 - \mu_1 \mu_2}{2\mu_2} \right)$$



POWER

Power defined as the time rate of work done, takes into account the duration in which work is done. To calculate power we make use of the following equation.

$$P = \frac{dW}{dt}$$

Instantaneous power of a force can be expressed by the following equation. Here velocity \vec{v} is the velocity of the point of application of the force \vec{F} .

$$P = \vec{F} \cdot \vec{v}$$

Work and Energy Theorem

Work energy theorem can be applied in similar fashion as it was applied to analyze translation motion of a single body or a system of several bodies.

The work energy theorem relates kinetic energy K_1 and K_2 of a body in its initial and final position with work $W_{1 \rightarrow 2}$ done by all the external forces acting on the body to carry it from the initial position to the final position according to the following equation.

$$W_{1 \rightarrow 2} = K_2 - K_1$$

This equation is applicable in all inertial as well as non inertial frames. To write equation of work energy theorem kinetic energy must be written relative to the frame under consideration. To calculate work consider only all the physical forces in inertial frame and all the physical forces as well as pseudo force in non-inertial frame and displacement of point of applications of these forces relative to the frame under consideration.

For a system of several bodies the corresponding equation of work energy theorem can be obtained by applying the theorem for each individual body and then adding all of them. In this way we obtain an equation of the following form.

$$\Sigma W_{1 \rightarrow 2} = K_{2,s} - K_{1,s}$$

Here the term $\Sigma W_{1 \rightarrow 2}$ equals to the work of all the forces acting on various bodies irrespective of whether the force are internal or external from point of view of the system under consideration. In systems of several bodies interconnected by links of constant length e.g. inextensible cords, rods etc. or body in direct contact the total work of internal forces vanishes. The work done by internal conservative forces can be accounted for by decrease in corresponding potential energies. The terms $K_{2,s}$ and $K_{1,s}$ are total kinetic energies of all the bodies in initial and final configurations of the system.

Conservation of Mechanical Energy

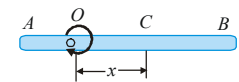
The work of conservative forces equals to decrease in potential energy. When a single rigid body moves or a system of rigid body changes its configuration under the action of conservative forces and non conservative forces are either not present or if present do no work, the work energy principle can be expressed as

$$U_{1,s} + K_{1,s} = U_{2,s} + K_{2,s}$$

The above equation expresses the law of conservation of mechanical energy and states that if a rigid body moves or a system consisting of several rigid bodies changes its configuration under action of conservative forces the mechanical energy i.e. sum of kinetic and potential energy remain constant; provided that non conservative, if present, do no work.

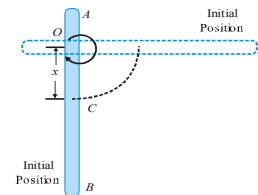
Though the work energy principle and the law of conservation of mechanical energy are equivalent, we prefer to use the former to account for non conservative forces easily

Ex. A uniform rod AB of mass m and length ℓ is pivoted at a point (O) to rotate in the vertical plane. The rod is held in horizontal position and released. Find the distance x of the pivot from the mass center (C) of the rod, so that angular speed ω of the rod as it passes through the vertical position is maximum.



Sol. The problem involves change in angular velocity with change in position, therefore demands application of principle of work and energy.

The rod when released rotates about a fixed horizontal axis passing through the point O . Its initial and final positions are shown in the adjacent figure. Moment of inertia of the rod about the pivot O can be calculated by theorem of parallel axes.



$I_o = I_C + mx^2 \rightarrow$ Substituting $\frac{1}{12} m\ell^2$ for I_C , we have

$$I_o = \frac{1}{12} m(\ell^2 + 12x^2) \quad \dots\dots\dots(1)$$

Kinetic energy in the initial position.

$$K = \frac{1}{2} I_o \omega^2 \rightarrow K_1 = 0 \quad \dots\dots\dots(2)$$

Kinetic energy in the final position.

$K = \frac{1}{2} I_o \omega^2 \rightarrow$ Substituting for I_o from eq. (1), we have

$$K_2 = \frac{1}{24} m(\ell^2 + 12x^2) \omega^2 \quad \dots\dots\dots(3)$$

Only gravity does work when the rod moves from the initial to final position.

$$W = \int \vec{F} \cdot d\vec{r} \rightarrow W_{1 \rightarrow 2} = mgx \quad \dots\dots\dots(4)$$

Substituting values from eq. (2), (3) and (4) in equation of work energy principle, we have

$$W_{1 \rightarrow 2} = K_2 - K_1 \rightarrow \omega = \sqrt{24gx / (\ell^2 + 12x^2)} \quad \dots\dots\dots(5)$$

The above equation expresses angular velocity of the rod when it passes the vertical position. For it to be maximum

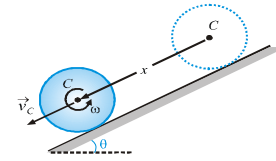
$$\frac{d\omega}{dx} = 0 \rightarrow x = \frac{\ell}{\sqrt{12}}$$

PHYSICS FOR JEE MAINS & ADVANCED

Ex. A uniform rigid body of mass m and round section of radius r rolls down a slope inclined at an angle θ to the horizontal. The radius of gyration of the body about its central axis of symmetry is k .

(a) Derive suitable expressions for angular velocity and velocity of its mass center after it covers a distance x .

(b) Obtain expression for its angular acceleration and acceleration of the mass center.



Sol. The problem involves change in angular velocity with change in position, therefore demands application of principle of work and energy

The geometrical center and mass center for uniform bodies coincide; therefore center C is the mass center.

In rolling the point of contact P must always be at instantaneous rest and angular velocity ω , velocity of center C , angular acceleration α and acceleration of the center must bear the following relations.

$$v_c = \omega r \text{ and } a_c = \alpha r \quad \dots\dots\dots(1)$$

(a) The rolling motion can be analyzed as superposition of translation of the mass center and simultaneous centroidal rotation.

Kinetic energy in the initial position.

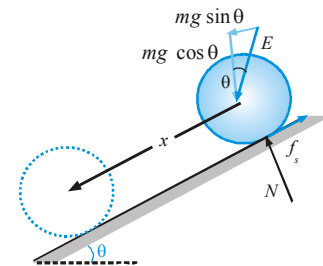
$$K_1 = 0 \quad \dots\dots\dots(2)$$

Kinetic energy in the final position.

$K = \frac{1}{2}mv_c^2 + \frac{1}{2}I_C\omega^2 \rightarrow$ Substituting for $I_C = mk^2$ and v_c from eq. (1), we have

$$K_2 = \frac{1}{2}m(r^2 + k^2)\omega^2 \quad \dots\dots\dots(3)$$

The forces acting on the body are its weight mg , the normal reaction N from the slope and the force of static friction f_s . These forces and displacement are shown in the adjacent figure. The normal reaction and the force of static friction do no work in rolling, it is the weight, which does work.



$$W = \int \vec{F} \cdot d\vec{r} \rightarrow W_{1 \rightarrow 2} = mgx \sin \theta \quad \dots\dots\dots(4)$$

Substituting values from eq. (2), (3) and (4) in equation of work energy principle, we have

$$W_{1 \rightarrow 2} = K_2 - K_1 \rightarrow \omega = \sqrt{\frac{2gx \sin \theta}{k^2 + r^2}}$$

Substituting v_c from eq. (1), we have

$$v_c = \sqrt{\frac{2gx \sin \theta}{(k^2/r^2) + 1}}$$

Acceleration a , velocity v and position coordinate x bear the relation $a = v(dv/dx)$. Therefore acceleration of mass center of the body.

$$a_c = v_c \frac{dv_c}{dx} \rightarrow a_c = \frac{g \sin \theta}{(k^2/r^2) + 1}$$

Substituting a_c in eq. (1), we have the angular acceleration of the body.

$$a_c = \alpha r \rightarrow \alpha = \frac{g \sin \theta}{(k^2/r) + r}$$

METHODS OF IMPULSE AND MOMENTUM

Methods of impulse and momentum describe what happens over a time interval. When motion of a body involves rotation we have to consider angular impulse as well as angular momentum. In this section we discuss concept of angular impulse, angular momentum of rigid body, angular impulse momentum principle and conservation of angular momentum.

Angular Impulse

Like impulse of a force angular impulse of a constant torque equals to product of the torque and concerned time interval and if the torque is not constant it must be integrated with time over the concerned time interval.

If torque $\vec{\tau}_o$ about an axis passing through O is constant, its angular impulse during a time interval from t_1 to t_2 denoted by $\vec{J}_{o,1 \rightarrow 2}$ is given by the following equation.

$$\vec{J}_{o,1 \rightarrow 2} = \vec{\tau}_o (t_2 - t_1)$$

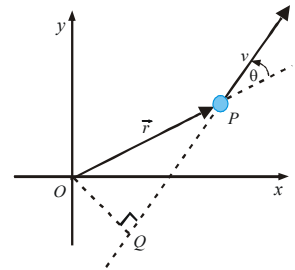
If torque $\vec{\tau}_o$ about an axis passing through O is time varying, its angular impulse during a time interval from t_1 to t_2 denoted by $\vec{J}_{o,1 \rightarrow 2}$ is given by the following equation.

$$\vec{J}_{o,1 \rightarrow 2} = \int_{t_1}^{t_2} \vec{\tau}_o dt$$

Angular momentum of a particle

Angular momentum \vec{L}_o about the origin O of a particle of mass m moving with velocity \vec{v} is defined as the moment of its linear momentum $\vec{p} = m\vec{v}$ about the point O .

$$\vec{L}_o = \vec{r} \times (m\vec{v})$$



Angular Momentum of a Rigid Body

Angular momentum is quantity of rotation motion in a body. The angular momentum of a system of particles is the sum of angular momentum all the particles within the system. A rigid body is an assemblage of large number of particles maintaining their mutual distances intact under all circumstances, therefore angular momentum of a rigid body must be sum of angular momenta of all of its particles.

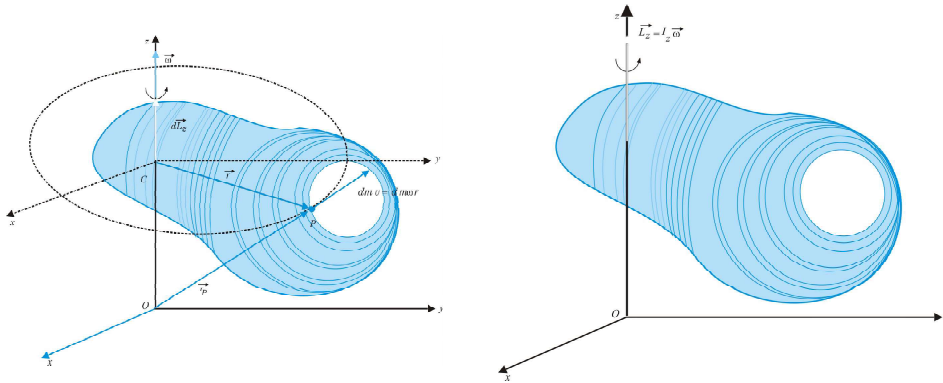
Angular Momentum about a point and about an axis

Angular momentum of a particle is not defined about an axis instead it is defined about a point. Therefore above idea of summing up angular momenta of all the particles about a point gives angular momentum of the rigid body about a point. But while dealing with fixed axis rotation or rotation about axis in translation we need angular momentum about an axis.

Angular momentum about an axis is calculated similar to torque about an axis. To calculate angular momentum of a particle of rigid body about an axis we take moment of momentum of the particle about the point where plane of motion of the point of application of the force intersects the axis.

In the following figure is shown angular momentum $d\vec{L}_z = \vec{r} \times (dm\vec{v}) = r^2 dm\omega$ of a particle P of a rigid body rotating about the z -axis. It is along the z -axis i.e. axis of rotation. In the next figure total angular momentum

$\vec{L}_z = \int d\vec{L}_z = I_z \vec{\omega}$ about the axis of rotation is shown. It is also along the axis of rotation.



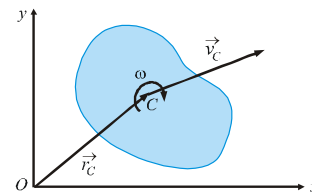
Angular Momentum in general plane motion

Angular momentum of a body in plane motion can also be written similar to torque equation or kinetic energy as sum of angular momentum due to translation of mass center and angular momentum of centroidal rotation about centroidal axis parallel to the original axis.

Consider a rigid body of mass M in plane motion. At the instant shown its mass center has velocity \vec{v} and it is rotating with angular velocity $\vec{\omega}$ about an axis perpendicular to the plane of the figure. Its angular momentum \vec{L}_o about an axis passing through the origin and parallel to the original is expressed by the following equation.

$$\vec{L}_o = \vec{r}_c \times (M\vec{v}_c) + I_c \vec{\omega}$$

The first term of the above equation represent angular momentum due to translation of the mass center and the second term represents angular momentum in centroidal rotation.

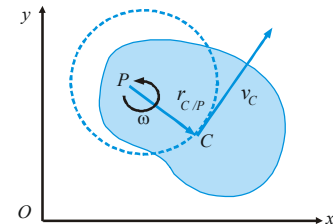


Angular momentum in rotation about fixed axis

Consider a body of mass M rotating with angular velocity ω about a fixed axis perpendicular to plane of the figure passing through point P . Making use of the parallel axis theorem $I_p = Mr_{C/P}^2 + I_c$ and equation $\vec{v}_c = \vec{\omega} \times \vec{r}_{C/P}$ we can express the angular momentum \vec{L}_p of the body about the fixed rotational axis.

$$\vec{L}_p = I_p \vec{\omega}$$

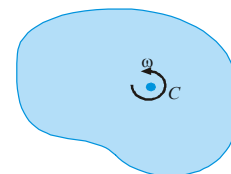
The above equation reveals that the angular momentum of a rigid body in plane motion can also be expressed in a single term due to rotation about the instantaneous axis of rotation.



Angular momentum in pure centroidal rotation

In pure centroidal rotation, mass center remains at rest, therefore angular momentum due to translation of the mass center vanishes.

$$\vec{L}_c = I_c \vec{\omega}$$



Rotational Equivalent of the Newton's Laws of Motion

Differentiating terms on both the sides of equation $\vec{L}_o = \vec{r}_c \times (M\vec{v}_c) + I_c\vec{\omega}$ with respect to time, and making substitution of $\vec{v}_c = d\vec{r}_c/dt$, $\vec{a}_c = d\vec{v}_c/dt$ and $\vec{\alpha} = d\vec{\omega}/dt$ we have

$$\frac{d\vec{L}_o}{dt} = \vec{v}_c \times (M\vec{v}_c) + \vec{r}_c \times M\vec{a}_c + I_c\vec{\alpha}$$

The first term on the right hand side vanishes, so we can write

$$\frac{d\vec{L}_o}{dt} = \vec{r}_c \times M\vec{a}_c + I_c\vec{\alpha}$$

Now comparing the above equation with torque equation $\Sigma\vec{\tau}_o = \vec{r}_c \times M\vec{a}_c + I_c\vec{\alpha}$, we have

$$\Sigma\vec{\tau}_o = \frac{d\vec{L}_o}{dt}$$

The above equation though developed for plane motion only yet is valid for rotation about an axis in rotation also. It states that the net torque about the origin of an inertial frame equals to the time rate of change in angular momentum about the origin and can be treated as a parallel to Newton's second law which states that net external force on a body equals to time rate of change in its linear momentum.

Angular Impulse Momentum Principle

Rearranging the terms and integrating both the sides obtained from previous equation, we can write

$$\Sigma \int_{t_1}^{t_2} \vec{\tau}_o dt = \vec{L}_{o2} - \vec{L}_{o1}$$

The left hand side of the above equation is the angular impulse of torque of all the external forces in the time interval in the time interval t_1 to t_2 .

$$\Sigma \vec{J}_{o,1 \rightarrow 2} = \vec{L}_{o2} - \vec{L}_{o1}$$

The idea expressed by the above equation is known as angular impulse momentum principle and states that increment in the angular momentum of a body about a point in a time interval equals to the net angular impulse of all the external forces acting on it during the concerned time interval.

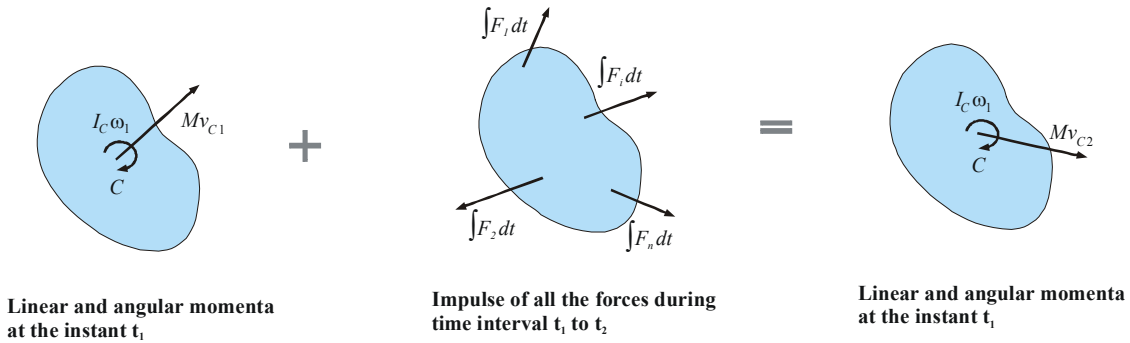
For the ease of application the above equation is rearranged as

$$\vec{L}_{o1} + \Sigma \vec{J}_{o,1 \rightarrow 2} = \vec{L}_{o2}$$

Like linear impulse momentum principle, the angular impulse momentum principle provides us solution of problems concerned with change in angular velocity in a time interval or change in angular velocity during very short interval interactions.

Method of Impulse Momentum Principle for Plane motion of a Rigid Body

Linear momentum and angular momentum serve as measures of amount of translation and rotation motion respectively. The external forces acting on a rigid body can change its state of translation as well as rotation motion which is reflected by change in linear as well as angular momentum according to the principles of linear impulse and momentum and angular impulse and momentum.



In the above figure is shown strategy to apply method of impulse and momentum. Consider a rigid body of mass M in plane motion. Its moment of inertial about the centroidal axis perpendicular to plane of motion is I_C . Let \vec{v}_{C1} and $\vec{\omega}_1$ represent velocity of its mass center and its angular velocity at the beginning of a time interval t_1 to t_2 . Under the action of several forces $\vec{F}_1, \vec{F}_2, \dots, \vec{F}_j, \dots, \vec{F}_n$ during the time interval its mass center velocity and angular velocity become \vec{v}_{C2} and $\vec{\omega}_2$ respectively. The adjacent figure shows strategy representing how to write equations for linear and angular impulse momentum principles.

While applying the principle it becomes simpler to consider translation of the mass center and centroidal rotation separately. Thus in an alternative way we apply linear impulse momentum principle for translation of the mass center and angular impulse momentum principle for centroidal rotation.

Translation of mass center: zLinear impulse momentum principle.

$$\vec{p}_1 + \sum \vec{I}_{mp1 \rightarrow 2} = \vec{p}_2$$

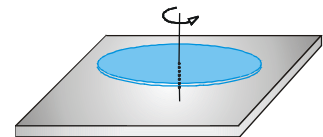
Here $\vec{p}_1 = M\vec{v}_{C1}$ and $\vec{p}_2 = M\vec{v}_{C2}$ represent linear momentums at the beginning and end of the time interval and $\sum \vec{I}_{mp1 \rightarrow 2}$ stands for impulse of all the external forces during the time interval.

Centroidal rotation: Angular impulse momentum principle.

$$\vec{L}_{C1} + \sum \vec{J}_{C,1 \rightarrow 2} = \vec{L}_{C2}$$

Here $\vec{L}_{C1} = I_C\vec{\omega}_1$ and $\vec{L}_{C2} = I_C\vec{\omega}_2$ represent angular momentums about the centroidal axis at the beginning and end of the time interval and $\sum \vec{J}_{C,1 \rightarrow 2}$ stands for angular impulse of all the external forces about the centroidal axis during the time interval.

Ex. A uniform disk of mass M and radius R rotating with angular velocity ω about a vertical axis passing through its center and perpendicular to its plane is placed gently on a rough horizontal ground, where coefficient of friction is μ . How long it will take to stop.



Sol. Refer the worked out example 8.12. The torque of friction forces is

$$\tau_c = \frac{2}{3} \mu MgR \quad \dots\dots\dots (1)$$

The angular impulse of the torque of friction is responsible to stop the disk. Applying angular impulse momentum principle, we have

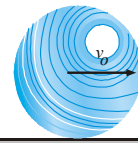
$$\vec{L}_{C1} + \sum \vec{J}_{C,1 \rightarrow 2} = \vec{L}_{C2} \rightarrow I_C\omega_o - \tau_c t = 0$$

Substituting $I_C = \frac{1}{2}MR^2$ and τ_c from eq. (1), we have

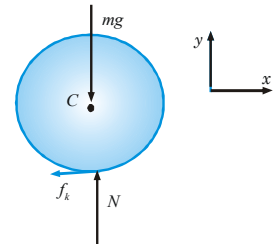
$$t = \frac{3R\omega_o}{4\mu g}$$

Ex. A uniform sphere of mass m and radius r is projected along a rough horizontal floor with linear velocity v_o and no angular velocity. The coefficients of kinetic and static frictions are represented by μ_s and μ_k respectively.

- (a) How long the sphere will slide on the floor before it starts rolling.
- (b) How far the sphere will slide on the floor before it starts rolling.
- (c) Find the linear and angular velocities of the sphere when it starts rolling.
- (d) Find the work done by frictional forces during the process and thereafter.



Sol. When the sphere touches the floor it is on translation motion. All the points including the bottom one are moving with the same velocity v_o . Thus the bottom point which makes the contact with the floor slide on it causing kinetic friction to act in backward direction. In the adjacent figure the forces acting on the sphere are shown. Here mg represent weight, N the normal reaction from the ground and f_k .



Since the sphere has no vertical component of acceleration, by applying Newton's law we have

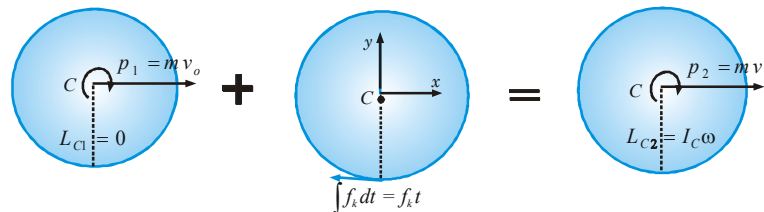
$$\sum F_y = 0 \rightarrow N = mg$$

The kinetic friction $f_k = \mu N = \mu mg$ (1)

The only force which applied torque about the centroidal axis is the kinetic friction. Angular impulse of torque of kinetic friction increases the angular velocity ω and impulse of kinetic friction decreases the mass center velocity v_C till both bear following condition required for rolling. Thereafter the sphere will continue to roll with the uniform velocity.

$$v_C = \omega r$$
(2)

In the adjacent figure of impulse momentum diagram the impulse of kinetic friction is shown



Translation of mass center: Applying linear impulse momentum principle in x direction, we have

$$\vec{p}_1 + \sum \vec{I}_{m p1 \rightarrow 2} = \vec{p}_2 \rightarrow p_1 - f_k t = p_2$$

Substituting p_1, p_2 and f_k from eq. (1), we have

$$v_C = v_o - \mu g t$$
(3)

Centroidal rotation: Angular impulse momentum principle about the centroidal axis.

$$\vec{L}_{C1} + \sum \vec{J}_{C,1 \rightarrow 2} = \vec{L}_{C2} \rightarrow 0 + f_k r t = I_C \omega$$

Substituting $\frac{2}{5} m r^2$ for I_C and f_k from eq. (1), we have

$$\omega = \frac{5 \mu g t}{2 r}$$
(4)

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- (a) Substituting values of v_c and ω from eq. (3) and (4) into eq. (2), we have

$$\text{Time when rolling starts } t = \frac{2v_o}{7\mu g} \quad \dots\dots\dots(5)$$

- (b) Eq. (3) reveals that the mass center is in uniformly retarded motion. So its displacement in time t , when it starts rolling is given by the following equation.

$$x = \frac{1}{2}(v_o + v_c)t \rightarrow \quad \text{Substituting values for } v_c \text{ and } t \text{ from eq. (3) and (5) respectively we have}$$

$$x = \frac{12v_o^2}{49\mu g} \quad \dots\dots\dots(6)$$

- (c) Linear and angular velocities of the sphere when it starts rolling can be obtained by substituting t from eq. (5) into (3) and (4) respectively.

$$\text{Linear velocity when rolling starts } v_c = \frac{5}{7}v_o \quad \dots\dots\dots(7)$$

$$\text{Angular velocity when rolling starts } \omega = \frac{5v_o}{7r} \quad \dots\dots\dots(8)$$

- (d) Work done by a force depends on displacement of point of application or displacement of the particle on which force is applied. The particle of the body in contact with the ground on which force of kinetic friction acts continuously changes; therefore it is recommended to calculate work done with the help of work energy theorem instead of using definition of work.

Kinetic energy in the initial position at the instant t_1 ,

$$K = \frac{1}{2}mv_c^2 + \frac{1}{2}I_C\omega^2 \rightarrow \quad K_1 = \frac{1}{2}mv_o^2 \quad \dots\dots\dots(9)$$

Since in the beginning angular velocity is zero.

Kinetic energy in the final position at the instant t_2 ,

$$K = \frac{1}{2}mv_c^2 + \frac{1}{2}I_C\omega^2 \rightarrow \quad \text{Substituting values of } v_c \text{ and } \omega \text{ from eq. (7) and (8) and } \frac{2}{5}mr^2 \text{ for } I_C, \text{ we can write}$$

$$K_2 = \frac{35mv_o^2}{98} \quad \dots\dots\dots(10)$$

Only force of kinetic friction does work during sliding. Denoting it by $W_{f1 \rightarrow 2}$ in the equation of work energy theorem, we have

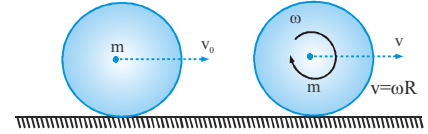
$$K_1 + W_{f1 \rightarrow 2} = K_2 \rightarrow \quad W_{f1 \rightarrow 2} = K_2 - K_1$$

Substituting values of K_1 and K_2 from eq. (9) and (10), we have

$$W_{f1 \rightarrow 2} = -\frac{1}{7}mv_o^2$$

Ex. A body of radius R and mass m is placed on horizontal rough surface with linear velocity v_0 , after some time it comes in the condition of pure rolling then determine :

- (i) Time t at which body starts pure rolling.
- (ii) Linear velocity of body at time t .
- (iii) Work done by frictional force in this time t .



Sol. For translatory motion $v = u + at$

Initial velocity $u = v_0$

Let after time t pure rolling starts and at this time t final velocity = v and acceleration = a

From FBD:

Normal Reaction $N = mg$

Friction force $f = \mu N = \mu mg \Rightarrow ma = \mu mg \quad [\because f = ma]$

Retardation $a = \mu g$

$v = v_0 - at$ (–ve sign for retardation)

$v = v_0 - \mu gt \quad \dots(i)$

For rotatory motion $\omega = \omega_0 + \alpha t$ (Initial angular velocity $\omega_0 = 0$)

$\Rightarrow \omega = \alpha t \quad \dots(ii)$

$\because \tau = I\alpha \Rightarrow \alpha = \frac{\tau}{I} = \frac{fR}{mK^2} = \frac{\mu mgR}{mK^2} = \frac{\mu gR}{K^2} \quad \dots(iii)$

From eqⁿ (ii) and eqⁿ (iii) $\omega = \frac{\mu gR}{K^2} t \quad \dots(iv)$

\because For pure rolling $v = \omega R \Rightarrow \omega = \frac{v}{R} \quad \dots(v)$

From eq. (iv) and (v) $\frac{v}{R} = \frac{\mu gR}{K^2} t$ or $v = \frac{\mu gR^2 t}{K^2} \quad \dots(vi)$

substitute v from eqⁿ (vi) into eqⁿ (i) $\frac{\mu gR^2 t}{K^2} = v_0 - \mu gt \Rightarrow t = \frac{v_0}{\mu g \left[1 + \frac{R^2}{K^2} \right]}$

Putting the value of t in equation (i) $v = v_0 - \mu g \frac{v_0}{\mu g \left[1 + \frac{R^2}{K^2} \right]} = v_0 - \frac{v_0}{1 + \frac{R^2}{K^2}} = \frac{v_0}{1 + \frac{R^2}{K^2}}$

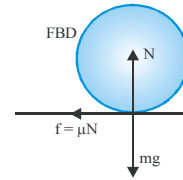
Work done in sliding by frictional force = Initial kinetic energy – Final kinetic energy

Work done by friction $W_f = \frac{1}{2} Mv_0^2 - \frac{1}{2} Mv^2 \left(1 + \frac{K^2}{R^2} \right) = \frac{1}{2} Mv_0^2 - \frac{1}{2} \frac{Mv_0^2}{\left(1 + \frac{R^2}{K^2} \right)} = \frac{Mv_0^2}{2 \left(1 + \frac{R^2}{K^2} \right)}$

Conservation of Angular Momentum

If angular impulse of all the external forces about an axis in time interval vanishes, the angular momentum of the system about the same axis in that time interval remain unchanged.

If $\Sigma \int_{t_1}^{t_2} \vec{\tau}_o dt = 0$, we have $\vec{L}_{o1} = \vec{L}_{o2}$



The condition of zero net angular impulse required for conservation of angular momentum can be fulfilled in the following cases.

- (i) If no external force acts, the angular impulse about all axes will be zero and hence angular momentum remains conserved about all axes.
- (ii) If net torque of all the external forces or torques of each individual force about an axis vanishes the angular momentum about that axes will be conserved.
- (iii) If all the external forces are finite in magnitude and the concerned time interval is infinitely small, the angular momentum remain conserved.
- (iv) If a system of rigid bodies changes its moment of inertia by changing its configuration due to internal forces only its angular momentum about any axes remains conserved. If we denote the moment of inertias in two configurations by I_1 and I_2 and angular velocities by ω_1 and ω_2 , we can write

$$I_1 \vec{\omega}_1 = I_2 \vec{\omega}_2$$

The principle of conservation of angular momentum governs a wide range of physical processes from subatomic to celestial world. The following examples explicate some of these applications.

Spinning Ice Skater

A spinning ice skater and ballet dancers can control her moment of inertia by spreading or bringing closer her hands and make use of conservation of angular momentum to perform their spins. In doing so no external forces is needed and if we ignore effects of friction from the ground and the air, the angular momentum can be assumed conserved. When she spreads her hand or leg away, her moment of inertia decreases therefore her angular velocity decreases and when she brings her hands or leg closer her moment of inertia increases therefore her angular velocity increases.



Student on rotating turntable

The student, the turntable and dumbbells make an isolated system on which no external torque acts, if we ignore friction in the bearing of the turntable and air friction. Initially the student has his arm stretched on rotating turntable. When he pulls dumbbells close to his body, angular velocity increases due to conservation of angular momentum.

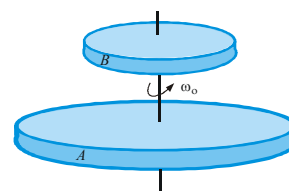


Larger moment of inertia and smaller angular velocity



Smaller moment of inertia and larger angular velocity

Ex. Consider the disk A of moment of inertia I_1 rotating freely in horizontal plane about its axis of symmetry with angular velocity ω_0 . Another disk B of moment of inertia I_2 held at rest above the disk A . The axis of symmetry of the disk B coincides with that of the disk A as shown in the figure. The disk B is released to land on the disk A . When sliding stops, what will be the angular velocity of both the disks?

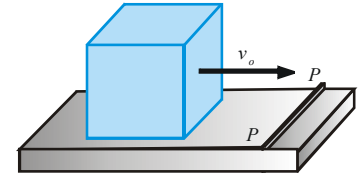


Sol. Both the disks are symmetric about the axis of rotation therefore does not require any external torque to keep the axis stationary. When the disk B lands on A slipping starts. The force of friction provides an internal torque to system of both the disk. It slows down rotation rate of A and increases that of B till both acquire same angular velocity ω .

Since there is no external torques on the system of both the disks about the axis of rotation, the total angular momentum of the system remains conserved. The total angular momentum of the system is the sum of angular momentum of both disks. Denoting the angular momentum of the disk A before B lands on it and long after slipping between them stops by symbols \vec{L}_{A1} , \vec{L}_{B1} , \vec{L}_{A2} and \vec{L}_{B2} respectively, we can express conservation of angular momentum by the following equation.

$$\vec{L}_{A1} + \vec{L}_{B1} = \vec{L}_{A2} + \vec{L}_{B2} \rightarrow I_1 \omega_0 + 0 = I_1 \omega + I_2 \omega \Rightarrow \omega = \frac{I_1 \omega_0}{I_1 + I_2}$$

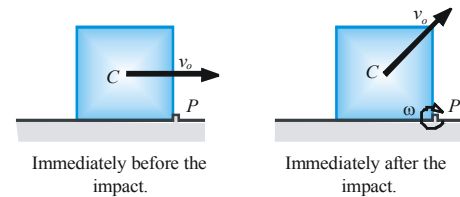
Ex. A cube of mass m and edge length ℓ can slide freely on a smooth horizontal floor. Moving on the floor with velocity v_o , it strikes a long obstruction PP of small height. The obstruction is parallel to the leading bottom edge of the cube. The leading bottom edge gets pivoted with the obstruction and the cube starts rotating. Determine angular velocity of the cube immediately after the impact.



Sol. Before the impact, there is no external force in the horizontal direction and the cube slides with uniform velocity v_o , and during the impact reaction forces of the obstruction stops its leading bottom edge and cause it to rotate about its leading bottom.

During the impact external forces acting on the cube are its weight, the normal reaction from the ground and reaction from the obstruction. The weight and the normal reaction from the ground both are finite in magnitude and the impact ends in infinitesimally small time interval so their impulses and angular impulses about any axes are negligible. It is the reaction from the obstruction which has finite impulse during the impact. Its horizontal component changes the momentum of the cube during the impact, but its angular impulse about the obstruction is zero, therefore the angular momentum of the cube about an axis coincident with the leading bottom edge remain conserved.

Let the velocity of the mass center and angular velocity of the cube immediately after the impact are v_{C_o} and ω_o . These velocities are shown in the adjacent figure.



We denote the angular momentum of the cube about axis coincident with the obstruction edge before and after the impact by L_{P_1} and L_{P_2} .

Applying principle of conservation of angular momentum about an axis coincident with the obstruction, we have

$$\vec{L}_{P_1} = \vec{L}_{P_2} \rightarrow \vec{r}_{C/P} \times m\vec{v}_o = I_P \vec{\omega}$$

Using theorem of parallel axes for moment of inertia I_P about the leading bottom edge, we get

$$I_P = I_C + mr_{C/P}^2 = \frac{2}{3} m\ell^2. \text{ Substituting this in the above equation, we have}$$

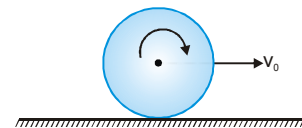
$$\text{Angular velocity immediately after the impact } \omega_o = \frac{3v_o}{4\ell}$$

Angular momentum of a body in combined translational and rotational motion

Suppose a body is rotating about an axis passing through its centre of mass with an angular velocity ω_{cm} and moving translationally with a linear velocity v . Then, the angular momentum of the body about a point P outside the body in the lab frame is given by, $\vec{L}_P = \vec{L}_{cm} + \vec{r} \times \vec{p}_{cm}$ where r is the position vector of the centre of mass with respect to point P . Hence, $\vec{L}_P = I_{cm} \vec{\omega} + \vec{r} \times m\vec{v}_{cm}$

Ex. A solid sphere rolls without slipping on a rough surface and the centre of mass has constant speed v_o . If mass of the sphere is m and its radius is R , then find the angular momentum of the sphere about the point of contact.

Sol. $\therefore \vec{L}_P = \vec{L}_{cm} + \vec{r} \times \vec{p}_{cm} = I_{cm} \vec{\omega} + \vec{R} \times m\vec{v}_{cm}$



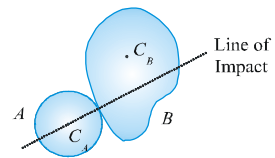
Since sphere is in pure rolling motion hence

$$\omega = v_o R \Rightarrow \vec{L}_P = \left(\frac{2}{5} MR^2 \frac{v_o}{R} \right) (-\vec{k}) + Mv_o R = \frac{7}{5} Mv_o R (-\vec{k})$$

ECCENTRIC IMPACT

In eccentric impact the line of impact which is the common normal drawn at the point of impact does not pass through mass center of at least one of the colliding bodies. It involves change in state of rotation motion of either or both the bodies.

Consider impact of two A and B such that the mass center C_B of B does not lie on the line of impact as shown in figure. If we assume bodies to be frictionless their mutual forces must act along the line of impact. The reaction force of A on B does not pass through the mass center of B as a result state of rotation motion of B changes during the impact.



Problems of Eccentric Impact

Problems of eccentric impact can be divided into two categories. In one category both the bodies under going eccentric impact are free to move. No external force act on either of them. Their mutual forces are responsible for change in their momentum and angular momentum. In another category either or both of the bodies are hinged.

Eccentric Impact of bodies free to move

Since no external force acts on the two body system, we can use principle of conservation of linear momentum, principle of conservation of angular momentum about any point and concept of coefficient of restitution.

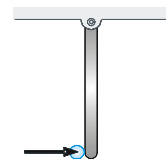
The coefficient of restitution is defined for components of velocities of points of contacts of the bodies along the line of impact.

While applying principle of conservation of angular momentum care must be taken in selecting the point about which we write the equation. The point about which we write angular momentum must be at rest relative to the selected inertial reference frame and as far as possible its location should be selected on line of velocity of the mass center in order to make zero the first term involving moment of momentum of mass center.

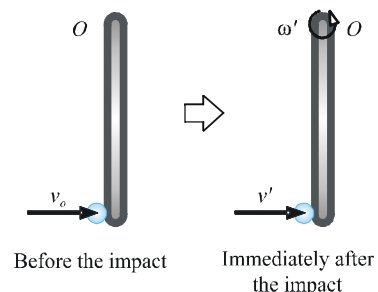
Eccentric Impact of hinged bodies

When either or both of the bodies are hinged the reaction of the hinge during the impact act as external force on the two body system, therefore linear momentum no longer remain conserved and we cannot apply principle of conservation of linear momentum. When both the bodies are hinged we cannot also apply conservation of angular momentum, and we have to use impulse momentum principle on both the bodies separately in addition to making use of coefficient of restitution. But when one of the bodies is hinged and other one is free to move, we can apply conservation of angular momentum about the hinge.

Ex. A uniform rod of mass M and length ℓ is suspended from a fixed support and can rotate freely in the vertical plane. A small ball of mass m moving horizontally with velocity v_0 strikes elastically the lower end of the rod as shown in the figure. Find the angular velocity of the rod and velocity of the ball immediately after the impact.



Sol. The rod is hinged and the ball is free to move. External forces acting on the rod ball system are their weights and reaction from the hinge. Weight of the ball as well as the rod are finite and contribute negligible impulse during the impact, but impulse of reaction of the hinge during impact is considerable and cannot be neglected. Obviously linear momentum of the system is not conserved. The angular impulse of the reaction of hinge about the hinge is zero. Therefore angular momentum of the system about the hinge is conserved. Let velocity of the ball after the impact becomes v'_B and angular velocity of the rod becomes ω' .



We denote angular momentum of the ball and the rod about the hinge before the impact by L_{B1} and L_{R1} and after the impact by L_{B2} and L_{R2} .

Applying conservation of angular momentum about the hinge, we have

$$\vec{L}_{B1} + \vec{L}_{R1} = \vec{L}_{B2} + \vec{L}_{R2} \rightarrow mv_o\ell + 0 = mv'_B\ell + I_o\omega'$$

Substituting $\frac{1}{3}M\ell^2$ for I_o , we have

$$3mv'_B + M\ell\omega' = 3mv_o \quad \dots\dots\dots(1)$$

The velocity of the lower end of the rod before the impact was zero and immediately after the impact it becomes $\ell\omega'$ towards right. Employing these facts we can express the coefficient of restitution according to eq.

$$e = \frac{v'_{Qn} - v'_{Pn}}{v_{pn} - v_{qn}} \rightarrow \ell\omega' - v'_B = ev_o \quad \dots\dots\dots(2)$$

From eq. (1) and (2), we have

Velocity of the ball immediately after the impact $v'_B = \frac{(3m - eM)v_o}{3m + M}$

Angular velocity of the rod immediately after the impact $\omega' = \frac{3(1 + e)mv_o}{(3m + M)\ell}$

Ex. A uniform rod AB of mass M and length ℓ is kept at rest on a smooth horizontal plane. A particle P of mass m_o moving perpendicular to the rod with velocity v_o strikes the rod at one of its ends as shown in the figure. Derive suitable expressions for the coefficient of restitution, velocity of mass center of the rod and angular velocity of the rod immediately after the impact. Assume it is the coefficient of restitution.

Sol.

Both the bodies can move freely in the horizontal plane, therefore no horizontal external force acts on the particle-rod system. The linear momentum as well as

angular momentum about any axis normal to the plane is conserved.

Let the velocity of the particle, angular velocity of the rod and velocity of the mass center of the rod immediately after the impact are v'_p towards right, ω' in clockwise sense and v'_c towards right as shown in the adjacent figure.

Using relative motion equation, we can

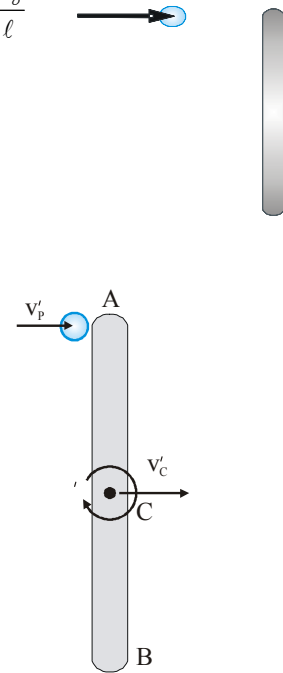
express the velocity of the end A of the rod.

$$v'_A = v'_C + \frac{1}{2}\omega'\ell \quad \dots\dots\dots(1)$$

We denote linear momentum of the particle and rod before the impact by \vec{p}_{p1} and \vec{p}_{c1} and immediately after the impact by \vec{p}_{p2} and \vec{p}_{c2} respectively.

Applying conservation of linear momentum, we have

$$\vec{p}_{p2} + \vec{p}_{c2} = \vec{p}_{p1} + \vec{p}_{c1} \rightarrow mv'_p + Mv'_c = mv_o \quad \dots\dots\dots(2)$$



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The above equation shows that the mass center of the rod will move toward the right. If we write angular momentum of the rod about a stationary point O , which is in line with the velocity v'_C , the first term involving moment of momentum of rod vanishes and only angular momentum due to its centroidal rotation remains in the expression.

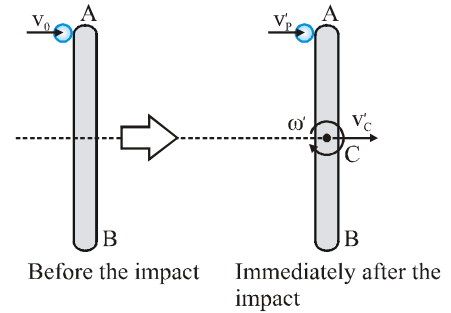
We denote angular momentum of the particle and the rod about the point O before the impact by L_{P1} and L_{R1} and after the impact by L_{P2} and L_{R2} .

Applying conservation of angular momentum about the hinge, we have

$$\vec{L}_{P2} + \vec{L}_{R2} = \vec{L}_{P1} + \vec{L}_{R1} \rightarrow \frac{1}{2} m v'_p \ell + I_C \omega' = \frac{1}{2} m v_o \ell$$

Substituting $\frac{1}{12} M \ell^2$ for I_C we have

$$6 m v'_p + M \ell \omega' = 6 m v_o \tag{3}$$



The velocity of the end A of the rod before the impact was zero and immediately after the impact it becomes v'_A towards right. Employing these facts we can express the coefficient of restitution as

$$e = \frac{v'_{Qn} - v'_{Pn}}{v_{pn} - v_{Qn}} \rightarrow v'_A - v'_p = e v_o$$

Substituting v'_A from eq. (1), we have

$$2 v'_C + \omega' \ell - 2 v'_p = 2 e v_o \tag{4}$$

Eq. (2), (3) and (4) involves three unknowns v'_C , ω' and v'_p , which can be obtained by solving these equation.

Velocity of the ball immediately after the impact
$$v'_p = \left(\frac{4m - eM}{4m + M} \right) v_o$$

Velocity of mass center of rod immediately after the impact
$$v'_C = \left\{ \frac{m(1 + e)}{4m + M} \right\} v_o$$

Angular velocity of the rod immediately after the impact
$$\omega' = \left\{ \frac{6m(1 + e)}{4m + M} \right\} \frac{v_o}{\ell}$$

1. **Angular velocity** $\vec{\omega} = \frac{d\vec{\theta}}{dt}$
2. **Angular acceleration** $\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\theta}}{dt^2}$
3. **Angular momentum** $\vec{L} = \vec{r} \times \vec{p} = I\vec{\omega}$
4. **Torque** $\vec{\tau} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt}$
5. **Rotational Kinetic energy** $K = \frac{1}{2}I\omega^2 = \frac{L^2}{2I}$
6. **Rotational Power** $P = \vec{\tau} \cdot \vec{\omega}$

7. **For constant angular acceleration**

$$\omega = \omega_0 + \alpha t, \theta = \omega_0 t + \frac{1}{2}\alpha t^2, \omega^2 = \omega_0^2 + 2\alpha\theta, \theta_n = \omega_0 + \frac{\alpha}{2}(2n-1)$$

8. **Moment of Inertia**

(a) A tensor but for fixed axis it is a scalar

(b) For discrete distribution of mass $I = m_1 r_1^2 + m_2 r_2^2 + \dots = \sum m_i r_i^2$

(c) For continuous distribution of mass $I = \int dl = \int dm r^2$

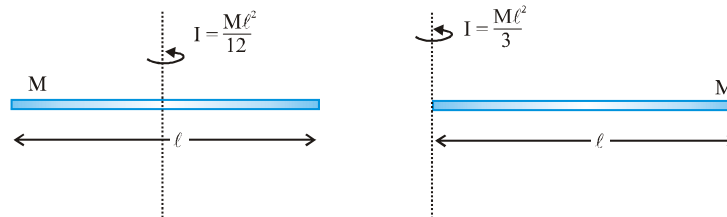
9. **Radius of gyration** $k = \sqrt{\frac{I}{M}}$

10. **Theorem regarding moment of inertia**

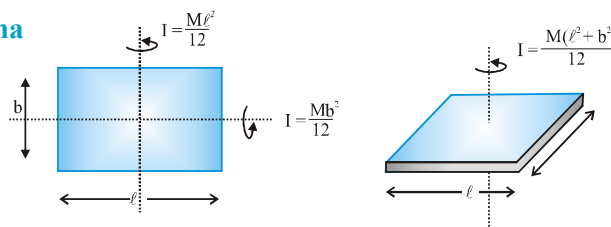
(a) Theorem of parallel axes $I_{\text{axis}} = I_{\text{cm}} + md^2$ where d is the perpendicular distance between parallel axes

(b) Theorem of perpendicular axes $I_2 = I_x + I_y$

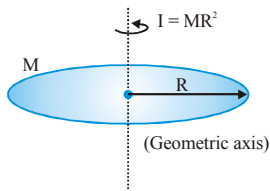
11. **Rod**



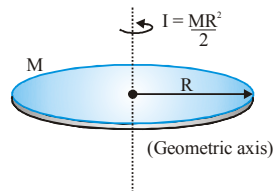
12. **Rectangular Lamina**



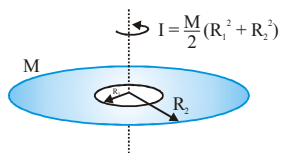
13. Ring



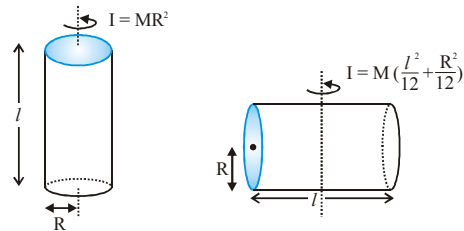
14. Disc



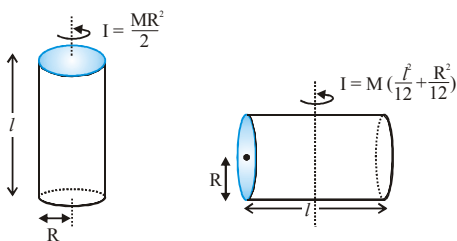
15. Circular Hollow Disk



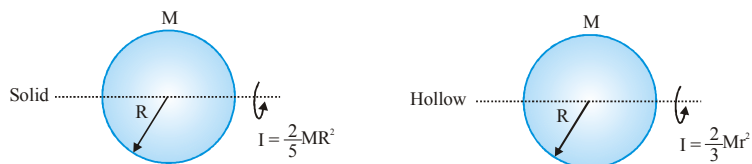
16. Hollow cylinder



17. Solid cylinder



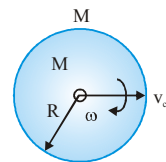
18. Solid & Hollow sphere



19. Rolling motion

$$\text{Total kinetic energy} = \frac{1}{2} M v_{CM}^2 + \frac{1}{2} I_{cm} \omega^2$$

$$\text{Total angular momentum} = M v_{CM} R + I_{cm} \omega$$



20. Pure rolling (or rolling without slipping) on stationary surface

Condition : $v_{cm} = R \omega$

In accelerated motion $a_{cm} = R \alpha$

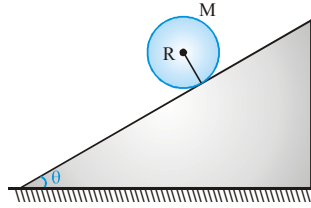
If $v_{cm} > R \omega$ then rolling with forward slipping,

If $v_{cm} < R \omega$ then rolling backward slipping

Total kinetic energy in pure rolling

$$K_{total} = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} (M k^2) \left(\frac{v_{cm}^2}{R^2} \right) = \frac{1}{2} M v_{cm}^2 \left(1 + \frac{k^2}{R^2} \right)$$

21. Pure rolling motion on an inclined plane



$$\text{Acceleration } a = \frac{g \sin \theta}{1 + k^2 / R^2}$$

$$\text{Minimum frictional coefficient } \mu_{\min} = \frac{\tan \theta}{1 + k^2 / R^2}$$

$$\text{Torque } \vec{\tau} = I\vec{\alpha} = I \frac{d\vec{\omega}}{dt} = \frac{d(I\vec{\omega})}{dt} = \frac{d\vec{L}}{dt} \quad \text{or} \quad \frac{d\vec{J}}{dt}$$

$$\text{Change in angular momentum } \Delta\vec{L} = \vec{\tau}\Delta t$$

$$\text{Work done by a torque} \quad W = \int \vec{\tau} \cdot d\vec{\theta}$$

22. A ladder is more apt to slip, when you are high up on it than when you just being to climb because at the high up on a ladder the torque is large and on climbing up the torque is small.
23. When a sphere is rolls on a horizontal table, it slows down and eventually stops because when the sphere rolls on the table, both the sphere and the surface deform near the contact. As a result the normal force does not pass through the centre and provide an angular deceleration.
24. The spokes near the top of a rolling bicycle wheel are more blurred than those near the bottom of the wheel because the spokes near the top of wheel are moving faster than those near bottom of the wheel.
25. Instantaneous angular velocity is a vector quantity because infinitesimal angular displacement is a vector.
26. The relative angular velocity between any two points of a rigid body is zero at any instant.
27. All particles of a rigid body, which do not lie on an axis of rotation move on circular paths with centres at an axis of rotation.
28. Instantaneous axis of rotation is stationary w.r.t. ground.
29. Many greater rivers flow towards the equator. The sediment that they carry increases the time of rotation of the earth about its own axis because the angular momentum of the earth about its rotation axis is conserved. The hard boiled egg and raw egg can be distinguished on the basis of spinning of both.