Ellipse

Ellipse is the locus of a point in a plane which moves in such a way that the ratio of the distance from a fixed point (focus) in the same plane to its distance from a fixed straight line (directrix) is always constant, which is always less than unity.

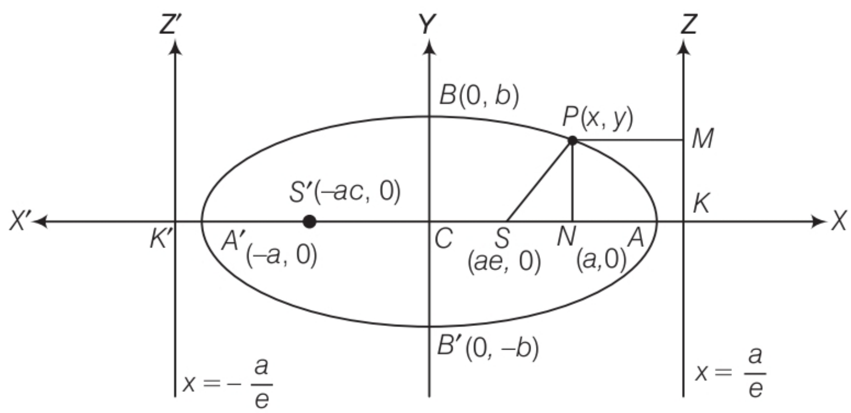
Major and Minor Axes

The line segment through the foci of the ellipse with its end points on the ellipse, is called its major axis.

The line segment through the centre and perpendicular to the major axis with its end points on the ellipse, is called its minor axis.

Horizontal Ellipse i.e. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (0 < b < a)$

If the coefficient of x^2 has the larger denominator, then its major axis lies along the X-axis, then it is said to be horizontal ellipse.



(i) Vertices $A(a, 0), A_1(-a, 0)$

(ii) Centre $O\,(0, 0)$

(iii) Length of major axis, $AA_1 = 2a$; Length of minor axis, $BB_1 = 2b$

(iv) Foci are $S(ae, 0)$ and $S_1(-ae, 0)$

(v) Equation of directrices are $l : x = \frac{a}{e}, l'; x = -\frac{a}{e}$

(vi) Length of latusrectum, $LL_1 = L' L_1' = \dfrac{2b^2}{a}$

(vii) Eccentricity, $e = \sqrt{1 - \dfrac{b^2}{a^2}} < 1$

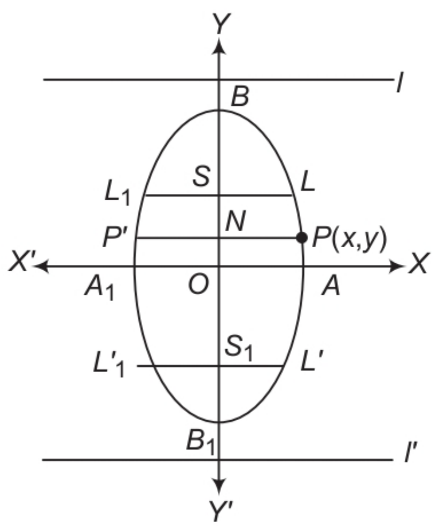
(viii) Focal distances of point $P(x, y)$ are SP and S_1P i.e. $|a - ex|$ and $|a + ex|$. Also, $SP + S_1P = 2a$ = major axis.

(ix) Distance between foci = $2ae$

(x) Distance between directrices = $\dfrac{2a}{e}$

Vertical Ellipse i.e. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, (0 < a < b)$

If the coefficient of x^2 has the smaller denominator, then its major axis lies along the Y-axis, then it is said to be vertical ellipse.



(i) Vertices $B(0, b), B_1(0, -b)$

(ii) Centre $O(0, 0)$

(iii) Length of major axis $BB_1 = 2b$, Length of Minor axis $AA_1 = 2a$

(iv) Foci are $S(0, ae)$ and $S_1(0, -ae)$

(v) Equation of directrices are $l : y = \dfrac{b}{e}$; $l' : y = -\dfrac{b}{e}$

(vi) Length of latusrectum $LL_1 = L' L_1' = \dfrac{2a^2}{b}$

(vii) Eccentricity $e = \sqrt{1 - \dfrac{a^2}{b^2}} < 1$

(viii) Focal distances of point $P(x, y)$ are SP and S_1P,
i.e. $|b - ex|$ and $|b + ex|$.
Also, $SP + S_1P = 2b =$ major axis.

(ix) Distance between foci $= 2be$

(x) Distance between directrices $= \dfrac{2b}{e}$

Parametric Equation

The equation $x = a\cos\phi$, $y = b\sin\phi$, taken together are called the parametric equation of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where ϕ is any parameter.

Special Form of Ellipse

If centre of the ellipse is (h, k) and the direction of the axes are parallel to the coordinate axes, then its equation is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$.

Ordinate and Double Ordinate

Let P be any point on the ellipse and PN be perpendicular to the major axis AA', such that PN produced meets the ellipse at P'. Then, PN is called the ordinate of P and PNP' is the double ordinate of P.

Position of a Point with Respect to an Ellipse

The point (x_1, y_1) lies outside, on or inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ according as } \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > 0, = \text{ or } < 0.$$

Auxiliary Circle

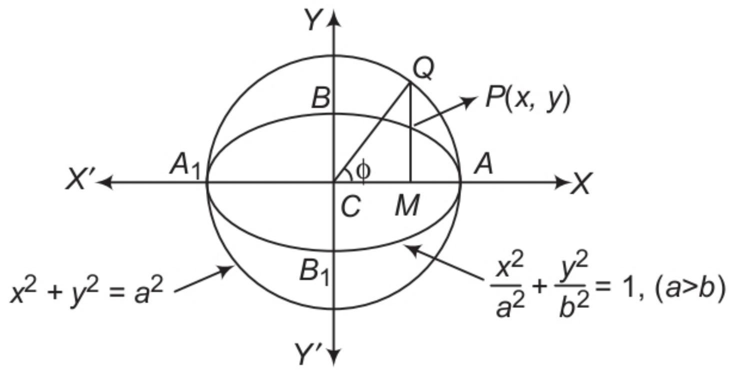
The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, becomes $x^2 + y^2 = a^2$, if $b = a$.

This is called auxiliary circle of the ellipse. i.e. the circle described on the major axis of an ellipse as diameter is called auxiliary circle.

Eccentric Angle of a Point

Let P be any point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Draw PM perpendicular from P on the major axis of the ellipse and produce MP to the auxiliary circle in Q. Join CQ.

The $\angle ACQ = \phi$ is called the eccentric angle of the point P on the ellipse.



Equation of Tangent

(i) **Point Form** The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ or $T = 0$.

(ii) **Parametric Form** The equation of the tangent to the ellipse at the point $(a \cos\theta, b \sin\theta)$ is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$.

(iii) **Slope Form** The equation of the tangent of slope m to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $y = mx \pm \sqrt{a^2m^2 + b^2}$ and the coordinates of the point of contact are $\left(\pm \frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \mp \frac{b^2}{\sqrt{a^2m^2 + b^2}}\right)$.

(iv) **Point of Intersection of Two Tangents** The equation of the tangents to the ellipse at points $P(a \cos\theta_1, b \sin\theta_1)$ and $Q(a \cos\theta_2, b \sin\theta_2)$ are $\frac{x}{a}\cos\theta_1 + \frac{y}{b}\sin\theta_1 = 1$

and $\frac{x}{a}\cos\theta_2 + \frac{y}{b}\sin\theta_2 = 1$ and these two intersect at the point

$$\left(\frac{a\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}, \frac{b\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}\right)$$

(v) **Pair of Tangents** The combined equation of the pair of tangents drawn from a point (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

is $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2$ i.e. $SS_1 = T^2$

Director Circle

The locus of the point of intersection of perpendicular tangents to an ellipse is a director circle. If equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then equation of director circle is $x^2+y^2=a^2+b^2$.

Equation of Chord

Let $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ be any two points of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(i) The equation of the chord joining these points will be

$$(y-b\sin\theta)=\frac{b\sin\phi-b\sin\theta}{a\cos\phi-a\cos\theta}(x-a\cos\theta)$$

or $$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$$

(ii) The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \text{ or } T=0.$$

(iii) The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ bisected at the point (x_1, y_1) is given by

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

or $$T=S_1$$

Equation of Normal

(i) **Point Form** The equation of the normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

(ii) **Parametric Form** The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(a\cos\theta, b\sin\theta)$ is

$$ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2$$

(iii) **Slope Form** The equation of the normal of slope m to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are given by $y = mx \pm \frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$

and the coordinates of the point of contact are

$$\left(\pm\frac{a^2}{\sqrt{a^2+b^2m^2}}, \pm\frac{b^2m}{\sqrt{a^2+b^2m^2}}\right)$$

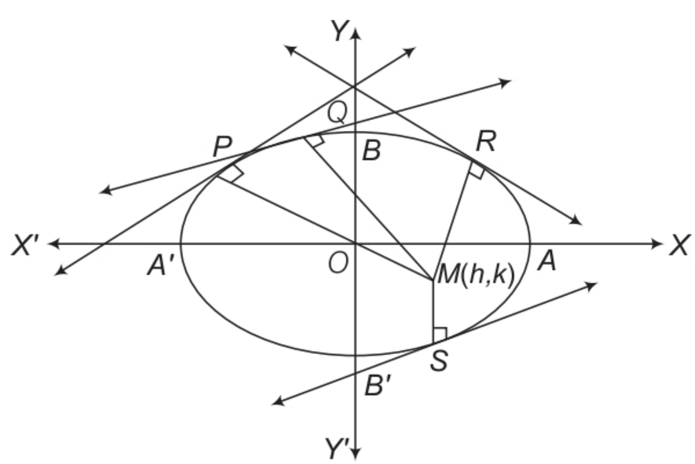
(iv) **Point of Intersection of Two Normals** Point of intersection of the normal at points $(a\cos\theta_1, b\sin\theta_1)$ and $(a\cos\theta_2, b\sin\theta_2)$ are given by

$$\left(\frac{a^2-b^2}{a}\cos\theta_1\cos\theta_2\frac{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}, \frac{-(a^2-b^2)}{b}\sin\theta_1\sin\theta_2\frac{\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}\right)$$

(v) If the line $y = mx + c$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $c^2 = \frac{m^2(a^2-b^2)^2}{a^2+b^2m^2}$

Conormal Points

The points on the ellipse, the normals at which the ellipse passes through a given point are called conormal points.



Here, P, Q, R and S are the conormal points.

(i) The sum of the eccentric angles of the conormal points on the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an odd multiple of π.

(ii) If $\theta_1, \theta_2, \theta_3$ and θ_4 are eccentric angles of four points on the ellipse, the normals at which are concurrent, then

(a) $\Sigma \cos(\theta_1+\theta_2)=0$

(b) $\Sigma \sin(\theta_1+\theta_2)=0$

(iii) If θ_1, θ_2 and θ_3 are the eccentric angles of three points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, such that

$$\sin(\theta_1+\theta_2)+\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)=0,$$

then the normals at these points are concurrent.

(iv) If the normal at four points $P(x_1, y_1), Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are concurrent, then

$$(x_1+x_2+x_3+x_4)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}\right)=4$$

Conjugate Points and Conjugate Lines

Two points are said to be conjugate points with respect to an ellipes, if each lies on the polar of the other.

Two lines are said to be conjugate lines with respect to an ellipse, if each passes through the pole of the other.

Diameter and Conjugate Diameter

The locus of the mid-point of a system of parallel chords of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is called a diameter, whose equation of diameter is

$$y=-\frac{b^2}{a^2 m}x.$$

Two diameters of an ellipse are said to be conjugate diameters, if each bisects the chords parallel to the other.

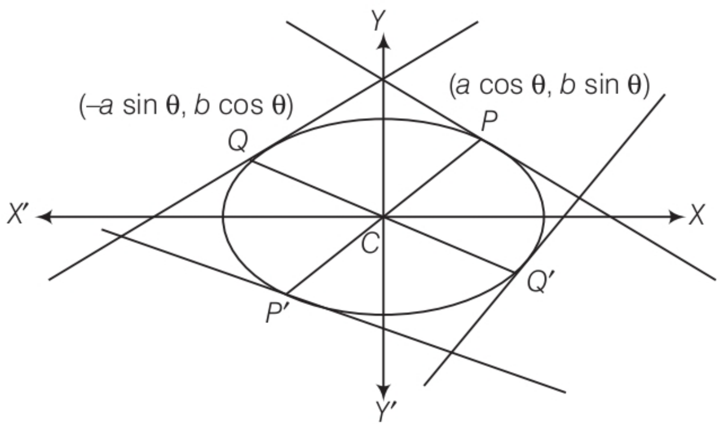
Properties of Conjugate Diameters

(i) The eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by a right angle.

(ii) The sum of the squares of any two conjugate semi-diameters of an ellipse is constant and equal to the sum of the squares of the semi-axis of the ellipse i.e. $CP^2 + CD^2 = a^2 + b^2$.

(iii) If PCP', QCQ' are two conjugate semi-diameters of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and S, S_1 be two foci of an ellipse, then

$$SP \times S_1P = CQ^2$$



(iv) The tangent at the ends of a pair of conjugate diameters of an ellipse form a parallelogram.

(v) The area of the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant and is equal to the product of the axes.

Important Points on Ellipse

(i) The line $y = mx + c$ touches the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ if } c^2 = a^2m^2 + b^2$$

(ii) The tangent and normal at any point of an ellipse bisect the external and internal angles between the focal radii to the point.

(iii) If SM and $S'M'$ are perpendiculars from the foci upon the tangent at any point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $SM \times S'M' = b^2$ and M, M' lie on the auxiliary circle.

(iv) If the tangent at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the major axis in T and minor axis in T', then $CN \times CT = a^2, CN' \times CT' = p^2$, where N and N' are the foot of the perpendiculars from P on the respective axis.

(v) The common chords of an ellipse and a circle are equally inclined to the axes of the ellipse.

Contd. ...

(vi) Maximum four normals can be drawn from a point to ellipse.

(vii) Polar of the point (x_1, y_1) with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

Here, point (x_1, y_1) is the pole of $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ with respect to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(viii) The pole of the line $lx+my+n=0$ with respect to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $P\left(\frac{-a^2l}{n}, \frac{-b^2m}{n}\right)$.

(ix) Two tangents can be drawn from a point P to an ellipse. These tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the ellipse.

(x) Tangents at the extremities of latusrectum of an ellipse intersect on the corresponding directrix.

(xi) Locus of mid-point of focal chords of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{ex}{a}$.

(xii) Point of intersection of the tangents at two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whose eccentric angles differ by a right angle lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$.

(xiii) Locus of mid-point of normal chords of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2\left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right)=(a^2-b^2)^2$$

(xiv) Eccentric angles of the extremities of latusrectum of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $\tan^{-1}\left(\pm\frac{b}{ae}\right)$.

(xv) The straight lines $y=m_1x$ and $y=m_2x$ are conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if $m_1 m_2=-\frac{b^2}{a^2}$.

(xvi) The normal at point P on an ellipse with foci S, S_1 is the internal bisector of $\angle SPS_1$.