Hyperbola

A hyperbola is the locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point in the same plane to its distance from a fixed line is always constant, which is always greater than unity.

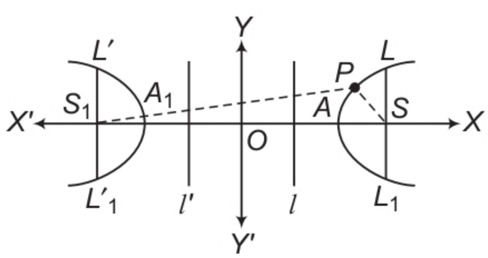
The fixed point is called the focus and the fixed line is directrix and the ratio is the eccentricity.

Transverse and Conjugate Axes

(i) The line through the foci of the hyperbola is called its transverse axis.

(ii) The line through the centre and perpendicular to the transverse axis of the hyperbola is called its conjugate axis.

Hyperbola of the Form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



(i) Centre : $O(0, 0)$

(ii) Foci : $S(ae, 0), S_1(-ae, 0)$

(iii) Vertices : $A(a, 0), A_1(-a, 0)$

(iv) Equation of directrices $l : x = \frac{a}{e}, l' : x = -\frac{a}{e}$

(v) Length of latusrectum : $LL_1 = L'L'_1 = \frac{2b^2}{a}$

(vi) Length of transverse axis : $2a$

(vii) Length of conjugate axis : $2b$

(viii) Eccentricity $e = \sqrt{1 + \left(\frac{b}{a}\right)^2}$

or $b^2 = a^2(e^2 - 1)$

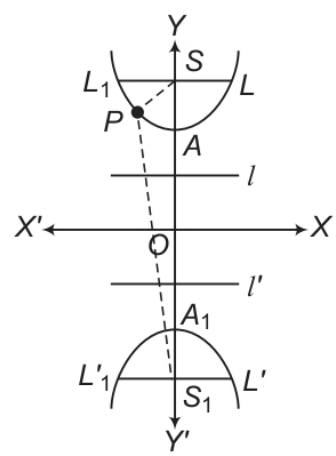
(ix) Distance between foci $= 2ae$

(x) Distance between directrices $= \frac{2a}{e}$

(xi) Coordinates of ends of latusrectum $= \left(\pm ae, \pm \frac{b^2}{a}\right)$

(xii) Focal radii $|SP| = |ex_1 - a|$ and $|S_1P| = |ex_1 + a|$

Conjugate Hyperbola $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



(i) Centre : $O(0, 0)$

(ii) Foci : $S\ (0, be), S_1\ (0, -be)$

(iii) Vertices : $A(0, b), A_1(0, -b)$

(iv) Equation of directrices

$$l : y = \frac{b}{e}, l' : y = -\frac{b}{e}$$

(v) Length of latusrectum :

$$LL_1 = L' L_1' = \frac{2a^2}{b}$$

(vi) Length of transverse axis : $2b$.

(vii) Length of conjugate axis : $2a$.

(viii) Eccentricity $e = \sqrt{1 + \left(\frac{a}{b}\right)^2}$

(ix) Distance between foci $= 2be$

(x) Distance between directrices $= \frac{2b}{e}$

(xi) Coordinates of ends of latusrectum $= \left(\pm \frac{a^2}{b}, \pm be\right)$

(xii) Focal radii $|SP| = |ey_1 - b|$ and $|S_1P| = |ey_1 + b|$

Focal Distance of a Point

The distance of a point on the hyperbola from the focus is called its focal distance.

The difference of the focal distances of any point on a hyperbola is constant and is equal to the length of transverse axis of the hyperbola i.e.

$$|S_1P - SP| = 2a$$

where, S and S_1 are the foci and P is any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Equation of Hyperbola in Different Forms

(i) If the centre of the hyperbola is (h, k) and the directions of the axes are parallel to the coordinate axes, then the equation of the hyperbola, whose transverse and conjugate axes are $2a$ and $2b$ is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.

(ii) If a point $P(x, y)$ moves in the plane of two perpendicular straight lines $a_1 x + b_1 y + c_1 = 0$ and $b_1x - a_1y + c_2 = 0$ in such a way that

$$\frac{\left(\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}}\right)^2}{a^2} - \frac{\left(\frac{b_1x - a_1y + c_2}{\sqrt{a_1^2 + b_1^2}}\right)^2}{b^2} = 1$$

Then, the locus of P is hyperbola whose transverse axis lies along $b_1x - a_1y + c_2 = 0$ and conjugate axis along the line $a_1x + b_1y + c_1 = 0$. The length of transverse and conjugate axes are $2a$ and $2b$, respectively.

Parametric Equations

(i) Parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are

$$x = a\sec\theta,\ y = b\tan\theta$$

or $$x = a\cosh\theta,\ y = b\sinh\theta$$

(ii) The equations $x = a\left(\frac{e^{\theta} + e^{-\theta}}{2}\right)$, $y = b\left(\frac{e^{\theta} - e^{-\theta}}{2}\right)$ are also the parametric equations of the hyperbola.

Tangent Equation of Hyperbola

(i) **Point Form** The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ or $T=0$.

(ii) **Parametric Form** The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $(a\sec\theta, b\tan\theta)$ is $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$.

(iii) **Slope Form** The equation of the tangents of slope m to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are given by $y=mx\pm\sqrt{a^2m^2-b^2}$.

The coordinates of the point of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right).$$

(iv) The tangent at the points $P(a\sec\theta_1, b\tan\theta_1)$ and $Q(a\sec\theta_2, b\tan\theta_2)$ intersect at the point

$$\left(\frac{a\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}, \frac{b\sin\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}\right)$$

(v) Two tangents drawn from P are real and distinct, coincident or imaginary according as the roots of the equation $m^2(h^2-a^2)-2khm+k^2+b^2=0$ are real and distinct, coincident or imaginary.

(vi) The line $y=mx+c$ touches the hyperbola, if $c^2=a^2m^2-b^2$ and the point of contacts $\left(\pm\frac{a^2m}{c}, \pm\frac{b^2}{c}\right)$, where $c=\sqrt{a^2m^2-b^2}$.

(vii) Maximum two tangents can be drawn from a point to a hyperbola.

(viii) The combined equation of the pairs of tangent drawn from a point $P(x_1, y_1)$ lying outside the hyperbola $S\equiv\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $SS_1=T^2$.

i.e. $$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2$$

Equation of Chord

(i) Equations of chord joining two points $P(a\sec\theta_1, b\tan\theta_1)$ and $Q(a\sec\theta_2, b\tan\theta_2)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$y - b\tan\theta_1 = \frac{b\tan\theta_2 - b\tan\theta_1}{a\sec\theta_2 - a\sec\theta_1}\cdot(x - a\sec\theta_1)$$

or $$\frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

(ii) Equations of chord of contact of tangents drawn from a point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ or $T=0$.

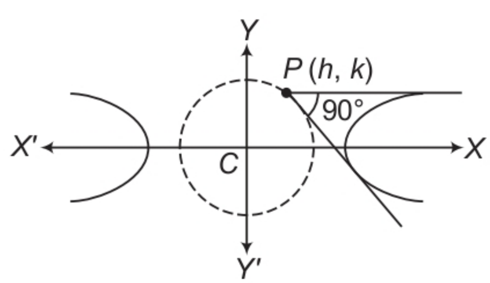
(iii) The equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ bisected at point (x_1, y_1) is given by

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1 = \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$$

or $$T = S_1$$

Director Circle

The locus of the point of intersection of perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is called a director circle. The equation of director circle is $x^2+y^2=a^2-b^2$.



Note Director circle of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is exist only when $a^2 > b^2$.

Normal Equation of Hyperbola

(i) **Point Form** The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$.

(ii) **Parametric Form** The equation of the normal at $(a\sec\theta, b\tan\theta)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $ax\cos\theta + by\cot\theta = a^2 + b^2$.

(iii) **Slope Form** The equation of the normal of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by

$$y = mx \mp \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2 m^2}}$$

The coordinates of the point of contact are

$$\left(\pm\frac{a^2}{\sqrt{a^2 - b^2 m^2}}, \mp\frac{b^2 m}{\sqrt{a^2 - b^2 m^2}}\right).$$

(iv) The line $y = mx + c$ will be normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if

$$c^2 = \frac{m^2 (a^2 + b^2)^2}{a^2 - b^2 m^2}$$

(v) Maximum four normals can be drawn from a point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Conormal Points

Points on the hyperbola, the normals at which passes through a given point are called conormal points.

(i) The sum of the eccentric angles of conormal points is an odd multiple of π.

(ii) If $\theta_1, \theta_2, \theta_3$ and θ_4 are eccentric angles of four points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the normal at which they are concurrent, then

(a) $\sum \cos(\theta_1 + \theta_2) = 0$ (b) $\sum \sin(\theta_1 + \theta_2) = 0$

(iii) If θ_1, θ_2 and θ_3 are the eccentric angles of three points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, such that

$$\sin(\theta_1 + \theta_2) + \sin(\theta_2 + \theta_3) + \sin(\theta_3 + \theta_1) = 0.$$

Then, the normals at these points are concurrent.

(iv) If the normals at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are concurrent, then

$$(x_1 + x_2 + x_3 + x_4)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right) = 4$$

and $$(y_1 + y_2 + y_3 + y_4)\left(\frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} + \frac{1}{y_4}\right) = 4.$$

Conjugate Points and Conjugate Lines

(i) Two points are said to be conjugate points with respect to a hyperbola, if each lies on the polar of the other.

(ii) Two lines are said to be conjugate lines with respect to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if each passes through the pole of the other.

Diameter and Conjugate Diameter

(i) **Diameter** The locus of the mid-points of a system of parallel chords of a hyperbola is called a diameter.

The equation of the diameter bisecting a system of parallel chords of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = \frac{b^2}{a^2 m} x$

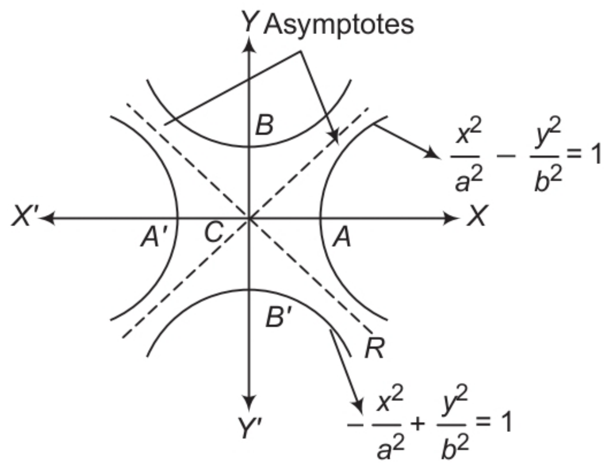
(ii) **Conjugate Diameter** The diameters of a hyperbola are said to be conjugate diameter, if each bisect the chords parallel to the other.

The diameters $y = m_1 x$ and $y = m_2 x$ are conjugate, if $m_1 m_2 = \frac{b^2}{a^2}$.

Note If a pair of diameters is conjugate with respect to a hyperbola, they are conjugate with respect to its conjugate hyperbola also.

Asymptote

An asymptote to a curve is a straight line, at a finite distance from the origin, to which the tangent to a curve tends as the point of contact goes to infinity.



(i) The equation of two asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are

$$y = \pm \frac{b}{a}x \text{ or } \frac{x}{a} \pm \frac{y}{b} = 0$$

(ii) The combined equation of the asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

(iii) When $b = a$, i.e. the asymptotes of rectangular hyperbola $x^2 - y^2 = a^2$ are $y = \pm x$ which are at right angle.

(iv) A hyperbola and its conjugate hyperbola have the same asymptotes.

(v) The equation of the pair of asymptotes differ the hyperbola and the conjugate hyperbola by the same constant only i.e.

Hyperbola – Asymptotes = Asymptotes – Conjugate hyperbola

(vi) The asymptotes pass through the centre of the hyperbola.

(vii) The bisectors of angle between the asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are the coordinate axes.

(viii) The angle between the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $2\tan^{-1}\left(\frac{b}{a}\right)$ or $2\sec^{-1}(e)$.

Rectangular Hyperbola

A hyperbola whose asymptotes include a right angle is said to be rectangular hyperbola or we can say that, if the lengths of transverse and conjugate axes of any hyperbola be equal, then it is said to be a rectangular hyperbola.

i.e. In a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $b = a$, then it said to be rectangular hyperbola. The eccentricity of a rectangular hyperbola is always $\sqrt{2}$.

Rectangular Hyperbola of the Form $x^2 - y^2 = a^2$

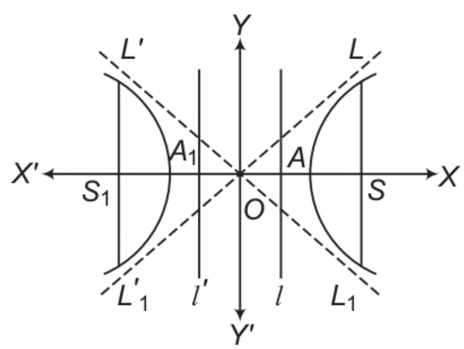
(i) Asymptotes are perpendicular lines i.e. $x \pm y = 0$

(ii) Eccentricity $e = \sqrt{2}$

(iii) Centre $(0, 0)$

(iv) Foci $(\pm\sqrt{2}\,a, 0)$

(v) Vertices $A(a, 0)$ and $A_1(-a, 0)$



(vi) Equation of directrices $x = \pm \frac{a}{\sqrt{2}}$

(vii) Length of latusrectum $= 2a$

(viii) Parametric form $x = a\sec\theta, y = a\tan\theta$

(ix) Equation of tangent, $x\sec\theta - y\tan\theta = a$

(x) Equation of normal, $\frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 2a$

Rectangular Hyperbola of the Form $xy = c^2$

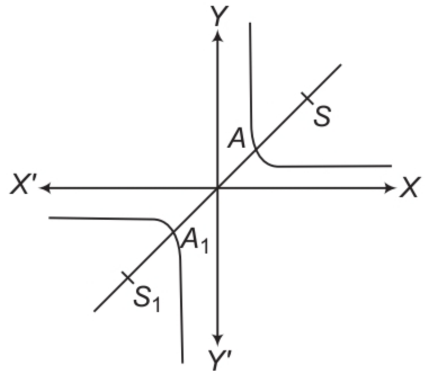
(i) Asymptotes are perpendicular lines i.e. $x = 0$ and $y = 0$

(ii) Eccentricity $e = \sqrt{2}$

(iii) Centre $(0, 0)$

(iv) Foci $S(\sqrt{2}c, \sqrt{2}c), S_1(-\sqrt{2}c, -\sqrt{2}c)$

(v) Vertices $A(c, c), A_1(-c, -c)$



(vi) Equations of directrices $x + y = \pm \sqrt{2}\, c$

(vii) Length of latusrectum $= 2\sqrt{2}\, c$

(viii) Parametric form $x = ct, y = \dfrac{c}{t}$

Equation of Tangent of Rectangular Hyperbola $xy = c^2$

(i) **Point Form** The equation of tangent at (x_1, y_1) to the rectangular hyperbola is $xy_1 + yx_1 = 2c^2$ or $\dfrac{x}{x_1} + \dfrac{y}{y_1} = 2.$

(ii) **Parametric Form** The equation of tangent at $\left(ct, \dfrac{c}{t}\right)$ to the hyperbola is $\dfrac{x}{t} + yt = 2c.$

(iii) Tangent at $P\left(ct_1, \dfrac{c}{t_1}\right)$ and $Q\left(ct_2, \dfrac{c}{t_2}\right)$ to the rectangular hyperbola intersect at $\left(\dfrac{2ct_1t_2}{t_1 + t_2}, \dfrac{2c}{t_1 + t_2}\right).$

(iv) The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the rectangular hyperbola is $xy_1 + yx_1 = 2c^2.$

Normal Equation of Rectangular Hyperbola $xy = c^2$

(i) **Point Form** The equation of the normal at (x_1, y_1) to the rectangular hyperbola is $xx_1 - yy_1 = x_1^2 - y_1^2.$

(ii) **Parametric Form** The equation of the normal at $\left(ct, \dfrac{c}{t}\right)$ to the rectangular hyperbola $xy = c^2$ is $xt^3 - yt - ct^4 + c = 0.$

(iii) The equation of the normal at $\left(ct, \frac{c}{t}\right)$ is a fourth degree equation in t. So, in general maximum four normals can be drawn from a point to the hyperbola $xy = c^2$.

Important Results about Hyperbola

(i) The point (x_1, y_1) lies outside, on or inside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 <, =$ or > 0

(ii) The equation of the chord of the hyperbola $xy = c^2$ whose mid-point is (x_1, y_1) is

$$xy_1 + yx_1 = 2x_1y_1 \text{ or } T = S_1$$

(iii) Equation of the chord joining t_1, t_2 on $xy = t^2$ is

$$x + yt_1t_2 = c(t_1 + t_2)$$

(iv) If a triangle is inscribed in a rectangular hyperbola, then its orthocentre lies on the hyperbola.

(v) Any straight line parallel to an asymptotes of a hyperbola intersects the hyperbola at only one point.