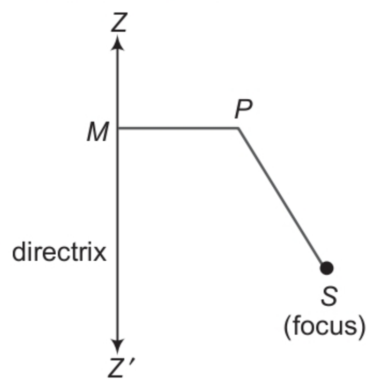
Parabola

Conic Section

A conic is the locus of a point whose distance from a fixed point bears a constant ratio to its distance from a fixed line. The fixed point is the focus S and the fixed line is directrix l.



The constant ratio is called the eccentricity denoted by e.

(i) If $0 < e < 1$, conic is an ellipse.

(ii) $e = 1$, conic is a parabola.

(iii) $e > 1$, conic is a hyperbola.

General Equation of Conic

If fixed point of curve is (x_1, y_1) and fixed line is $ax + by + c = 0$, then equation of the conic is

$$(a^2 + b^2)\,[(x - x_1)^2 + (y - y_1)^2] = e^2(ax + by + c)^2$$

which on simplification takes the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

where a, b, c, f, g and h are constants.

A second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents

(i) **a pair of straight lines**, if $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(ii) **a pair of parallel** (or coincident) **straight lines**, if $\Delta = 0$ and $h^2 = ab$.
(iii) **a pair of perpendicular straight lines**, if $\Delta = 0$ and $a + b = 0$
(iv) **a point**, if $\Delta = 0$ and $h^2 < ab$
(v) **a circle**, if $a = b \neq 0$, $h = 0$ and $\Delta \neq 0$
(vi) **a parabola**, if $h^2 = ab$ and $\Delta \neq 0$
(vii) **a ellipse**, if $h^2 < ab$ and $\Delta \neq 0$
(viii) **a hyperbola**, if $h^2 > ab$ and $\Delta \neq 0$
(ix) **a rectangular hyperbola**, if $h^2 > ab$, $a + b = 0$ and $\Delta \neq 0$

Important Terms Related to Parabola

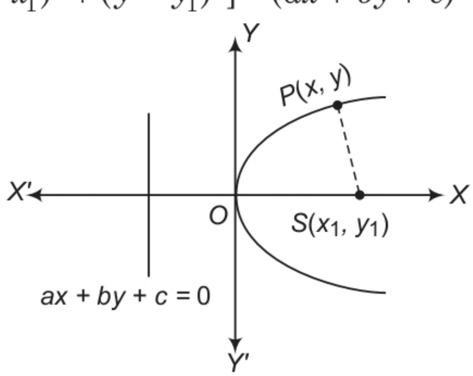
(i) **Axis** A line perpendicular to the directrix and passes through the focus.
(ii) **Vertex** The intersection point of the conic and axis.
(iii) **Centre** The point which bisects every chord of the conic passing through it.
(iv) **Focal Chord** Any chord passing through the focus.
(v) **Double Ordinate** A chord perpendicular to the axis of a conic.
(vi) **Latusrectum** A double ordinate passing through the focus of the parabola.
(vii) **Focal Distance** The distance of a point $P(x, y)$ from the focus S is called the focal distance of the point P.

Parabola

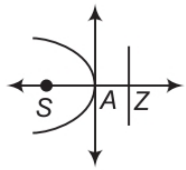
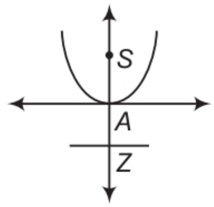
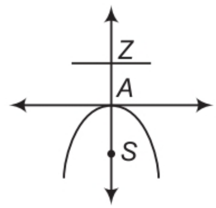
A parabola is the locus of a point which moves in a plane such that its distance from a fixed point in the plane is always equal to its distance from a fixed straight line in the same plane.

If focus of a parabola is $S(x_1, y_1)$ and equation of the directrix is $ax + by + c = 0$, then the equation of the parabola is

$$(a^2 + b^2)[(x - x_1)^2 + (y - y_1)^2] = (ax + by + c)^2$$



Standard Forms of a Parabola and Related Terms

Terms	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
				
Vertex	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$	$A(0, 0)$
Focus	$S(a, 0)$	$S(-a, 0)$	$S(0, a)$	$S(0, -a)$
Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Equation of directrix	$x + a = 0$	$x - a = 0$	$y + a = 0$	$y - a = 0$
Eccentricity	$e = 1$	$e = 1$	$e = 1$	$e = 1$
Extremities of latusrectum	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
Length of latusrectum	$4a$	$4a$	$4a$	$4a$
Equation of tangent at vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
Parametric equation	$\begin{cases} x = at^2 \\ y = 2at \end{cases}$	$\begin{cases} x = -at^2 \\ y = 2at \end{cases}$	$\begin{cases} x = 2at \\ y = at^2 \end{cases}$	$\begin{cases} x = 2at \\ y = -at^2 \end{cases}$
Focal distance of any point $P(h, k)$ on the parabola	$h + a$	$a - h$	$k + a$	$a - k$
Equation of latusrectum	$x - a = 0$	$x + a = 0$	$y - a = 0$	$y + a = 0$

Other Forms of a Parabola

If the vertex of the parabola is at a point $A(h, k)$ and its latusrectum is of length $4a$, then its equation is

(i) $(y - k)^2 = 4a(x - h)$, if its axis is parallel to OX i.e. parabola opens rightward.

(ii) $(y - k)^2 = -4a(x - h)$, if its axis is parallel to OX' i.e. parabola opens leftward.

(iii) $(x-h)^2 = 4a(y-k)$, if its axis is parallel to OY i.e. parabola opens upward.

(iv) $(x-h)^2 = -4a(y-k)$, if its axis is parallel to OY' i.e. parabola opens downward.

(v) The general equation of a parabola whose axis is parallel to X-axis, is $x = ay^2 + by + c$ and the general equation of a parabola whose axis is parallel to Y-axis, is $y = ax^2 + bx + c$.

Position of a Point

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 >, =, < 0$.

Chord

Joining any two points on a curve is called chord.

(i) **Parametric Equation of a Chord** Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be any two points on the parabola $y^2 = 4ax$, then the equation of the chord is

$$(y - 2at_1) = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2)$$

or $$y(t_1 + t_2) = 2x + 2at_1t_2$$

(ii) Let $P(at^2, 2at)$ be the one end of a focal chord PQ of the parabola $y^2 = 4ax$, then the coordinates of the other end Q are

$$\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$$

(iii) If l_1 and l_2 are the length of the focal segments, then length of the latusrectum = 2 (harmonic mean of focal segment)

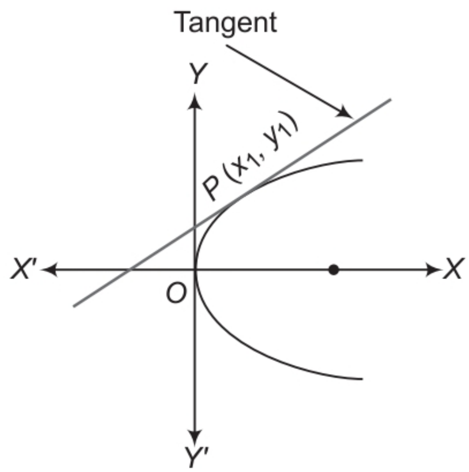
i.e. $$4a = \frac{4l_1l_2}{l_1 + l_2}$$

(iv) For a chord joining points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ and passing through focus, then $t_1t_2 = -1$.

(v) Length of the focal chord having t_1 and t_2 as end points is $a(t_2 - t_1)^2$.

Equation of Tangent

A line which touch only one point of a parabola.



Point Form

The equation of the tangent to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is given by $yy_1 = 2a(x + x_1)$.

Slope Form

(a) The equation of the tangent of slope m to the parabola $y^2 = 4ax$ is $y = mx + \dfrac{a}{m}$

(b) The equation of the tangent of slope m to the parabola $(y - k)^2 = 4a(x - h)$ is given by $(y - k) = m(x - h) + \dfrac{a}{m}$

The coordinates of the point of contact are $\left(h + \dfrac{a}{m^2}, k + \dfrac{2a}{m}\right)$.

Parametric Form

The equation of the tangent to the parabola $y^2 = 4ax$ at a point $(at^2, 2at)$ is $yt = x + at^2$.

Condition of Tangency

(i) The line $y = mx + c$ touches a parabola, iff $c = \dfrac{a}{m}$ and the point of contact is $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$.

(ii) The straight line $lx + my + n = 0$ touches $y^2 = 4ax$, if $nl = am^2$ and $x\cos\alpha + y\sin\alpha = p$ touches $y^2 = 4ax$, if $p\cos\alpha + a\sin^2\alpha = 0$.

Point of Intersection of Two Tangents

Let two tangents at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ intersect at R. Then, their point of intersection is $R(at_1t_2, a(t_1 + t_2))$ i.e. (GM of abscissa, AM of ordinate).

Angle between Two Tangents

Angle θ between tangents at two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is given by

$$\tan\theta = \left|\frac{t_2 - t_1}{1 + t_1 t_2}\right|$$

Important Results on Tangents

(i) The tangent at any point on a parabola bisects the angle between the focal distance of the point and the perpendicular on the directrix from the point.

(ii) The tangent at the extremities of a focal chord of a parabola intersect at right angle on the directrix.

(iii) The portion of the tangent to a parabola cut off between the directrix and the curve subtends a right angle at the focus.

(iv) The perpendicular drawn from the focus on any tangent to a parabola intersect it at the point where it cuts the tangent at the vertex.

(v) The orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.

(vi) The circumcircle formed by the intersection points of tangents at any three points on a parabola passes through the focus of the parabola.

(vii) The tangent at any point of a parabola is equally inclined to the focal distance of the point and the axis of the parabola.

(viii) The length of the subtangent at any point on a parabola is equal to twice the abscissa of the point.

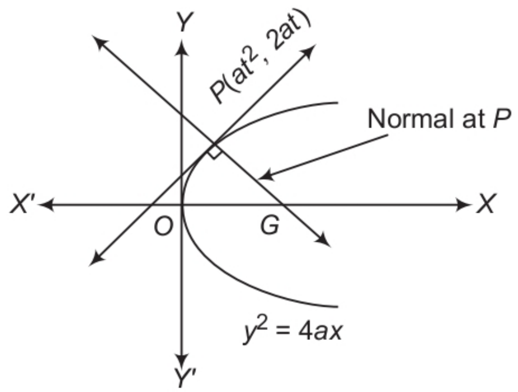
(ix) Two tangents can be drawn from a point to a parabola. Two tangents are real and distinct or coincident or imaginary according as given point lies outside, on or inside the parabola.

(x) The straight line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points. These two points are real and distinct, if $c > \frac{a}{m}$, points are real and coincident, if $c = \frac{a}{m}$, points are imaginary, if $c < \frac{a}{m}$.

(xi) Area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

Equation of Normal

A line which is perpendicular to the tangent at the point of contact with parabola.



Point Form

The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is given by $y - y_1 = -\frac{y_1}{2a}(x - x_1)$.

Parametric Form

The equation of the normal to the parabola $y^2 = 4ax$ at point $(at^2, 2at)$ is given by $y + tx = 2at + at^3$.

Slope Form

The equation of the normal to the parabola $y^2 = 4ax$ in terms of its slope m is given by $y = mx - 2am - am^3$ at point $(am^2, -2am)$.

Important Results on Normals

(i) If the normal at the point $P(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ at $(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$.

(ii) The tangent at one extremity of the focal chord of a parabola is parallel to the normal at other extremity.

(iii) The normal at points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ to the parabola $y^2 = 4ax$ intersect at the point

$$[2a + a(t_1^2 + t_2^2 + t_1t_2) - at_1t_2(t_1 + t_2)].$$

(iv) If the normal at points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ meet on the parabola, then $t_1t_2 = 2$.

(v) If the normal at two points P and Q of a parabola $y^2 = 4ax$ intersect at a third point R on the curve, then the product of the ordinates of P and Q is $8a^2$.

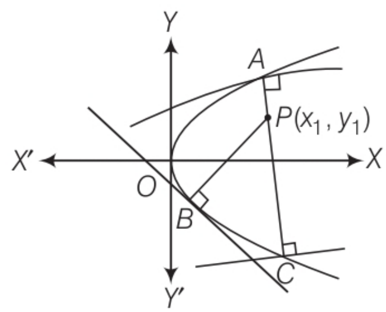
(vi) If the normal chord at a point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ subtends a right angle at the vertex of the parabola, then $t^2 = 2$.

(vii) The normal chord of a parabola at a point whose ordinate is equal to the abscissa, subtends a right angle at the focus.

(viii) The normal at any point of a parabola is equally inclined to the focal radius of the point and the axis of the parabola.

(ix) Maximum three distinct normals can be drawn from a point to a parabola.

(x) **Conormal Points** The points on the parabola at which the normals pass through a common point are called conormal points. The conormal points are called the feet of the normals.



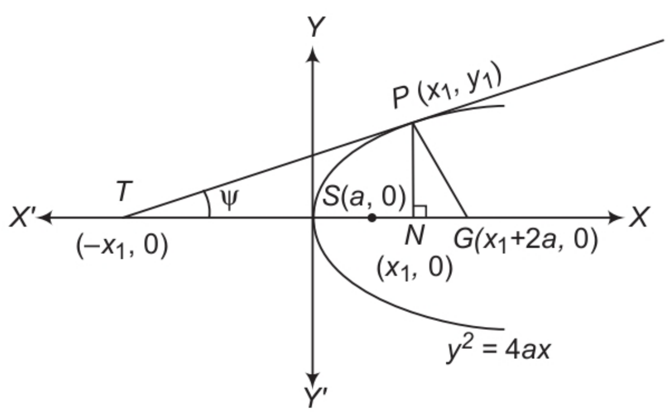
Points *A, B* and *C* are called conormal points.

(a) The algebraic sum of the slopes of the normals at conormals point is 0.

(b) The sum of the ordinates of the conormal points is 0.

(c) The centroid of the triangle formed by the conormal points on a parabola lies on its axis.

Length of Tangent and Normal



(i) The length of the tangent $= PT = PN \operatorname{cosec} \psi = y_1 \operatorname{cosec} \psi$

(ii) The length of subtangent $= NT = PN \cot \psi = y_1 \cot \psi$

(iii) The length of normal $= PG = PN \sec \psi = y_1 \sec \psi$

(iv) The length of subnormal $= NG = PN \tan \psi = y_1 \tan \psi$

Equation of the Chord Bisected at a Given Point

The equation of the chord of the parabola $y^2 = 4ax$ which is bisected at (x_1, y_1) is $yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$, or $T = S_1$

where, $S_1 = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$.

Equation of Diameter

The locus of mid-point of a system of parallel chords of a conic is known its diameter.

The diameter bisecting chords of slope m to the parabola $y^2 = 4ax$ is $y = \frac{2a}{m}$.

Pair of Tangents

The combined equation of the pair of tangents drawn from a point to a parabola $y^2 = 4ax$ is given by

$$SS_1 = T^2$$

where, $S = y^2 - 4ax, S_1 = y_1^2 - 4ax_1$

and $T = [yy_1 - 2a(x + x_1)]$

Chord of Contact

The chord of contact of tangents drawn from a point (x_1, y_1) to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$.

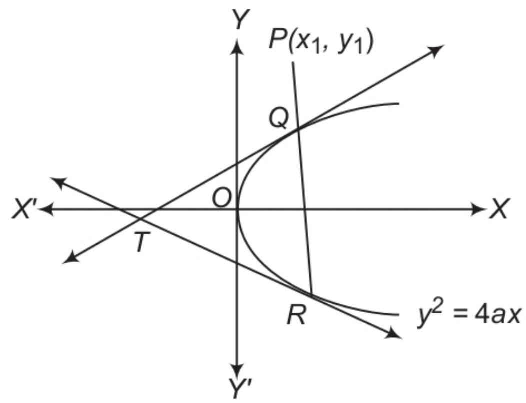
Director Circle

The locus of the point of intersection of perpendicular tangents to a parabola is known as director circle.

The director circle of a parabola is same as its directrix.

Pole and Polar

Let P be a point lying within or outside a given parabola. Suppose any straight line drawn through P intersects the parabola at Q and R. Then, the locus of the point of intersection of the tangents to the parabola at Q and R is called the polar of the given point P with respect to the parabola and the point P is called the pole of the polar.



(i) The polar of a point $P(x_1, y_1)$ with respect to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$ or $T = 0$.

(ii) Any tangent is the polar of its point of contact.

(iii) Pole of $lx + my + n = 0$ with respect to $y^2 = 4ax$ is $\left(\frac{n}{l}, -\frac{2am}{l}\right)$.

(iv) Pole of the chord joining (x_1, y_1) and (x_2, y_2) is $\left(\frac{y_1y_2}{4a}, \frac{y_1 + y_2}{2}\right)$.

(v) If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of Q will passes through P. Here, P and Q are called **conjugate points**.

(vi) If the pole of a line $a_1x + b_1y + c_1 = 0$ lies on another line $a_2x + b_2y + c_2 = 0$, then the pole of the second line will lies on the first line. Such lines are called **conjugate lines**.

(vii) The point of intersection of the polar of two points Q and R is the pole of QR.

(viii) The tangents at the ends of any chord of the parabola meet on the diameter which bisect the chord.

Important Points to be Remembered

(i) For the ends of latusrectum of the parabola $y^2 = 4ax$, the values of the perimeter are ± 1.

(ii) The circles described on focal radii of a parabola as diameter touches the tangent at the vertex.

(iii) The circles described on any focal chord of a parabola as diameter touches the directrix.

(iv) If y_1, y_2, y_3 are the ordinates of the vertices of a triangle inscribed in the parabola $y^2 = 4ax$, then its area is $\frac{1}{8a}|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$.