DCAM classes

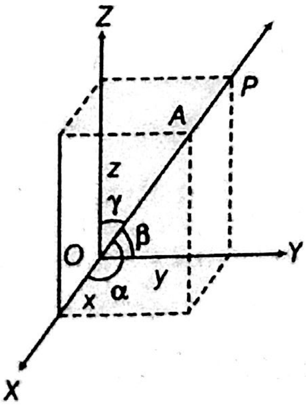
Dynamic Classes for Academic Mastery

Three Dimensional Geometry

[TOPIC 1] Direction cosines and Lines

1.1 Direction Cosines of a Line

If a directed line $\overrightarrow{OP}$ makes angles α, β and γ with positive direction of X-axis, Y-axis and Z-axis respectively, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$, are called direction cosines of a line. They are denoted by l, m and n. Therefore, $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$. Also, sum of squares of direction cosines of a line is always 1, i.e. $l^2 + m^2 + n^2 = 1$ or $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.



NOTE (i) Direction cosines of a directed line are unique.

(ii) Direction cosines of X, Y and Z-axes are $(1,0,0), (0,1,0)$ and $(0,0,1)$.

1.2 Direction Ratios of a Line

Any three numbers which are proportional to the direction cosines of a line, are called direction ratios of a line.

1. If a, b and c are direction ratios of a line, then $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$ (say), k being a constant.
   $\Rightarrow l = ak, m = bk$ and $n = ck$
2. If a, b and c are direction ratios of a line, then its direction cosines are
   $l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$.
3. Direction ratios of a line PQ passing through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1, y_2 - y_1$ and $z_2 - z_1$ and direction cosines are $\frac{x_2 - x_1}{|\overrightarrow{PQ}|}, \frac{y_2 - y_1}{|\overrightarrow{PQ}|}, \frac{z_2 - z_1}{|\overrightarrow{PQ}|}$.

NOTE (i) Direction ratios may be taken as $(x_1 - x_2), (y_1 - y_2)$ and $(z_1 - z_2)$.

(ii) Direction ratios of two parallel lines are proportional, as two parallel lines have same set of direction cosines.

(iii) Direction ratios of a line are not unique.

1.3 Straight line

A straight line is a curve, such that all the points on the line segment joining any two points of it lies on it.

Equation of a Line through a Given Point and Parallel to a Given Vector

1. **Vector form** The vector equation of a line passing through a point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$ where, $\vec{r}$ is the position vector of an arbitrary point on the line and λ is some real number.
2. **Cartesian form** The cartesian equation of a line passing through a point $A(x_1, y_1, z_1)$ and having direction ratios a, b and c is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.

   If l, m and n are the direction cosines of the line, then the equation of the line is
   $$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

Equation of Line Passing through Two Given Points

1. **Vector form** The vector equation of a line passing through two points A and B with position vector $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, $\lambda \in R$.
2. **Cartesian form** The cartesian equation of a line passing through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is
   $$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

Angle between Two Lines

1. **Vector form** Angle between the lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ is given as
   $$\cos\theta = \left| \frac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}||\vec{b_2}|} \right|$$
   where, θ is the acute angle between the lines.
2. **Cartesian form** Angle between the lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is given as
   $$\cos\theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$
   $$\text{or } \sin\theta = \frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}.$$
   Also, angle (θ) between two lines with direction cosines, l_1, m_1, n_1 and l_2, m_2, n_2 is given by $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$
   $$\text{or } \sin\theta = \sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2}.$$

Condition of Perpendicularity

Two lines are said to be perpendicular,

in **vector form**, if $\vec{b_1} \cdot \vec{b_2} = 0$;

in **cartesian form**, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ [direction ratio form]

or $l_1l_2 + m_1m_2 + n_1n_2 = 0$. [direction cosine form]

Condition that Two Lines are Parallel

Two lines are parallel, in **vector form** $\vec{b_1} = \lambda \vec{b_2}$;

in **cartesian form** if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ [direction ratio form]

or $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$. [direction cosine form]

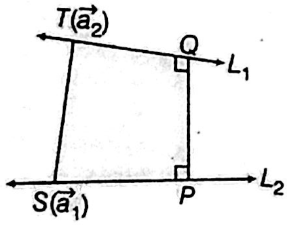
Skew lines

Lines which are neither parallel nor intersecting lines, are called **skew lines**. In fact, such lines are non-coplanar.

Shortest Distance bewteen two lines

For skew lines, the line of the shortest distance will be perpendicular to both the lines.

In the figure given below, i.e. PQ is the line of shortest distance.



1. **Vector form** If the lines are $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$. Then, shortest distance

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|.$$

2. **Cartesian form** If the lines are

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$,

then shortest distance,

$$d = \left| \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}} \right|$$

NOTE The condition for two given lines to be intersect, is $(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = 0$

or $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0,$

i.e., the shortest distance between two lines is zero.

Distance between Two Parallel Lines

If two lines l_1 and l_2 are parallel, then they are coplanar. Let the lines be $\vec{r} = \vec{a}_1 + \lambda \vec{b}$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}$, then the distance between parallel lines is $\left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$.

NOTE If two lines are parallel, then they both have same or proportional direction ratios.

Distance between Two Points

The distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Mid-point of a Line

The mid-point of a line joining points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

NOTE If P is a point and P' is the image of the point with respect to the line, then mid-point of P and P' satisfy the equation of line.

[TOPIC 2] Plane

2.1 Plane

A plane is a surface such that a line segment joining any two points of it lies completely on it. A straight line which is perpendicular to every line lying on a plane, is called a **normal** to the plane.

Equation of a Plane in Normal form

1. **Vector form** The equation of plane in normal form is given by $\vec{r} \cdot \hat{n} = d$, where $\hat{n}$ is a unit vector normal to the plane and d is distance of plane from the origin.
2. **Cartesian form** The equation of plane is given by $lx + my + nz = p$, where l, m and n are direction cosines of unit normal vector $(\hat{n})$ to the plane and p is a distance of a plane from origin.

NOTE Equation of plane XY, YZ and ZX are $z = 0$, $x = 0$ and $y = 0$, respectively.

Equation of a Plane Perpendicular to a Given Vector and Passing Through a Given Point

1. **Vector form** Let a plane passes through a point A with position vector $\vec{a}$ and perpendicular to the vector $\vec{n}$, then vector equation of plane is given by $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$ or $\vec{r} \cdot \vec{n} = d$, where $d = \vec{a} \cdot \vec{n}$.

NOTE Equation of a plane passing through the origin and perpendicular to given vector $\vec{n}$ is $\vec{r} \cdot \vec{n} = 0$.

2. **Cartesian form** Equation of plane passing through point (x_1, y_1, z_1) is given by $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where, a, b and c are the direction ratios of normal to the plane.

NOTE Every equation of the form, $\vec{r} \cdot \vec{n} = d$ or $\vec{r} \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = d$ or $ax + by + cz = d$ always represents a plane, where a, b, c are the direction ratios of the normal to this plane. Here, d is not a distance of the plane form the origin. For finding the distance of the plane from the origin, convert the equation in normal form.

Equation of Plane Passing through three Non-collinear Points

1. **Vector form** If $\vec{a}, \vec{b}$ and $\vec{c}$ are the position vectors of three given points, then equation of a plane passing through three non-collinear points is $(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})]=0$.
where, $\vec{r}$ is the position vector of any point on the plane.
2. **Cartesian form** If (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are three non-collinear points, then equation of the plane is
$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0.$$
If above points are collinear, then
$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=0.$$

Equation of Plane in Intercept Form

If a, b and c are x-intercept, y-intercept and z-intercept, respectively made by the plane on the coordinate axes, then equation of plane is
$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

Equation of Plane Passing through the Line of Intersection of Two Given Planes

1. **Vector form** If equation of the planes are $\vec{r}\cdot\vec{n_1}=d_1$ and $\vec{r}\cdot\vec{n_2}=d_2$, then equation of any plane passing through the intersection of given planes is $\vec{r}\cdot(\vec{n_1}+\lambda\vec{n_2})=d_1+\lambda d_2$
where, λ is a constant and calculated from given condition.
2. **Cartesian form** If the equation of planes are $a_1x+b_1y+c_1z=d_1$ and $a_2x+b_2y+c_2z=d_2$, then equation of any plane passing through the intersection of given planes is
$(a_1x+b_1y+c_1z-d_1)+\lambda(a_2x+b_2y+c_2z-d_2)=0$
where, λ is a constant and calculated from given condition.

NOTE The intersection of two planes always gives a line.

Equation of a Plane Parallel to a Given Plane

1. **Vector form** The vector equation of a plane parallel to the given plane
$\vec{r}\cdot\vec{n}=d_1$ is $\vec{r}\cdot\vec{n}=d_2$, where d_2 is a constant determined by the given condition.
2. **Cartesian form** The cartesian equation of a plane parallel to the given plane
$ax+by+cz+d_1=0$ is $ax+by+cz+d_2=0$,
where d_2 is a constant determined by the given condition.

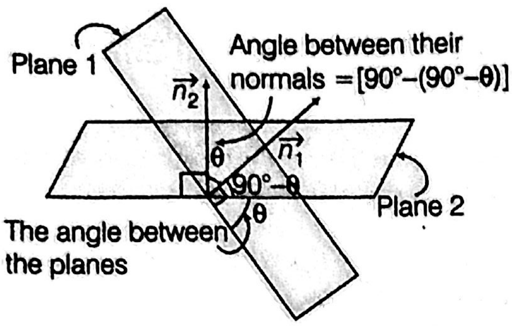
Condition for Coplanarity of Two Lines

1. **Vector form** Two lines $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ are coplanar, if $(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})=0$
$\because$ $(\vec{a_2}-\vec{a_1})$ is perpendicular to $(\vec{b_1}\times\vec{b_2})$.
2. **Cartesian form** Two lines
$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$
and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ are coplanar, if
$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0.$$

2.2 Angle between Two Planes

The angle between two planes is defined as the angle between their normals. Let θ be the acute angle between two planes.

1. **Vector form** If $\vec{n_1}$ and $\vec{n_2}$ are normals to the planes $\vec{r}\cdot\vec{n_1}=d_1$ and $\vec{r}\cdot\vec{n_2}=d_2$, then angle between the normals to the planes drawn from some common points is
$$\cos\theta=\left|\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}\right|.$$



NOTE Two planes are perpendicular to each other, if $\vec{n_1} \cdot \vec{n_2} = 0$ and two planes are parallel to each other, if $\vec{n_1} = \lambda \vec{n_2}$, where λ is a scalar.

2. **Cartesian form** If the two planes are $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$, then $\cos\theta = \left|\dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$.

NOTE Two planes are perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and parallel, if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

2.3 Distance of a Point from a Plane

1. **Vector form** The distance of a point whose position vector is $\vec{a}$ from the plane $\vec{r} \cdot \hat{n} = d$ is $|\vec{a} \cdot \hat{n} - d|$.

NOTE (i) If the equation of the plane is in the form $\vec{r} \cdot \vec{n} = d$, where $\vec{n}$ is normal to the plane, then the perpendicular distance is $\dfrac{|\vec{a} \cdot \vec{n} - d|}{|\vec{n}|}$.

(ii) The length of the perpendicular from origin O to the plane $\vec{r} \cdot \vec{n} = d$ is $\dfrac{|d|}{|\vec{n}|}$. $[\because \vec{a} = 0]$

2. **Cartesian form** The perpendicular distance of the point (x_1, y_1, z_1) from the plane $ax + by + cz = d$ is

$$d = \left|\frac{ax_1 + by_1 + cz_1 - d}{\sqrt{a^2 + b^2 + c^2}}\right|.$$

NOTE The length of the perpendicular from the origin to the plane $ax + by + cz + d = 0 = \dfrac{|d|}{\sqrt{a^2 + b^2 + c^2}}$.

2.4 Angle between a Line and a Plan

The angle between a line and a plane is the complement of the angle between the line and normal to the plane.

1. **Vector form** If the equation of line is $\vec{r} = \vec{a} + \lambda \vec{b}$ and the equation of plane is $\vec{r} \cdot \vec{n} = d$, then the angle θ between the line and the normal to the plane is $\cos\theta = \left|\dfrac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|}\right|$.

Let ϕ be the angle between the line and the plane, then it is equal to $90° - \theta$.

$$\Rightarrow \quad \sin\phi = \sin(90° - \theta) = \cos\theta$$

$$\therefore \quad \sin\phi = \left|\frac{\vec{b} \cdot \vec{n}}{|\vec{b}||\vec{n}|}\right|$$

2. **Cartesian form** If a_1, b_1 and c_1 are the DR's of line and $a_2x + b_2y + c_2z + d = 0$ is the equation of plane, then

$$\sin\phi = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|,$$

where ϕ denotes the angle between the line and the plane.

Remember Points

1. If a line is parallel to the plane, then normal to the plane is perpendicular to the line,
i.e. $a_1a_2 + b_1b_2 + c_1c_2 = 0$.
2. If a line is perpendicular to the plane, then DR's of line are proportional to the normal of the plane,
i.e. $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$
(where, a_1, b_1 and c_1 are the DR's of a line and a_2, b_2 and c_2 are the DR's of normal to the plane).
3. If P' is the image of a point P with respect to the plane, then mid-point of P and P' satisfy the equation of given plane.