Application of Derivatives

Derivatives as the Rate of Change

If a variable quantity y is some function of time t i.e. $y = f(t)$, then small change in time Δt have a corresponding change Δy in y.

Thus, the average rate of change $= \dfrac{\Delta y}{\Delta t}$.

When limit $\Delta t \to 0$ is applied, the rate of change becomes instantaneous and we get the rate of change with respect to t at any instant y, i.e. $\lim\limits_{\Delta t \to 0} \dfrac{\Delta y}{\Delta t} = \dfrac{dy}{dt}$.

Similarly, the differential coefficient of y with respect to x i.e. $\dfrac{dy}{dx}$ is nothing but the rate of change of y relative to x.

Derivative as the Rate of Change of Two Variables

Let two variables are varying with respect to another variable t, i.e. $y = f(t)$ and $x = g(t)$.

Then, rate of change of y with respect to x is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Note $\dfrac{dy}{dx}$ is positive, if y increases as x increases and is negative, if y decreases as x increases.

Marginal Cost

Marginal cost represents the instantaneous rate of change of the total cost with respect to the number of items produced at an instant. If $C(x)$ represents the cost function for x units produced, then marginal cost, denoted by MC, is given by

$$\text{MC} = \frac{d}{dx}\{C(x)\}.$$

Marginal Revenue

Marginal revenue represents the rate of change of total revenue with respect to the number of items sold at an instant. If $R(x)$ represents the revenue function for x units sold, then marginal revenue, denoted by MR, is given by

$$\text{MR} = \frac{d}{dx}\{R(x)\}.$$

Note Total cost = Fixed cost + Variable cost i.e. $C(x) = f(c) + v(x)$.

Tangents and Normals

A tangent is a straight line, which touches the curve $y = f(x)$ at a point. A normal is a straight line perpendicular to a tangent to the curve $y = f(x)$ intersecting at the point of contact.

Slope of Tangent and Normal

(i) If the tangent at P is perpendicular to X-axis or parallel to Y-axis, then $\theta = \frac{\pi}{2} \Rightarrow \tan\theta = \infty \Rightarrow \left(\frac{dy}{dx}\right)_P = \infty$.

(ii) If the tangent at P is perpendicular to Y-axis or parallel to X-axis, then $\theta = 0 \Rightarrow \tan\theta = 0 \Rightarrow \left(\frac{dy}{dx}\right)_P = 0$.

(iii) Slope of the normal at $P = \dfrac{-1}{\text{Slope of the tangent at } P}$

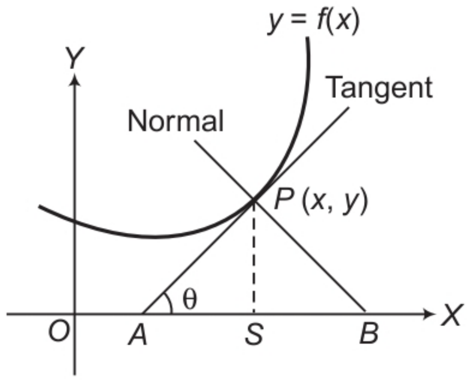
$$= \frac{-1}{\left(\frac{dy}{dx}\right)_P} = -\left(\frac{dx}{dy}\right)_P$$

(iv) If $\left(\frac{dy}{dx}\right)_P = 0$, then normal at (x, y) is parallel to Y-axis and perpendicular to X-axis.

(v) If $\left(\frac{dy}{dx}\right)_P = \infty$, then normal at (x, y) is parallel to X-axis and perpendicular to Y-axis.

Equation of Tangents and Normals

The derivative of the curve $y = f(x)$ is $f'(x)$ which represents the slope of tangent and equation of the tangent to the curve at P is $Y - y = \frac{dy}{dx}(X - x)$, where (x, y) is an arbitrary point on the tangent.



The equation of normal at (x, y) to the curve is

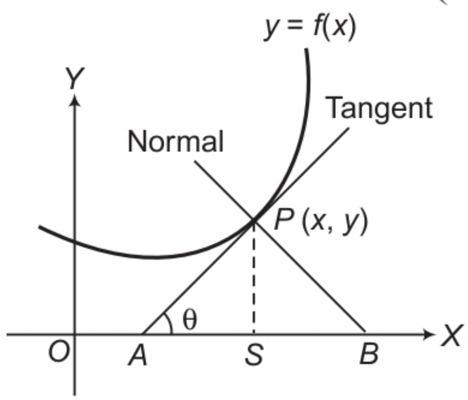
$$Y - y = -\frac{dx}{dy}(X - x)$$

(i) If $\left(\frac{dy}{dx}\right)_{(x,\, y)} = 0$, then the equations of the tangent and normal at (x, y) are $(Y - y) = 0$ and $(X - x) = 0$, respectively.

(ii) If $\left(\frac{dy}{dx}\right)_{(x,\, y)} = \pm\infty$, then the equation of the tangent and normal at (x, y) are $(X - x) = 0$ and $(Y - y) = 0$, respectively.

Length of Tangent and Normal

(i) Length of tangent, $PA = y \operatorname{cosec}\theta = \dfrac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\left(\frac{dy}{dx}\right)}$



(ii) Length of normal, $PB = y\sec\theta = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

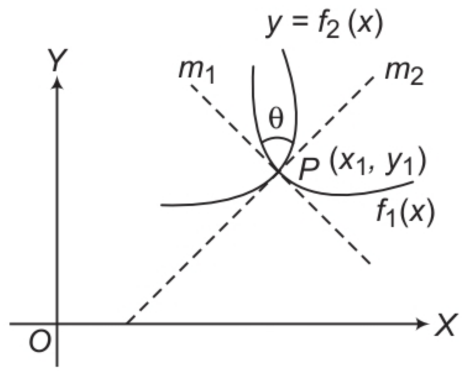
(iii) Length of subtangent, $AS = y\cot\theta = \dfrac{y}{(dy/dx)}$

(iv) Length of subnormal, $BS = y\tan\theta = y\left(\frac{dy}{dx}\right)$

Angle of Intersection of Two Curves

Let $y = f_1(x)$ and $y = f_2(x)$ be the two curves, meeting at some point $P(x_1, y_1)$, then

The angle between the two curves at $P(x_1, y_1)$ = the angle between the tangents to the curves at $P(x_1, y_1)$.



The other angle between the tangents is $(180 - \theta)$. Generally, the smaller of these two angles is taken to be the angle of intersection.

∴ The angle of intersection of two curves is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

where, $m_1 = \left(\frac{df_1}{dx}\right)_{(x_1, y_1)}$ and $m_2 = \left(\frac{df_2}{dx}\right)_{(x_1, y_1)}$

(i) If $\theta = \frac{\pi}{2}$, $m_1 m_2 = -1 \Rightarrow \left(\frac{df_1}{dx}\right)_{(x_1, y_1)} \left(\frac{df_2}{dx}\right)_{(x_1, y_1)} = -1$

such curves are called **orthogonal curves**.

(ii) If $\theta = 0$, $m_1 = m_2 \Rightarrow \left(\frac{df_1}{dx}\right)_{(x_1, y_1)} = \left(\frac{df_2}{dx}\right)_{(x_1, y_1)}$

such curves are tangential at (x_1, y_1).

Rolle's Theorem

Let f be a real function defined in the closed interval $[a, b]$, such that

(i) f is continuous in the closed interval $[a, b]$.

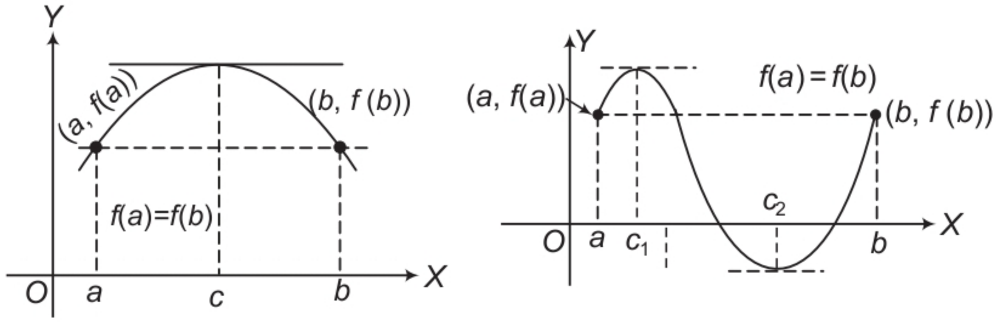
(ii) $f(x)$ is differentiable in the open interval (a, b).

(iii) $f(a) = f(b)$

Then, there is some point c in the open interval (a, b), such that $f'(c) = 0$.

Geometrically

Under the assumptions of Rolle's theorem, the graph of $f(x)$ starts at point $(a, f(a))$ and ends at point $(b, f(b))$ as shown in figures.



The conclusion is that there is atleast one point c between a and b, such that the tangent to the graph at $(c, f(c))$ is parallel to the X-axis.

Algebraic Interpretation of Rolle's Theorem

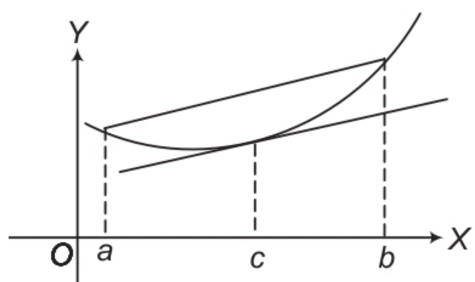
Between any two roots of a polynomial $f(x)$, there is always a root of its derivative $f'(x)$.

Lagrange's Mean Value Theorem

Let f be a real function, continuous on the closed interval $[a, b]$ and differentiable in the open interval (a, b). Then, there is atleast one point c in the open interval (a, b), such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrically For any chord of the curve $y = f(x)$, there is a point on the graph, where the tangent is parallel to this chord.



Remarks In the particular case, when $f(a) = f(b)$,

the expression $\frac{f(b) - f(a)}{b - a}$ becomes zero,

i.e. when $f(a) = f(b), f'(c) = 0$ for some c in (a, b), Thus, the Rolle's theorem becomes a particular case of the Lagrange's mean value theorem.

Approximations and Errors

1. Let $y = f(x)$ be a given function and Δx denotes a small increment in x, corresponding which y increases by Δy. Then, for small increments, we assume that

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} \qquad \text{[symbol} \approx \text{ stands for "approximately equal to"]}$$

$$\therefore \quad \Delta y = \frac{dy}{dx} \Delta x$$

For approximations of y, $\Delta y \approx dy$

Then, $dy = \left(\frac{dy}{dx}\right) \Delta x$

Thus, $y + \Delta y = f(x + \Delta x) = f(x) + \left(\frac{dy}{dx}\right) \Delta x$

2. Let Δx be the **error** in the measurement of independent variable x and Δy is corresponding error in the measurement of dependent variable y.

Then, $$\Delta y = \left(\frac{dy}{dx}\right) \Delta x$$

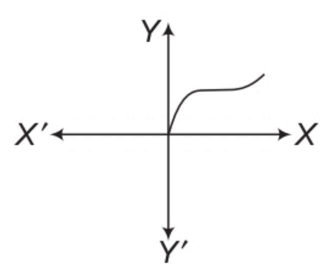
Δy = Absolute error in measurement of y

$\frac{\Delta y}{y}$ = Relative error in measurement of y

$\frac{\Delta y}{y} \times 100$ = Percentage error in measurement of y

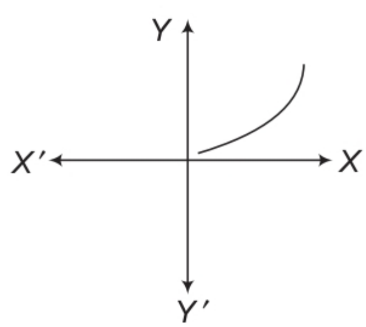
Increasing Function
(Non-decreasing Function)

A function f is called an increasing function in domain D, if $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2), \forall \; x_1, x_2 \in D$.



Strictly Increasing Function

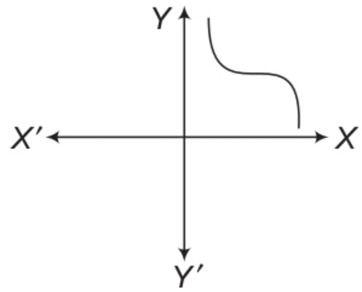
$f(x)$ is said to be strictly increasing in D, if for every $x_1, x_2 \in D$; $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.



Decreasing Function (Non-increasing Function)

A function f is called a decreasing function in domain D,

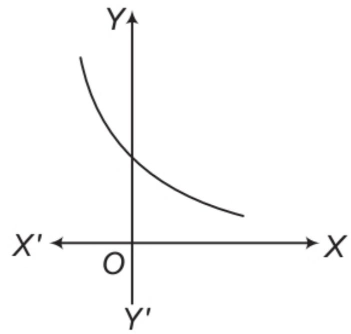
if $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_1), \forall\, x_1, x_2 \in D$.



Strictly Decreasing Function

$f(x)$ is said to be strictly decreasing in D,

if for every $x_1, x_2 \in D, x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.



Important Points to be Remembered

(i) A function $f(x)$ is said to be increasing (decreasing) at point x_0, if there is an interval $(x_0 - h, x_0 + h)$ containing x_0, such that $f(x)$ is increasing (decreasing) on $(x_0 - h, x_0 + h)$.

(ii) A function $f(x)$ is said to be increasing on $[a, b]$, if it is increasing on (a, b) and it is also increasing at $x = a$ and $x = b$.

(iii) Let f be a differentiable real function defined on an open interval (a, b).

(a) If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is strictly increasing on (a, b).

(b) If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is strictly decreasing on (a, b).

(iv) Let f be a function defined on (a, b).

(a) If $f'(x) > 0$ for all $x \in (a, b)$ except for a finite number of points, where $f'(x) = 0$, then $f(x)$ is increasing on (a, b).

(b) If $f'(x) < 0$ for all $x \in (a, b)$ except for a finite number of points, where $f'(x) = 0$, then $f(x)$ is decreasing on (a, b).

Monotonic Function

If a function is either increasing or decreasing on an interval (a, b), then it is said to be a monotonic function.

Note If a function is increasing in some interval I_1 and decreasing in some interval I_2, then that function is not monotonic function.

Properties of Monotonic Functions

(i) If $f(x)$ is strictly increasing (decreasing) function on an interval $[a, b]$, then f^{-1} exist and also a strictly increasing (decreasing) function.

(ii) If $f(x)$ and $g(x)$ are strictly increasing (or decreasing) function on $[a, b]$, then $gof(x)$ and $fog(x)$ (provided they exists) is strictly increasing function on $[a, b]$.

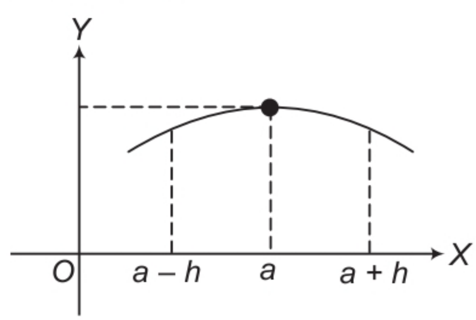
(iii) If one of the two functions $f(x)$ and $g(x)$ is strictly increasing and other a strictly decreasing, then $gof(x)$ and $fog(x)$ (provided they exists) is strictly decreasing on $[a, b]$.

(iv) If $f(x)$ is continuous on $[a, b]$, and differentiable on (a, b) such that $(f'(c) > 0)$ for each $c \in (a, b)$ is strictly increasing function on $[a, b]$.

(v) If $f(x)$ is continuous on $[a, b]$ such that $f'(c) < 0$ for each $c \in (a, b)$, then $f(x)$ is strictly decreasing function on $[a, b]$.

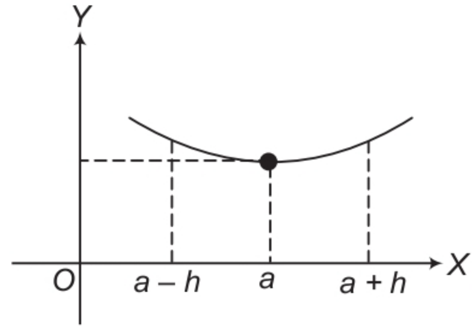
Maxima and Minima of Functions

Local Maximum (Maxima) A function $y = f(x)$ is said to have a **local maximum** at a point $x = a$. If $f(x) \leq f(a)$ for all $x \in (a - h, a + h)$, where h is very small positive quantity.



The point $x = a$ is called **a point of local maximum** of the function $f(x)$ and $f(a)$ is known **as the local maximum value** of $f(x)$ at $x = a$.

Local Minimum (Minima) A function $y = f(x)$ is said to have a **local minimum** at a point $x = a$, if $f(x) \geq f(a)$ for all $x \in (a - h, a + h)$, where h is very small positive quantity.



The point $x = a$ is called a **point of local minimum** of the function $f(x)$ and $f(a)$ is known as the **local minimum value** of $f(x)$ at $x = a$.

Note **Extreme value** A function $f(x)$ is said to have an extreme value in domain, if there exists a point c in interval such that $f(c)$ is either a local maximum value or local minimum value in the interval.

Properties of Maxima and Minima

(i) If $f(x)$ is continuous function in its domain, then atleast one maxima and one minima must lie between two different values of x on which functional values are equal.

(ii) Maxima and minima occur alternately, *i.e.*, between two maxima there is one minima and *vice-versa*.

(iii) If $f(x) \to \infty$ as $x \to a$ or b and $f'(x) = 0$ only for one value of x (say c) between a and b, then $f(c)$ is necessarily the minimum and the least value.

(iv) If $f(x) \to -\infty$ as $x \to a$ or b and $f'(c) = 0$ only for one value of x (say c) between a and b, then $f(c)$ is necessarily the maximum and the greatest value.

Critical Points of a Function

Points where a function $f(x)$ is not differentiable and points where its derivative (differentiable coefficient) is zero are called the critical points of the function $f(x)$.

Maximum and minimum values of a function $f(x)$ can occur only at critical points. However, this does not mean that the function will have maximum or minimum values at all critical points. Thus, the points where maximum or minimum value occurs are necessarily critical points but a function may or may not have maximum or minimum value at a critical point.

Important Points to be Remembered

(i) If $f(x)$ be a differentiable functions, then $f'(x)$ vanishes at every local maximum and at every local minimum.

(ii) The converse of above is not true, i.e. every point at which $f'(x)$ vanishes need not be a local maximum or minimum. e.g. if $f(x) = x^3$, then $f'(0) = 0$, but at $x = 0$ the function has neither maxima nor minima. In general these points are **point of inflection**.

(iii) A function may attain an extreme value at a point without being derivable at that point. e.g. $f(x) = |x|$ has a minima at $x = 0$ but $f'(0)$ does not exist.

(iv) A function $f(x)$ can has several local maximum and local minimum values in an interval. Thus, the maximum and minimum values of $f(x)$ defined above are not necessarily the greatest and the least values of $f(x)$ in a given interval.

(v) A local value at some point may even be greater than a local values at some other point.

Methods to Find a Local Maximum and Local Minimum

1. First Derivative Test

Let $f(x)$ be a differentiable function on an interval I and $a \in I$. Then,

(i) Point a is a local maximum of $f(x)$, if

(a) $f'(a) = 0$

(b) $f'(x) > 0$, if $x \in (a - h, a)$ and $f'(x) < 0$, if $x \in (a, a + h)$, where h is a small positive quantity.

(ii) Point a is a local minimum of $f(x)$, if

(a) $f'(a) = 0$

(b) $f'(a) < 0$, if $x \in (a - h, a)$ and $f'(x) > 0$, if $x \in (a, a + h)$, where h is a small positive quantity.

(iii) If $f'(a) = 0$ but $f'(x)$ does not changes sign in $(a - h, a + h)$, for any positive quantity h, then $x = a$ is neither a point of local minimum nor a point of local maximum.

2. Second Derivative Test

Let $f(x)$ be a differentiable function on an interval I. Let $a \in I$ is such that $f''(x)$ is continuous at $x = a$. Then,

(i) $x = a$ is a point of local maximum, if $f'(a) = 0$ and $f''(a) < 0$.

(ii) $x = a$ is a point of local minimum, if $f'(a) = 0$ and $f''(a) > 0$.

(iii) If $f'(a) = f''(a) = 0$, but $f'''(a) \neq 0$, if exists, then $x = a$ is neither a point of local maximum nor a point of local minimum and is called **point of inflection**.

(iv) If $f'(a) = f''(a) = f'''(a) = 0$ and $f^{iv}(a) < 0$, then it is a local maximum. And if $f^{iv}(a) > 0$, then it is a local minimum.

3. *n*th Derivative Test

Let f be a differentiable function on an interval I and let a be an interior point of I such that

$f'(a) = f''(a) = f'''(a) = \ldots = f^{n-1}(a) = 0$ and $f^n(a)$ exists and is non-zero.

(i) If n is even and $f^n(a) < 0 \Rightarrow x = a$ is a point of local maximum.

(ii) If n is even and $f^n(a) > 0 \Rightarrow x = a$ is a point of local minimum.

(iii) If n is odd, then $x = a$ is neither a point of local maximum nor a point of local minimum.

Concept of Global Maximum/Minimum

Let $y = f(x)$ be a given function with domain D.

Let $[a, b] \subseteq D$, then global maximum/minimum of $f(x)$ in $[a, b]$ is basically the greatest/least value of $f(x)$ in $[a, b]$.

Global maxima/minima in $[a, b]$ would always occur at critical points of $f(x)$ with in $[a, b]$ or at end points of the interval.

Global Maximum/Minimum in [*a*, *b*]

In order to find the global maximum and minimum of $f(x)$ in $[a, b]$, find out all critical points $c_1, c_2, \ldots, c_n$ of $f(x)$ in $[a, b]$ (*i.e.*, all points at which $f'(x) = 0$) or $f'(x)$ not exists and let $f(c_1), f(c_2), \ldots, f(c_n)$ be the values of the function at these points.

Then, $M_1 \rightarrow$ Global maxima or greatest value.

and $M_2 \rightarrow$ Global minima or least value.

where $M_1 = \max\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

and $M_2 = \min\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

Then, M_1 is the greatest value or global maxima in $[a, b]$ and M_2 is the least value or global minima in $[a, b]$.

Important Points to be Remembered

(i) **To Find Range of a Continuous Function** Let $f(x)$ be a continuous function on $[a,b]$, such that its least value in $[a,b]$ is m and the greatest value in $[a,b]$ is M. Then, range of value of $f(x)$ for $x \in [a,b]$ is $[m, M]$.

(ii) **To Check for the Injectivity of a Function** A strictly monotonic function is always one-one (injective).

Hence, a function $f(x)$ is one-one in the interval $[a,b]$, if $f'(x) > 0, \forall x \in [a,b]$ or $f'(x) < 0, \forall x \in [a,b]$.

(iii) The points at which a function attains either the local maximum value or local minimum value are known as the **extreme points** or **turning points** and both local maximum and local minimum values are called the extreme values of $f(x)$.

Thus, a function attains an extreme value at $x = a$, if $f(a)$ is either a local maximum value or a local minimum value. Consequently at an extreme point 'a', $f(x) - f(a)$ keeps the same sign for all values of x in a deleted nbd of a.

(iv) A necessary condition for $f(a)$ to be an extreme value of a function $f(x)$ is that $f'(a) = 0$ in case it exists. It is not sufficient. i.e. $f'(a) = 0$ does not necessarily imply that $x = a$ is an extreme point. There are functions for which the derivatives vanish at a point but do not have an extreme value. e.g. the function $f(x) = x^3, f'(0) = 0$ but at $x = 0$ the function does not attain an extreme value.

(v) Geometrically the above condition means that the tangent to the curve $y = f(x)$ at a point where the ordinate is maximum or minimum is parallel to the X-axis.

(vi) All x, for which $f'(x) = 0$, do not give us the extreme values. The values of x for which $f'(x) = 0$ are called **stationary values** or **critical values** of x and the corresponding values of $f(x)$ are called stationary or turning values of $f(x)$.