System of Particles and Rotational Motion

1. Rigid Body

A body is said to be a rigid body when it has a perfectly definite shape and size.

e.g. A wheel can be considered as rigid body by ignoring a little change in its shape.

2. Rotational Motion (Fixed Axis of Rotation)

In pure rotational motion, every particle of the rigid body moves in circles of different radii about a fixed line, which is known as axis of rotation.

e.g. A potter's wheel, a merry-go-round etc.

Note *In precession, one end of axis of rotation is fixed and other end rotates about a circular path.*

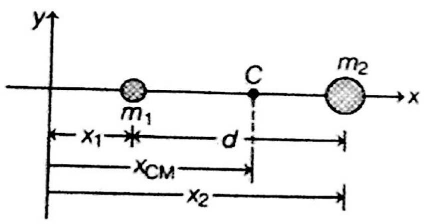
3. Centre of Mass

A point at which the entire mass of the body or system of bodies is supposed to be concentrated is known as the centre of mass.

For a System of two Particles The centre of mass of the system at point which is a distance x_{CM} from origin is given by

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

According to the figure,



For a System of *n* Particles Suppose a system having masses $m_1, m_2, m_3, \ldots, m_n$ occupying *x*-coordinates $x_1, x_2, x_3, \ldots, x_n$.

i.e. x_{CM} = *x*-coordinates of centre of mass of system

$$= \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots + m_n x_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

$$\text{Centre of mass, } x_{CM} = \frac{\sum_{i=1}^{n} m_i x_i}{\Sigma m_i}$$

Note (i) *If particles are distributed in three-dimensional space, then the centre of mass has 3-coordinates, which are*

$$x_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i; \quad y_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i;$$

$$z_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

where, $M = m_1 + m_2 + m_3 + \ldots = \sum_{i=1}^{n} m_i$ *is the total mass of the system. The index i runs from 1 to n,* m_i *is the mass of the ith particle and the position of the ith particle is given by* (x_i, y_i, z_i).

(ii) **Relation between position vectors of particles and centre of mass** $\mathbf{R} = \dfrac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{m}$

where $\mathbf{r}_i = (x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}})$ *be the position vector of the ith particle and* $\mathbf{R}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$ *be the position vector of the centre of mass.*

4. Centre of Mass of Rigid Continuous Bodies

For a real body which is a continuous distribution of matter, point masses are then differential mass elements *dm* and centre of mass is given as

$$x_{CM} = \frac{1}{M}\int x\,dm, \quad y_{CM} = \frac{1}{M}\int y\,dm$$

and $$z_{CM} = \frac{1}{M}\int z\,dm$$

where, *M* is total mass of that real body.

If we choose the origin of coordinates axes at centre of mass, then

$$\int x\,dm = \int y\,dm = \int z\,dm = 0$$

5. Motion of Centre of Mass

Velocity about centre of mass, $\mathbf{v}_{CM} = \dfrac{\sum_{i=1}^{n} m_i \mathbf{v}_i}{M}$

where, $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$, rate of change of position vector is velocity.

Acceleration about centre of mass, $\mathbf{a}_{CM} = \dfrac{\sum_{i=1}^{n} m_i \mathbf{a}_i}{M}$

But, $m_i\mathbf{a}_i$ is the resultant force on the *i*th particle, so

$$M\mathbf{a}_{CM} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots + \mathbf{F}_n$$

$$M\mathbf{a}_{CM} = \mathbf{F}_{net}$$

6. Linear Momentum of a System of Particle

The total momentum of a system of particles is equal to the product of the total mass and velocity of its centre of mass.

∴ Total linear momentum $\mathbf{p} = M\mathbf{v}_{CM}$

7. Moment of Force (Torque)

Torque is also known as moment of force. We can define the torque for a particle about a point as the vector product of position vector of the point where the force acts and with the force itself. Let us consider a particle *P* and force **F** acting on it.

$$\text{Torque, } \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

The magnitude of torque $|\boldsymbol{\tau}|$ is

$$\tau = rF\sin\theta = Fr\sin\theta$$

$$\tau = Fr_{\perp}$$

Here $r_{\perp}$ is the perpendicular distance of the line of action of **F** from the origin.

8. Angular Momentum of a Particle

Angular momentum (**L**) can be defined as moment of linear momentum about a point.

The angular momentum of a particle of mass m moving with velocity $\mathbf{v}$ (having a linear momentum, $\mathbf{p} = m\mathbf{v}$) about a point O is defined by the following vector product, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$

or Angular momentum, $\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$

- Angular momentum will be zero ($L = 0$), if
  $p = 0$ or $r = 0$ or $\theta = 0°, 180°$
- Angular momentum is a vector quantity and its direction could be found out with the help of cross-product.
- The SI unit of angular momentum is kg-m^{2}s^{-1}.

9. Relation between Torque (τ) and Angular Momentum (L)

$$\frac{dL}{dt} = \tau$$

Above equation gives Newton's second law of motion in angular form, i.e. rate of change of angular momentum is equal to the torque applied.

10. Couple

A pair of equal and opposite forces with parallel lines of action are known as a couple. A couple produces rotation without translation.

11. Principle of Momentum

When an object is in rotational equilibrium, then algebraic sum of all torques acting on it is zero. Clockwise torques are taken negative and anti-clockwise torques are taken positive.

12. Centre of Gravity

If a body is supported on a point such that total gravitational torque about this point is zero, then this point is called centre of gravity of the body.

13. Moment of Inertia

For a rotating body, its moment of inertia is

$$I = \sum_{i=1}^{n} m_i r_i^2$$

Hence, moment of inertia of a rigid body about a fixed axis is defined as the sum of the products of the masses of the particles constituting the body and the squares of their respective distances from the axis of rotation.

Moment of inertia, $I = mR^2$

The SI unit of moment of inertia is kg-m^2 and its dimensions are $[ML^2]$.

14. Relation between Angular Momentum and Moment of Inertia

For a rigid body (about an fixed axis),

$$L = \text{sum of angular momenta of all particles}$$
$$= m_1 v_1 r_1 + m_2 v_2 r_2 + \ldots\ldots$$
$$= m_1 r_1^2 \omega + m_2 r_2^2 \omega + \ldots\ldots \qquad [\because v = \omega r]$$
$$= (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots\ldots)\omega$$

i.e. $L = I\omega$

where, $I = m_1 r_1^2 + m_2 r_2^2 + \cdots$, is called moment of inertia which is discussed in coming sections.

15. Radius of Gyration

The radius of gyration of a body about an axis may be defined as the distance from the axis of a mass point whose mass is equal to the mass of whole body and whose moment of inertia is equal to the moment of inertia of the body about the axis.

When square of radius of gyration is multiplied with the mass of the body, then it gives the moment of inertia of the body about the given axis, i.e. $I = MK^2$. Radius of gyration, $K = \sqrt{\frac{I}{M}}$

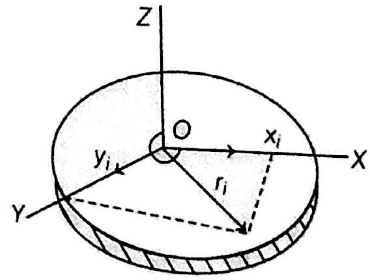
where, K is radius of gyration of the body.

For rotating body, $K = \sqrt{\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n}}$

Hence, radius of gyration of a rotating body about a given axis is equal to root mean square distance of constituent particles from the given axis.

16. Theorem of Perpendicular Axes

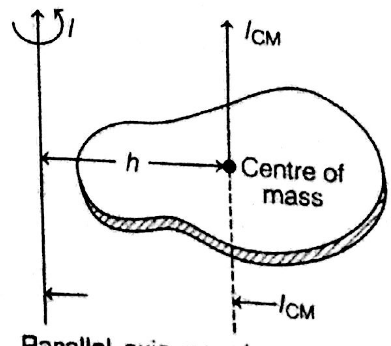
It states that the moment of inertia of a planar body (lamina) about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body $I_{ZZ} = I_{XX} + I_{YY}$



Perpendicular axis showing a planar body where x and y are perpendicular axis and z-axis is perpendicular to the plane.

17. Theorem of Parallel Axes

It states that the moment of inertia of a body about an axis is equal to sum of moments of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of distance between the two parallel axes.



Parallel axis passing through the centre of mass

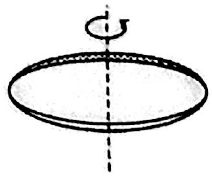
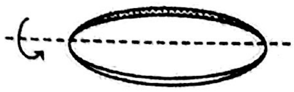
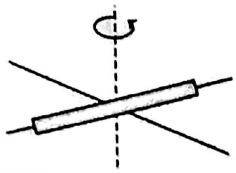
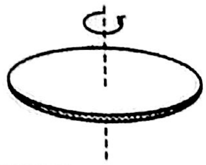
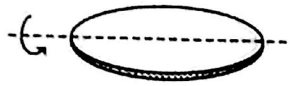
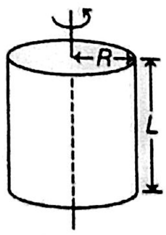
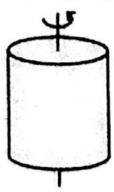
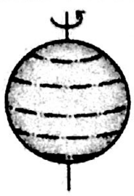
Moment of inertia of lamina about an axis is I and by the theorem, $I = I_{CM} + Mh^2$

where, M = mass of lamina

h = distance between two parallel axes.

Note *Theorem of parallel axes is applicable only for 2D bodies.*

Moment of Inertia in Some Standard Cases

Body	Axis of Rotation	Figure	Moment of Inertia	K	K^2/R^2
Thin circular ring, radius R	About an axis passing through CG and perpendicular to its plane		MR^2	R	1
Thin circular ring, radius R	About its diameter		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$	$\frac{1}{2}$
Thin rod, length L	Perpendicular to rod, at mid-point		$\frac{1}{12}ML^2$	$\frac{L}{\sqrt{12}}$	
Circular disc, radius R	Perpendicular to disc at centre		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$	$\frac{1}{2}$
Circular disc, radius R	Diameter		$\frac{1}{4}MR^2$	$\frac{R}{2}$	$\frac{1}{4}$
Hollow cylinder, radius R	About its own axis		MR^2	R	1
Solid cylinder, radius R	About its own axis		$\frac{MR^2}{2}$	$\frac{R}{\sqrt{2}}$	$\frac{1}{2}$
Solid sphere, radius R	About its diametric axis		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$	$\frac{2}{5}$

18. Kinetic Equations of Rotational Motion about a Fixed Axis

The three equations of rotational motion are

(i) $\omega = \omega_0 + \alpha t$ (ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ (iii) $\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0)$

where, θ_0 and ω_0 are the initial angular displacement and initial angular velocity of the body respectively, α is uniform angular acceleration.

19. Dynamics of Rotational Motion about a Fixed Axis

From the given table, we compare linear motion and rotational motion about a fixed axis, i.e. z-axis.

Comparison of Translational and Rotational Motion

Pure Translational	Pure Rotational
Position, x	Angular position, θ
Velocity, $v = \frac{dx}{dt}$	Angular velocity, $\omega = \frac{d\theta}{dt}$
Acceleration, $a = \frac{dv}{dt}$	Angular acceleration, $\alpha = \frac{d\omega}{dt}$
Mass, m	Rotational inertia, I
Newton's second law, $F = ma$	Newton's second law, $\tau = I\alpha$
Work done, $W = \int F\,dx$	Work done, $W = \int \tau\,d\theta$
Kinetic energy, $K = \frac{1}{2}mv^2$	Kinetic energy, $K = \frac{1}{2}I\omega^2$
Power, $P = Fv$	Power, $P = \tau\omega$
Linear momentum, $p = mv$	Angular momentum, $L = I\omega$

20. Relation between Torque and Moment of Inertia

$$\text{Torque, } \tau = I\alpha = dL/dt$$

It is similar to the expression of Newton's second law for translational motion with constant mass.

Note *If $\tau = 0$, then $L =$ constant, this is called conservation of angular momentum.*

21. Rolling Motion

The rolling motion can be regarded as the combination of pure rotation and pure translation. It is also one of the most common motions observed in daily life.

22. Kinetic Energy of a Rolling Body

The kinetic energy of a body rolling without slipping is the sum of kinetic energies of translational and rotational motion.

$$\therefore \quad KE_{rolling} = KE_{rotation} + KE_{translation} = \frac{1}{2}I\omega^2 + \frac{1}{2}mv_{CM}^2$$

$$= \frac{1}{2}mv_{CM}^2\left[1 + \frac{K^2}{R^2}\right]$$

$$[\because v_{CM} = R\omega \text{ and } I = mK^2]$$