Probability

[TOPIC 1] Conditional Probability and Independent Events

1.1 Experiment and Events

1. **Experiment** An operation which can produce some well defined outcomes is called an experiment.
2. **Random Experiment** An experiment is called random experiment, if it satisfies following conditions
   (i) It has more than one possible outcome.
   (ii) It is not possible to predict the outcome in advance.
3. **Outcomes and Sample Space** A possible result of a random experiment is called its outcome. The set of all possible outcomes of a random experiment is called its sample space.
4. **Trial** The number of times the experiment is repeated is called the number of **trials**.
5. **Event** A subset of the sample space associated with a random experiment is called an event or a **case**.
   e.g. In tossing a coin, getting either head or tail is an event.
6. **Impossible and Sure Events** The empty set ϕ and the sample space S describe events. The empty set ϕ is called an impossible event and whole sample space S is called the sure event.
7. **Simple event** If an event has only one sample point of a sample space, then it is called a simple or elementary event.
8. **Compound event** If an event has more than one sample point, then it is called a compound event.
9. **Equally Likely Events** The given events are said to be equally likely, if none of them is expected to occur in preference to the other.
   e.g. In throwing an unbiased die, all the six faces are equally likely to come.
10. **Mutually Exclusive Events** A set of events is said to be mutually exclusive, if the happening of one excludes the happening of the other, i.e. if no two or more of them can occur simultaneously, if A and B are mutually exclusive, then $(A \cap B) = \phi$.
    e.g. In throwing a die, all the 6 faces numbered 1 to 6 are mutually exclusive, since if anyone of these faces comes, then the possibility of others in the same trial is ruled out.
11. **Exhaustive Events** A set of events is said to be exhaustive, if the performance of the experiment always results in the occurrence of atleast one of them.
    If $E_1, E_2, \ldots, E_n$ are exhaustive events, then $E_1 \cup E_2 \cup \ldots \cup E_n = S$.
    e.g. When a die is thrown, events {1, 2}, {3, 4}, {5, 6} form an exhaustive set of events.

12. **Complement of an Event** Let A be an event in a sample space S, then complement of A is the set of all sample points of the sample space, which are not in A and it is denoted by A' or $\overline{A}$, i.e. $A' = \{n : n \in S, n \notin A\}$.

13. **Probability of an Event** If there are n elementary equally likely events associated with a random experiment and m of them are favourable to the happening of an event A, then the **probability** of happening of A is given by

$$P(A) = \frac{\text{Number of elementary events favourable to event } A}{\text{Total number of elementary events to the experiment}} = \frac{n(A)}{n(S)} = \frac{m}{n}$$

NOTE (i) $0 \le P(A) \le 1$

(ii) Probability of impossible event is zero.

(iii) Probability of certain event (sure event) is 1.

(iv) $P(A \cup A') = P(S)$ (v) $P(A) + P(A') = 1$

(vi) $P(A \cap A') = P(\phi)$ (vii) $P(A')' = P(A)$

(viii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

1.2. Conditional Probability

Let E and F be two events associated with the same sample space of a random experiment. Then, probability of occurrence of event E, when the event F has already occurred, is called conditional probability of event E over F and it is denoted by $P(E/F)$.

$$P(E/F) = \frac{P(E \cap F)}{P(F)}, \text{ where } P(F) \neq 0.$$

Similarly, conditional probability of event F over E is given as

$$P(F/E) = \frac{P(F \cap E)}{P(E)}, \text{ where } P(E) \neq 0.$$

Properties of Conditional Probability

If E and F are two events of sample space S and G is an event of S such that $P(G) \neq 0$, then

(i) $P((E \cup F)/G) = P(E/G) + P(F/G) - P((E \cap F)/G)$

(ii) $P((E \cup F)/G) = P(E/G) + P(F/G)$, if E and F are disjoint events.

(iii) $P(E'/G) = 1 - P(E/G)$

(iv) $P(S/G) = P(G/G) = 1$

1.3 Multiplication Theorem

If E and F are two events associated with a sample space S, then the probability of simultaneous occurrence of the events E and F is

$$P(E \cap F) = P(E) \cdot P(F/E), \text{ where } P(E) \neq 0$$

or $P(E \cap F) = P(F) \cdot P(E/F)$, where $P(F) \neq 0$.

This result is known as multiplication rule of probability.

Multiplication Rule for More than Two Events

If E, F and G are three events of sample space, then

$$P(E \cap F \cap G) = P(E) \cdot P\left(\frac{F}{E}\right) \cdot P\left(\frac{G}{E \cap F}\right).$$

1.4 Independent Events

Two events E and F are said to be independent, if probability of occurrence or non-occurrence of one of the events is not affected by that of the other. For any two independent events E and F, we have the relation

(i) $P(E \cap F) = P(E) \cdot P(F)$

(ii) $P(F/E) = P(F)$, $P(E) \neq 0$

(iii) $P(E/F) = P(E)$, $P(F) \neq 0$

Also, their complements are independent events,

i.e. $P(\overline{E} \cap \overline{F}) = P(\overline{E}) \cdot P(\overline{F})$

NOTE If E and F are dependent events, then $P(E \cap F) \neq P(E) \cdot P(F)$

1.5 Mutual Independent

Three events E, F and G are said to be mutually independent, if

(i) $P(E \cap F) = P(E) \cdot P(F)$

(ii) $P(F \cap G) = P(F) \cdot P(G)$

(iii) $P(E \cap G) = P(E) \cdot P(G)$

(iv) $P(E \cap F \cap G) = P(E) \cdot P(F) \cdot P(G)$

If atleast one of the above is not true for three given events, then we say that the events are not mutually independent.

NOTE Independent and mutually exclusive events do not have same meaning.

[TOPIC 2] Baye's Theorem and Probability Distributions

2.1 Partition of a Sample Space

A set of events $E_1, E_2, \ldots, E_n$ is said to represent a partition of the sample space S, if it satisfies the following conditions:

(i) $E_i \cap E_j = \phi; i \neq j; i, j = 1, 2, \ldots, n$

(ii) $E_1 \cup E_2 \cup \ldots \cup E_n = S$

(iii) $P(E_i) > 0, \forall i = 1, 2, \ldots, n$

NOTE The partition of a sample space is not unique. There can be several partitions of the same sample space.

2.2 Theorem of Total Probability

Let events $E_1, E_2, \ldots, E_n$ form a partition of the sample space S of an experiment. If A is any event associated with sample space S, then

$$P(A) = P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + \ldots + P(E_n) \cdot P(A/E_n)$$

$$= \sum_{j=1}^{n} P(E_j) \cdot P(A/E_j).$$

2.3 Baye's Theorem

If $E_1, E_2, ..., E_n$ are n non-empty events which constitute a partition of sample space S, i.e. $E_1, E_2, ..., E_n$ are pairwise disjoint, $E_1 \cup E_2 \cup ... \cup E_n = S$ and $P(E_i) > 0$, for all $i = 1,2,...,n$. Also, let A be any event of non-zero probability, then

$$P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)} \quad \forall\ i = 1, 2, 3, ..., n$$

2.4 Random Variable

A random variable is a real valued function, whose domain is the sample space of a random experiment. Generally, it is denoted by capital letter X.

NOTE More than one random variables can be defined on the same sample space.

Probability Distribution of a Random Variable

The system in which the values of a random variable are given along with their corresponding probabilities is called probability distribution of the random variable. Let X be a random variable which can take n values $x_1, x_2, ..., x_n$. Let $p_1, p_2, ..., p_n$ be the respective probabilities. Then, a probability distribution table is given as follows:

X	x_1	x_2	x_3	...	x_n
$P(x)$	p_1	p_2	p_3	...	p_n

where $p_i > 0, i = 1, 2, ..., n$
and $p_1 + p_2 + p_3 + ... + p_n = 1$.

Mean, Variance and Standard Deviation

Let X be a random variable taking values $x_1, x_2, ..., x_n$ with probabilities $p_1, p_2, ..., p_n$ respectively. Then, mean of a random variable X is $\sum_{i=1}^{n} x_i \cdot p_i$. It is also called expectation of X or expected value of X and it is denoted by $E(x)$ or μ.

Thus, $E(X) = \mu = \sum_{i=1}^{n} x_i p_i$.

Variance, denoted by $V(X)$ or σ_x^2 is given by

$$\sigma^2 = \sum_{i=1}^{n} x_i^2 \cdot p(x_i) - \left(\sum_{i=1}^{n} x_i \cdot p(x_i)\right)^2$$

or $\quad \sigma^2 = E(X^2) - [E(X)]^2$

or $\ V(X) = E(X - \mu)^2$ or $\sum_{i=1}^{n} (x_i - \mu)^2 p(x_i)$

Standard deviation is given by, $\sigma_x = \sqrt{V(x)}$

$$= \sqrt{\sum_{i=1}^{n} (x_i - \mu)^2 p(x_i)}$$