Vector Algebra

[TOPIC 1] Algebra of Vectors

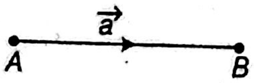
1.1 Scalars and Vectors

Those quantities which have only magnitude but no direction, are called **scalar quantities** or **scalars**. Those quantities which have magnitude as well as direction are called **vector quantities** or **vectors**.

Representation of Vector

A directed line segment has magnitude as well as direction, so it is called a vector and denoted as $\overrightarrow{AB}$ or simply as $\vec{a}$ and read as 'vector $\overrightarrow{AB}$' or 'vector $\vec{a}$'.

Here, the point A from where the vector $\overrightarrow{AB}$ starts is called its initial point and the point B where it ends is called its terminal point.



Magnitude of a Vector

The length of the vector $\overrightarrow{AB}$ or $\vec{a}$ is called magnitude of $\overrightarrow{AB}$ or $\vec{a}$ and it is represented by $|\overrightarrow{AB}|$ or $|\vec{a}|$ or a.

NOTE Since, the length is never negative, so the notation $|\vec{a}|<0$ has no meaning.

Position Vector

Let $O(0,0,0)$ be the origin and P be a point in space having coordinates (x, y, z) with respect to the origin O. Then, the vector $\overrightarrow{OP}$ or $\vec{r}$ is called the position vector of the point P with respect to O.

The magnitude of $\overrightarrow{OP}$ or $\vec{r}$ is given by

$$|\overrightarrow{OP}|=|\vec{r}|=\sqrt{x^2+y^2+z^2}.$$

NOTE Generally, the position vectors of points A, B, C, etc. with respect to origin are denoted by $\vec{a}, \vec{b}, \vec{c}$ etc., respectively

1.2 Direction Cosines

If the angles α, β and γ made by the vector $\overrightarrow{OP}$ with the positive directions of the coordinate axes OX, OY and OZ respectively, then cosine values of these angles, i.e. $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are known as the direction cosines of $\overrightarrow{OP}$ and are generally denoted by the letters l, m and n, respectively, i.e. $l=\cos\alpha, m=\cos\beta, n=\cos\gamma$

From the figure,

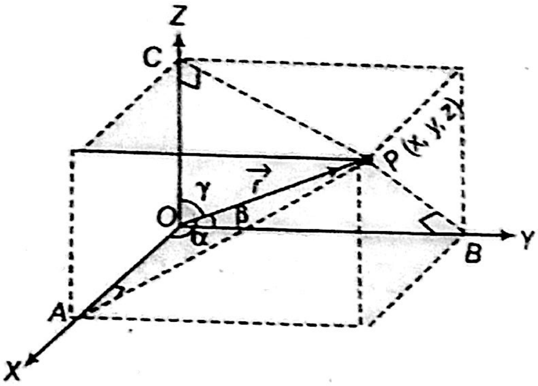
$$\cos\alpha=\frac{x}{|\vec{r}|}, \cos\beta=\frac{y}{|\vec{r}|} \text{ and } \cos\gamma=\frac{z}{|\vec{r}|}$$

$\therefore \quad l = \frac{x}{|\vec{r}|}, m = \frac{y}{|\vec{r}|}$ and $n = \frac{z}{|\vec{r}|}$

$\Rightarrow \quad x = l|\vec{r}|, y = m|\vec{r}|$ and $z = n|\vec{r}|$

Thus, the coordinates of the point P may be expressed as $(l|\vec{r}|, m|\vec{r}|, n|\vec{r}|)$

The numbers $l|\vec{r}|, m|\vec{r}|$ and $n|\vec{r}|$ are proportional to the direction cosines are called direction ratios of vector $\vec{r}$.



NOTE (i) In general, it may be noted that $l^2 + m^2 + n^2 = 1$ but $x^2 + y^2 + z^2 \neq 1$

(ii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then a_1, a_2 and a_3 are called direction ratios of $\vec{a}$.

(iii) If it is given that l, m and n are direction cosines of a vector, then $l\hat{i} + m\hat{j} + n\hat{k} = (\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}$ is the unit vector in the direction of that vector, where α, β and γ are the angles made by the vector with the positive direction of X, Y and Z-axes, respectively.

1.3 Types of Vectors

1. **Null vector or zero vector** A vector, whose initial and terminal points coincide, i.e. its magnitude is zero, is called a null vector and denoted as $\vec{0}$.

NOTE Zero vector cannot be assigned a definite direction as it has zero magnitude otherwise, it may be regarded as having any direction. The vectors $\vec{AA}$, $\vec{BB}$ represent the zero vector.

2. **Unit vector** A vector of unit length is called unit vector. The unit vector in the direction of $\vec{a}$ is $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

3. **Coinitial vectors** Two or more vectors having the same initial point are called coinitial vectors.

4. **Equal vectors** Two vectors are said to be equal, if they have equal magnitudes and same direction regardless of the position of their initial points.

NOTE If $\vec{a} = \vec{b}$, then $|\vec{a}| = |\vec{b}|$ but converse may not be true.

5. **Negative vector** A vector having the same magnitude but opposite in direction of the given vector, is called negative of the given vector. e.g. Vector $\vec{BA}$ is negative of the vector $\vec{AB}$ and written as $\vec{BA} = -\vec{AB}$.

6. **Collinear vectors** Two or more vectors are said to be collinear, if they are parallel to the same line, irrespective of their magnitudes and directions. e.g. $\vec{a}$ and $\vec{b}$ are collinear, when $\vec{a} = \lambda\vec{b}$, where λ is some scalar.

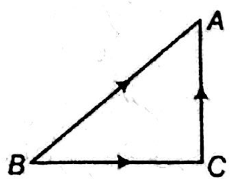
NOTE If the value of a vector depends only on its magnitude and direction and is independent of its position in the space, such vectors are called free vectors.

1.4 Addition of Vectors

1. Triangle law of vector addition

If two vectors are represented along two sides of a triangle taken in order, then their resultant is represented by its third side, i.e. in ΔABC, by triangle law of vector addition, we have

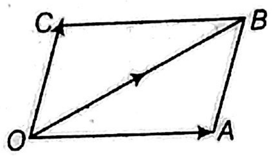
$$\vec{BC} + \vec{CA} = \vec{BA}.$$



NOTE The vector sum of three sides of a triangle taken in order is $\vec{0}$.

2. Parallelogram law of vector addition

If two vectors are represented along the two adjacent sides of a parallelogram, then their resultant is represented by the diagonal of the parallelogram througth the common point. In the parallelogram $OABC$, we have



$$\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB}.$$

NOTE Both laws of vector addition are equivalent to each other.

Properties of vector addition

(a) **Commutative** For vectors $\vec{a}$ and $\vec{b}$, we have $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

(b) **Associative** For vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, we have $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$.

(c) **Additive identity** For any vector $\vec{a}$, a zero vector $\vec{0}$ is its additive identity as $\vec{a} + \vec{0} = \vec{a}$.

(d) **Additive inverse** For a vector $\vec{a}$, a negative vector of $\vec{a}$ is its additive inverse as $\vec{a} + (-\vec{a}) = \vec{0}$.

NOTE The associative property of vector addition enables us to write the sum of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ as $\vec{a} + \vec{b} + \vec{c}$ without using brackets.

1.5 Multiplication of a Vector by a Scalar

Let $\vec{a}$ be a given vector and λ be a scalar, then multiplication of vector $\vec{a}$ by scalar λ, denoted as $\lambda\vec{a}$, is also a vector, collinear to the vector $\vec{a}$ whose magnitude is $|\lambda|$ times that of vector $\vec{a}$, i.e. $|\lambda\vec{a}| = |\lambda||\vec{a}|$ and its direction is same as $\vec{a}$, if $\lambda > 0$; opposite of $\vec{a}$, if $\lambda < 0$ and zero vector, if $\lambda = 0$.

NOTE For any scalar λ, $\lambda \cdot \vec{0} = \vec{0}$.

Properties of Scalar Multiplication

For vectors $\vec{a}$, $\vec{b}$ and scalars p, q, we have

(i) $p(\vec{a} + \vec{b}) = p\vec{a} + p\vec{b}$

(ii) $(p + q)\vec{a} = p\vec{a} + q\vec{a}$

(iii) $p(q\vec{a}) = (pq)\vec{a}$

NOTE To prove $\vec{a}$ is parallel to $\vec{b}$, we need to show that $\vec{a} = \lambda\vec{b}$, where λ is a scalar.

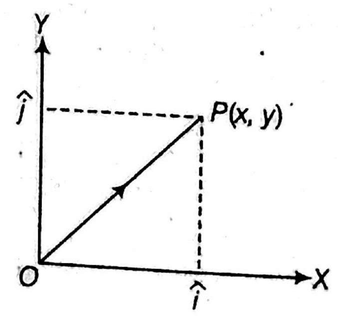
1.6 Components of a Vector

The position vector of any point $P(x, y, z)$ with reference to O is given by $\overrightarrow{OP}$ (or $\vec{r}$) $= x\hat{i} + y\hat{j} + z\hat{k}$, where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the X-axis, Y-axis and Z-axis, respectively. This form of any vector is called its **component form**.

Here, x, y and z are called the scalar components of $\vec{r}$ and $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the vector components of $\vec{r}$ along the respective axes.

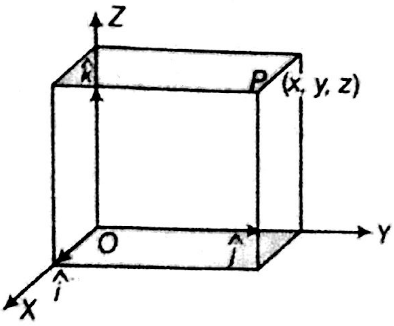
1. **Two dimensions** If a point P in a plane has coordinates (x, y), then $\overrightarrow{OP} = x\hat{i} + y\hat{j}$, where $\hat{i}$ and $\hat{j}$ are unit vectors along OX and OY-axes, respectively.

   Then, $|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$.



2. **Three dimensions** If a point P in a plane has coordinates (x, y, z), then $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$, where $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors along OX, OY and OZ-axes, respectively.

Then, $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$.



Important Facts

If we have two vectors $\vec{a}$ and $\vec{b}$, given in component form as $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and λ be any scalar, then

(i) $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

(ii) $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$

(iii) $\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

(iv) $\vec{a} = \vec{b}$ iff $a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$

(v) $\vec{a}$ and $\vec{b}$ are collinear (or parallel) iff

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \lambda \text{ (non-zero scalar)}$$

1.7 Vector Joining of Two Points

If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points, then the vector joining P_1 and P_2 is the vector $\overrightarrow{P_1P_2}$.

$$\overrightarrow{OP_2} - \overrightarrow{OP_1} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

i.e. $\overrightarrow{P_1P_2} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

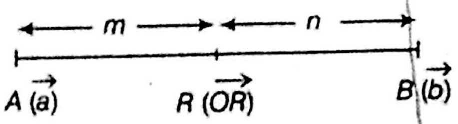
and $|\overrightarrow{P_1P_2}|$ the distance between two points,

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

1.8 Section Formula

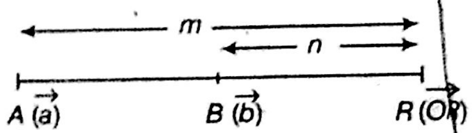
Position vector $\overrightarrow{OR}$ of point R which divides the line segment joining the points A and B with position vectors $\vec{a}$ and $\vec{b}$ respectively, internally in the ratio $m : n$ is given by

$$\overrightarrow{OR} = \frac{m\vec{b} + n\vec{a}}{m + n}.$$



For external division,

$$\overrightarrow{OR} = \frac{m\vec{b} - n\vec{a}}{m - n}.$$



NOTE Position vector $\overrightarrow{OR}$ of mid-point R of the line segment joining the points $A(\vec{a})$ and $B(\vec{b})$ is given by $\overrightarrow{OR} = \frac{\vec{a} + \vec{b}}{2}$.

[Topic 2] Product of Two Vectors and Scalar Triple Product

2.1 Dot (or scalar) Product of Two Vectors

If θ is the angle between two non-zero vectors $\vec{a}$ and $\vec{b}$, then the scalar or dot product, denoted by $\vec{a}\cdot\vec{b}$, is given by $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where $0 \le \theta \le \pi$.

NOTE (i) $\vec{a}\cdot\vec{b}$ is a real number.

(ii) If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then θ is not defined and in this case, we define $\vec{a}\cdot\vec{b}=0$

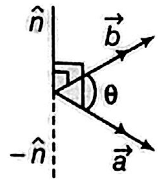
Properties of dot product of two vectors

1. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ [i.e. dot product is commutative]
2. $(\vec{a}\cdot\vec{0}) = 0$
3. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ [distributive property]
4. If $\vec{a}$ and $\vec{b}$ are perpendicular to each other, then $\vec{a}\cdot\vec{b}=0$, converse is also true.
5. Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ and projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.
6. Angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$ or $\theta = \cos^{-1}\left[\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}\right]$
7. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$
8. $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ or $|\hat{i}|^2 = |\hat{j}|^2 = |\hat{k}|^2 = 1$
9. $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
10. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

11. $(\lambda \cdot \vec{a}) \cdot \vec{b} = \lambda(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\lambda \cdot \vec{b})$, where λ is any scalar.

12. If $\theta = 0$, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$; if $\theta = \pi$, then $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$.

2.2 Vector (or Cross) Product of Two Vectors



If θ is the angle between two non-zero non-parallel vectors $\vec{a}$ and $\vec{b}$, then the cross product of vectors, denoted by $\vec{a} \times \vec{b}$, is given by

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \, \hat{n}$, such that $0 \le \theta \le \pi$

where, $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

NOTE (i) $\vec{a} \times \vec{b}$ is a vector quantity, whose magnitude is $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$.

(ii) If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then θ is not defined and in this case, we define $\vec{a} \times \vec{b} = \vec{0}$

Properties of cross product of two vectors

1. Unit vector $\hat{n}$ which is perpendicular to both the vectors $\vec{a}$ and $\vec{b}$, is given by

$$\hat{n} = \pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}.$$

2. Relation between dot and cross-product is $(\vec{a} \cdot \vec{b})^2 + |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2$.

3. Angle between two non-zero vectors is given by $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$ or $\theta = \sin^{-1}\left[\dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}\right]$.

4. $\vec{a} \times \vec{a} = \vec{0}$

5. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ [not commutative]

6. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ [distributive propty]

7. $\lambda(\vec{a} \times \vec{b}) = (\lambda \vec{a}) \times \vec{b} = \vec{a} \times (\lambda \vec{b})$

8. If $\vec{a}$ is parallel to $\vec{b}$, then $\vec{a} \times \vec{b} = \vec{0}$ and converse is also true.

or

Two non-zero vectors $\vec{a}$, $\vec{b}$ are collinear iff $\vec{a} \times \vec{b} = \vec{0}$

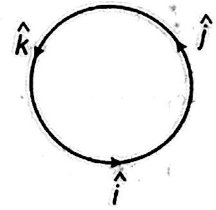
9. If $\theta = \dfrac{\pi}{2}$, then $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|$.

10. If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is given by $|\vec{a} \times \vec{b}|$.

11. If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle, then its area is given by $= \dfrac{1}{2}|\vec{a} \times \vec{b}|$.

12. If $\vec{d_1}$ and $\vec{d_2}$ represent the diagonals of a parallelogram, then its area is given by $\dfrac{1}{2}|\vec{d_1} \times \vec{d_2}|$.

13. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$
and $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$



14. $\hat{j} \times \hat{i} = -\hat{k}$, $\hat{k} \times \hat{j} = -\hat{i}$ and $\hat{i} \times \hat{k} = -\hat{j}$

15. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

16. If A, B and C are the position vectors of pane ABC, then the unit vector perpendicular o the plane ABC is $\dfrac{\vec{AB} \times \vec{AC}}{|\vec{AB} \times \vec{AC}|}$.

23 Scalar Triple Product of Vectors

Suppose $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors. Then, scalar product of $\vec{a}$ and $\vec{b} \times \vec{c}$, i.e. $\vec{a} \cdot (\vec{b} \times \vec{c})$ is called the scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ and it is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$, thus $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$

Properties of Scalar Triple Product

For any three vectors $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$, $\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ and $\vec{c} = a_3\hat{i} + b_3\hat{j} + c_3\hat{k}$,

1. $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$
2. If cyclic order of three vectors is unchanged, then scalar triple product remains unchanged, i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$.
3. If cyclic order of three vectors is changed, then scalar triple product changes in sign but not in magnitude, i.e.

   $$[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}]$$

   $$= -[\vec{c}\ \vec{b}\ \vec{a}] = -[\vec{a}\ \vec{c}\ \vec{b}].$$
4. $[\vec{a}\ \vec{a}\ \vec{b}] = [\vec{b}\ \vec{b}\ \vec{a}] = [\vec{c}\ \vec{c}\ \vec{b}] = 0$
5. $[k\vec{a}\ \vec{b}\ \vec{c}] = k[\vec{a}\ \vec{b}\ \vec{c}]$
6. $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ gives the volume of a parallelopiped formed by adjacent sides given by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$.
7. Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, if and only if $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.
8. Four points A, B, C and D are coplanar, iff $[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = 0$.