

## ● IMPULSE AND MOMENTUM ●

### INTRODUCTION

Until now, we have studied kinematics of particles, Newton’s laws of motion and methods of work and energy. Newton’s laws of motion describe relation between forces acting on a body at an instant and acceleration of the body at that instant. Therefore, it only helps us do analyze what is happening at an instant. The work kinetic energy theorem is obtained by integrating equation of motion ( $\vec{F} = m\vec{a}$ ) over a path. Therefore, methods of work and energy help us to in exploring change in speed over a position interval. Now, we direct our attention on another principle – principle of impulse and momentum. It is obtained when equation of motion ( $\vec{F} = m\vec{a}$ ) is integrated with respect to time. Therefore, this principle facilitates us with method to explore what is happening over a time interval.

### Impulse of a Force

Net force applied on a rigid body changes momentum i.e. amount of motion of that body. A net force for a longer duration cause more change in momentum than the same force acting for shorter duration. Therefore duration in which a force acts on a body together with magnitude and direction of the force decide effect of the force on the change in momentum of the body.

Linear impulse or simply impulse of a force is defined as integral of the force with respect to time.

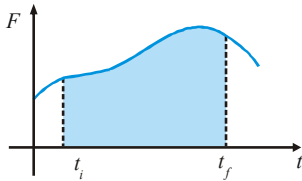
If a force  $\vec{F}$  acts on a body, its impulse in a time interval from  $t_i$  to  $t_f$  is given by the following equation.

$$\vec{I}_{mp} = \int_{t_i}^{t_f} \vec{F} dt$$

If the force is constant, its impulse equals to product of the force vector  $\vec{F}$  and time interval  $\Delta t$ .

$$\vec{I}_{mp} = \vec{F} (\Delta t)$$

For one-dimensional force, impulse equals to area between force-time graph and the time axis. In the given figure is shown how a force  $F$  along x-axis varies with time  $t$ . Impulse of this force in time interval  $t_i$  to  $t_f$  equals to area of the shaded portion.



If several forces  $\vec{F}_1, \vec{F}_2, \vec{F}_3, \dots, \vec{F}_n$  act on a body in a time interval, the total impulse  $\vec{I}_{mp}$  of all these forces equals to impulse of the net force.

$$\vec{I}_{mp} = \int_{t_i}^{t_f} \vec{F}_1 dt + \int_{t_i}^{t_f} \vec{F}_2 dt + \dots + \int_{t_i}^{t_f} \vec{F}_n dt = \int_{t_i}^{t_f} (\vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n) dt$$

Impulse is measured in newton-second.

Dimensions of impulse are  $MLT^{-1}$

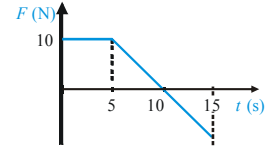
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**Ex.** Calculate impulse of force  $\vec{F} = (3t^2\hat{i} - (2t - 1)\hat{j} + 2\hat{k})$  N over the time interval from  $t = 1$  s to  $t = 3$  s.

**Sol.**  $\vec{I}_{mp} = \int_{t_i}^{t_f} \vec{F} dt \rightarrow \vec{I}_{mp} = \int_1^3 (3t^2\hat{i} - (2t - 1)\hat{j} + 2\hat{k}) dt = [t^3\hat{i} - (t^2 - t)\hat{j} + 2t\hat{k}]_1^3 = \hat{i}[t^3]_1^3 - \hat{j}[(t^2 - t)]_1^3 + \hat{k}[2t]_1^3$   
 $= (26\hat{i} - 6\hat{j} + 4\hat{k})$  N-s

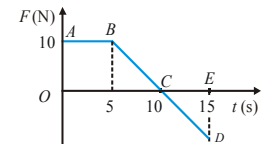
**Ex.** A one-dimensional force  $F$  varies with time according to the given graph. Calculate impulse of the force in following time intervals.

- (a) From  $t = 0$  s to  $t = 10$  s.
- (b) From  $t = 10$  s to  $t = 15$  s.
- (c) From  $t = 0$  s to  $t = 15$  s.



**Sol.** For one-dimensional force, impulse equals to area between force-time graph and the time axis.

- (a)  $I_{0 \rightarrow 10} = \text{Area of trapezium OABC} = 75$  N-s
- (b)  $I_{10 \rightarrow 15} = -\text{Area of triangle CDE} = -25$  N-s
- (c)  $I_{0 \rightarrow 15} = \text{Area of trapezium OABC} - \text{Area of triangle CDE} = 50$  N-s



### Impulse Momentum Principle :-

Consider body of mass  $m$  in translational motion. When it is moving with velocity  $\vec{v}$ , net external force acting on it is  $\vec{F}$ . Equation of motion as suggested by Newton's second law can be written in the form

$$\vec{F} dt = d(m\vec{v})$$

If the force acts during time interval from  $t_i$  to  $t_f$  and velocity of the body changes from  $\vec{v}_i$  to  $\vec{v}_f$ , integrating the above equation with time over the interval from  $t_i$  to  $t_f$ , we have

$$\int_{t_i}^{t_f} \vec{F} dt = m\vec{v}_f - m\vec{v}_i$$

Here left hand side of the above equation is impulse  $\vec{I}_{mp}$  of the net force  $\vec{F}$  in time interval from  $t_i$  to  $t_f$  and quantities  $m\vec{v}_i$  and  $m\vec{v}_f$  on the right hand side are linear momenta of the particle at instants  $t_i$  and  $t_f$ . If we denote them by symbols  $\vec{p}_i$  and  $\vec{p}_f$ , the above equation can be written as

$$\vec{I}_{mp} = \vec{p}_f - \vec{p}_i$$

The idea expressed by the above equation is known as *impulse momentum principle*. It states that change in the momentum of a body in a time interval equals to the impulse of the net force acting on the body during the concerned time interval.

For the ease of application to physical situations the above equation is rearranged as

$$\vec{p}_i + \vec{I}_{mp} = \vec{p}_f$$

This equation states that impulse of a force during a time interval when added to momentum of a body at the beginning of an interval of time we get momentum of the body at the end of the interval concerned.

Since impulse and momentum both are vector quantities, the impulse momentum theorem can be expressed by there scalar equations making use of Cartesian components.

$$p_{1x} + \sum I_{mp,x} = p_{2x}$$

$$p_{1y} + \sum I_{mp,y} = p_{2y}$$

$$p_{1z} + \sum I_{mp,z} = p_{2z}$$

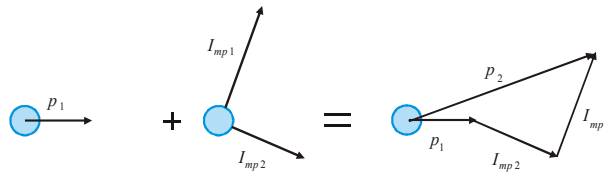
The impulse momentum principle is deduced here for a single body moving relative to an inertial frame, therefore impulses of only physical forces are considered. If we are using a non-inertial reference frame, impulse of corresponding pseudo force must also be considered in addition to impulse of the physical forces.

**How to apply Impulse Momentum Principle :-**

The impulse momentum principle is deduced here for a single body, therefore it is recommended at present to use it for a single body. To use this principle the following steps should be followed.

- (i) Identify the initial and final positions as position 1 and 2 and show momenta  $\vec{p}_1$  and  $\vec{p}_2$  of the body at these instants.
- (ii) Show impulse of each force acting on the body at an instant between positions 1 and 2.
- (iii) Use the impulse obtained in step (ii) and momenta obtained in step (i) into equation  $\vec{p}_i + \vec{I}_{mp} = \vec{p}_f$ .

Consider a particle moving with momentum  $\vec{p}_1$  in beginning. It is acted upon by two forces, whose impulses in a time interval are  $\vec{I}_{mp1}$  and  $\vec{I}_{mp2}$ . As a result, at the end of the time interval, momentum of the particle becomes  $\vec{p}_2$ . This physical situation is shown in the following diagram. Such a diagram is known as *impulse momentum diagram*.



**Ex.** A particle of mass 2 kg is moving with velocity  $\vec{v}_o = (2\hat{i} - 3\hat{j})$  m/s in free space. Find its velocity 3 s after a constant force  $\vec{F} = (3\hat{i} + 4\hat{j})$  N starts acting on it.

**Sol.**  $\vec{p}_f = \vec{p}_i + \vec{I}_{mp} \rightarrow m\vec{v}_f = m\vec{v}_o + \vec{F}\Delta t$

Substituting given values, we have

$$2\vec{v}_f = 2(2\hat{i} - 3\hat{j}) + (3\hat{i} + 4\hat{j}) \times 3 = 13\hat{i} + 6\hat{j}$$

$$\vec{v}_f = (6.5\hat{i} + 3\hat{j}) \text{ m/s}$$

**Ex.** A particle of mass 2 kg is moving in free space with velocity  $\vec{v}_o = (2\hat{i} - 3\hat{j} + \hat{k})$  m/s is acted upon by force

$\vec{F} = (2\hat{i} + \hat{j} - 2\hat{k})$  N. Find velocity vector of the particle 3 s after the force starts acting.

**Sol.**  $\vec{p}_f = \vec{p}_i + \vec{I}_{mp} \rightarrow m\vec{v}_f = m\vec{v}_o + \vec{F}\Delta t$

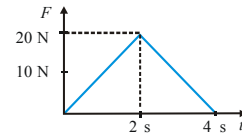
Substituting given values, we have

$$2\vec{v}_f = 2(2\hat{i} - 3\hat{j} + \hat{k}) + (2\hat{i} + \hat{j} - 2\hat{k}) \times 3 = 10\hat{i} - 3\hat{j} - 4\hat{k}$$

$$\vec{v}_f = (5\hat{i} - 1.5\hat{j} - 2\hat{k}) \text{ m/s}$$

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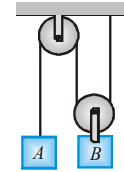
**Ex.** A box of mass  $m = 2$  kg resting on a frictionless horizontal ground is acted upon by a horizontal force  $F_x$  which varies as shown. Find speed of the particle when the force ceases to act.



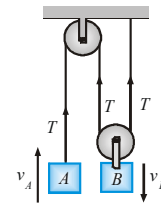
**Sol.**  $\vec{p}_f = \vec{p}_i + \vec{I}_{mp} \rightarrow m\vec{v}_f = m\vec{v}_i + \int_i^f \vec{F} dt$

$$2v = 2 \times 0 + \frac{1}{2} \times 20 \times 4 \Rightarrow v = 20 \text{ m/s}$$

**Ex.** Two boxes  $A$  and  $B$  of masses  $m$  and  $M$  interconnected by an ideal rope and ideal pulleys are held at rest as shown. When it is released, box  $B$  accelerates downwards. Find velocities of box  $A$  and  $B$  as function of time  $t$  after system has been released.



**Sol.** We first explore relation between accelerations  $a_A$  and  $a_B$  of the boxes  $A$  and  $B$ , which can be written either by using constrained relation or method of virtual work or by inspection.



$$v_A = 2v_B \quad \dots\text{(i)}$$

Applying impulse momentum principle to box A

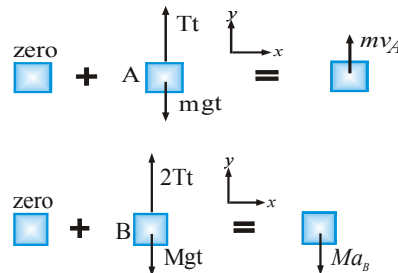
$$p_{2y} = p_{1y} + \sum I_{mp,y} \rightarrow Mv_A = 0 + Tt - mgt \quad \dots\text{(ii)}$$

Applying impulse momentum principle to box B

$$p_{2y} = p_{1y} + \sum I_{mp,y} \rightarrow mv_B = 0 + Mgt - 2Tt \quad \dots\text{(iii)}$$

From equations (i), (ii) and (iii), we have

$$v_A = 2 \left( \frac{M - 2m}{M + 4m} \right) gt \quad \text{and} \quad v_B = \left( \frac{M - 2m}{M + 4m} \right) gt$$



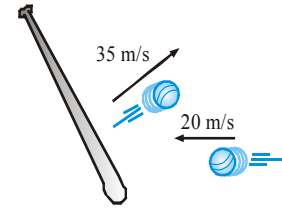
### Impulsive Motion

Sometimes a very large force acts for a very short time interval on a particle and produces finite change in momentum. Such a force is known as impulsive force and the resulting motion as impulsive motion. When a batsman hits a ball by bat, the contact between the ball and the bat lasts for a very small duration  $\Delta t$ , but the average value of the force  $F$  exerted by the bat on the ball is very large, and the resulting impulse  $F\Delta t$  is large enough to change momentum of the ball.



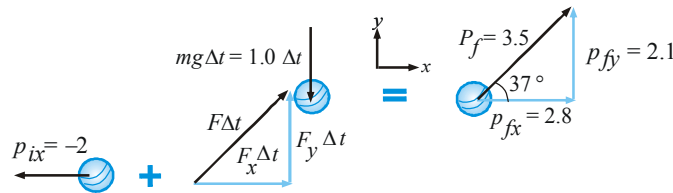
During an impulsive motion, some other forces of magnitudes very small in comparison to that of an impulsive force may also act. Due to negligible time interval of the impulsive motion, impulse of these forces becomes negligible. These forces are known as non-impulsive forces. Effect of non-impulsive forces during an impulsive motion is so small that they are neglected in analyzing impulsive motion of infinitely small duration. Non-impulsive forces are of finite magnitude and include weight of a body, spring force or any other force of finite magnitude. When duration of the impulsive motion is specified, care has to be taken in neglecting any of the non-impulsive force. In analyzing motion of the ball for very small contact duration (usually in milliseconds), impulse of the weight of the ball has to be neglected. Unknown reaction forces may be impulsive or non-impulsive; their impulse must therefore be included.

**Ex.** A 100 gm ball moving horizontally with 20 m/s is struck by a bat, as a result it starts moving with a speed of 35 m/s at an angle of 37° above the horizontal in the same vertical plane as shown in the figure.



- (a) Find the average force exerted by the bat if duration of impact is 0.30 s.
- (b) Find the average force exerted by the bat if duration of impact is 0.03 s.
- (c) Find the average force exerted by the bat if duration of impact is 0.003 s.
- (d) What do you conclude for impulse of weight of the ball as duration of contact decreases?

**Sol.** The impulse momentum diagram of the ball is shown in the figure below. Here  $F$ ,  $mg$ , and  $Dt$  represent the average value of the force exerted by the bat, weight of the ball and the time interval.



Applying principle of impulse and momentum in x- direction, we have

$$p_{fx} = p_{ix} + \sum I_{mp,x} \rightarrow -2 + F_x \Delta t = 2.8$$

$$F_x = \frac{4.8}{\Delta t} \text{ N} \quad \dots\text{(i)}$$

Applying principle of impulse and momentum in y- direction, we have

$$p_{fy} = p_{iy} + \sum I_{mp,y} \rightarrow 0.0 + F_y \Delta t - 1.0 \Delta t = 2.1$$

$$F_y = \left( \frac{2.1}{\Delta t} + 1.0 \right) \text{ N} \quad \dots\text{(ii)}$$

- (a) Substituting  $\Delta t = 0.30$  s, in equations (i) and (ii), we find  $\vec{F} = 16\hat{i} + 8\hat{j}$  N
- (b) Substituting  $\Delta t = 0.03$  s, in equations (i) and (ii), we find  $\vec{F} = 160\hat{i} + 71\hat{j}$  N
- (c) Substituting  $\Delta t = 0.003$  s, in equations 1 and 2, we find  $\vec{F} = 1600\hat{i} + 701\hat{j}$  N
- (d) It is clear from the above results that as the duration of contact between the ball and the bat decreases, effect of the weight of the ball also decreases as compared with that of the force of the bat and for sufficiently short time interval, it can be neglected.

**Momentum and Kinetic Energy**

A moving particle possesses momentum as well as kinetic energy. If a particle of mass  $m$  is moving with velocity  $v$ , magnitude of its momentum  $p$  and its kinetic energy  $K$  bear the following relation.

$$K = \frac{p^2}{2m} = \frac{1}{2} p v$$

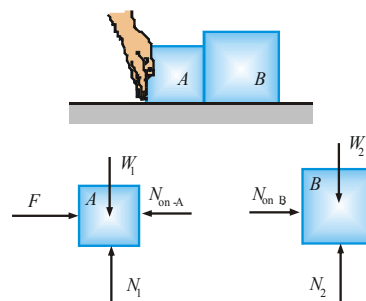
**Ex.** An object is moving so that its kinetic energy is 150 J and the magnitude of its momentum is 30.0 kg-m/s. With what velocity is it traveling?

**Sol.** 
$$K = \frac{p^2}{2m} = \frac{1}{2}pv \rightarrow v = \frac{2 \times 150}{30.0} = 10.0 \text{ m/s}$$

**Internal and external Forces and System of interacting Particles**

Bodies applying forces on each other are known as interacting bodies. If we consider them as a system, the forces, which they apply on each other, are known as internal forces and all other forces applied on them by bodies not included in the system are known as external forces.

Consider two blocks A and B placed on a frictionless horizontal floor. Their weights  $W_1$  and  $W_2$  are counterbalanced by normal reactions  $N_1$  and  $N_2$  on each of them from the floor. Push  $F$  by the hand is applied on A. The forces of normal reaction  $N_{on-A}$  and  $N_{on-B}$  constitute Newton’s third law action-reaction pair, therefore are equal in magnitude and opposite in direction. Among these forces weights  $W_1$  and  $W_2$  applied by the earth, normal reactions  $N_1$  and  $N_2$  applied by the ground and the push  $F$  applied by the hand are external forces and normal reactions  $N_{on-A}$  and  $N_{on-B}$  are internal forces.

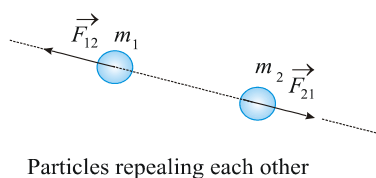
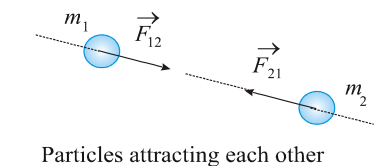


If the blocks are connected by a spring and the block A is either pushed or pulled, the forces  $W_1$ ,  $W_2$ ,  $N_1$  and  $N_2$  still remain external forces for the two block system and the forces, which the spring applies on each other are the internal forces. Here force of gravitational interaction between them being negligible has been neglected.

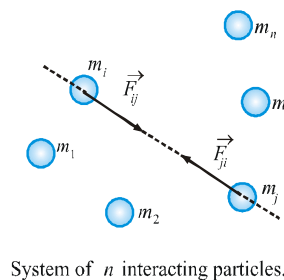
We can conceive a general model of two interacting particles.

In the figure is shown a system of two particles of masses  $m_1$  and  $m_2$ .

Particle  $m_1$  attracts  $m_2$  with a force  $\vec{F}_{12}$  and  $m_2$  attracts (or pulls)  $m_1$  with a force  $\vec{F}_{21}$ . These forces  $\vec{F}_{12}$  and  $\vec{F}_{21}$  are the internal forces of this two-particle system and are equal in magnitude and opposite in directions. Instead of attraction may repel each other. Such a system of two particles repelling each other is also shown.



In similar way we may conceive a model of a system of  $n$  interacting particles having masses  $m_1, m_2, \dots, m_i, \dots, m_j, \dots, m_n$  respectively. The forces of interaction  $\vec{F}_{ij}$  and  $\vec{F}_{ji}$  between  $m_i$  and  $m_j$  are shown in the figure. Similar to these other particles may also interact with each other. These forces of mutual interaction between the particles are internal forces of the system. Any of the two interacting particles always apply equal and opposite forces on each other. Here for simplicity only the forces  $\vec{F}_{ij}$  and  $\vec{F}_{ji}$  are shown.



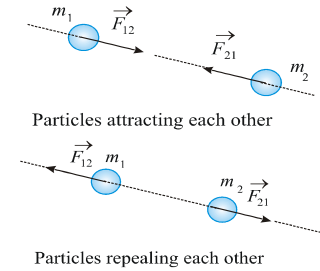
**Principle of Conservation of linear momentum**

The principle of conservation of linear momentum or simply conservation of momentum for two or more interacting bodies is one of guiding principles of the classical as well as the modern physics.

To understand this principle, we first discuss a system of two interacting bodies, and then extend the ideas developed to a system consisting of many interacting bodies.

Consider a system of two particles of mass  $m_1$  and  $m_2$ . Particle  $m_1$  attracts  $m_2$  with a force  $\vec{F}_{12}$  and  $m_2$  attracts  $m_1$  with a force  $\vec{F}_{21}$ .

These forces have equal magnitudes and opposite directions as shown in the figure. If the bodies are let free i.e. without any external force acting on

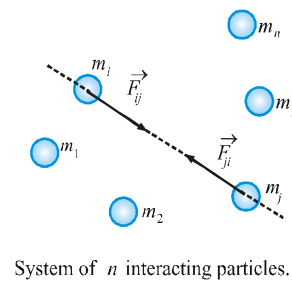


any of them, each of them move and gain momentum equal to the impulse of the force of interaction. Since equal and opposite interaction forces act on both of them for the same time interval, the momenta gained by them are equal in magnitude and opposite in direction resulting no change in total momentum of the system.

However, if an external force acts on any one of them or different forces with a nonzero resultant act on both of them, the total momentum of the bodies will certainly change. If the system undergoes an impulsive motion, total momentum will change only under the action of external impulsive force or forces. Internal impulsive forces also exist in pairs of equal and opposite forces and cannot change the total momentum of the system. Non-impulsive forces if act cannot change momentum of the system by appreciable amount. For example, gravity is a non-impulsive force, therefore in the process of collision between two bodies near the earth the total momentum remains conserved.

The total momentum of a system of two interacting bodies remains unchanged under the action of the forces of interaction between them. It can change only if a net impulse of external force is applied.

In similar way we may conceive a model of a system of n interacting particles having masses  $m_1, m_2, \dots, m_i, \dots, m_j, \dots$  and  $m_n$  respectively. The forces of interaction  $\vec{F}_{ij}$  and  $\vec{F}_{ji}$  between  $m_i$  and  $m_j$  are shown in the figure. Since internal forces exist in pairs of equal and opposite forces, in any time interval of concern each of them have a finite impulse but their total impulse is zero. Thus if the system is let free, in any time interval momentum of every individual particle changes but the total momentum of the system remains constant.



It can change only if external forces are applied to some or all the particles. Under the action of external forces, the change in total momentum of the system will be equal to the net impulse of all the external forces.

Thus, total momentum of a system of particles cannot change under the action of internal forces and if net impulse of the external forces in a time interval is zero, the total momentum of the system in that time interval will remain conserved.

$$\sum \vec{P}_{initial} = \sum \vec{P}_{final}$$

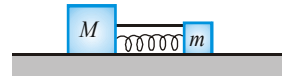
The above statement is known as the principle of conservation of momentum. It is applicable only when the net impulse of all the external forces acting on a system of particles becomes zero in a finite time interval. It happens in the following conditions.

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- (i) When no external force acts on any of the particles or bodies.
- (ii) When resultant of all the external forces acting on all the particles or bodies is zero.
- (iii) In impulsive motion, where time interval is negligibly small, the direction in which no impulsive forces act, total component of momentum in that direction remains conserved.

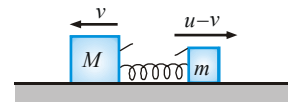
Since force, impulse and momentum are vectors, component of momentum of a system in a particular direction is conserved, if net impulse of all external forces in that direction vanishes.

**Ex.** Two blocks of masses  $m$  and  $M$  are held against a compressed spring on a frictionless horizontal floor with the help of a light thread. When the thread is cut, the smaller block leaves the spring with a velocity  $u$  relative to the larger block. Find the recoil velocity of the larger block.



**Sol.** When the thread is cut, the spring pushes both the block, and impart them momentum. The forces applied by the spring on both the block are internal forces of the two-block system. External forces acting on the system are weights and normal reactions on the blocks from the floor. These external forces have zero net resultant of the system. In addition to this fact no external force acts on the system in horizontal direction, therefore, horizontal component of the total momentum of the system remains conserved.

Velocities of both the objects relative to the ground (inertial frame) are shown in the adjoining figure.

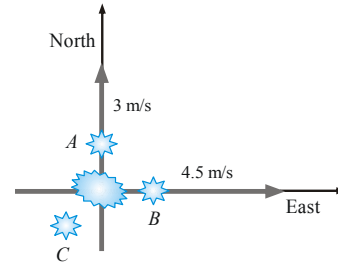


Since before the thread is cut system was at rest, its total momentum was zero. Principle of conservation of momentum for the horizontal direction yields

$$\sum_{i=1}^n p_{\text{horizontal}} = 0 \rightarrow -Mv + m(u - v) = 0$$

$$v = \frac{mu}{M + m}$$

**Ex.** A shell fired vertically up, when reaches its highest point, explodes into three fragments A, B and C of masses  $m_A = 4$  kg,  $m_B = 2$  kg and  $m_C = 3$  kg. Immediately after the explosion, A is observed moving with velocity  $v_A = 3$  m/s towards north and B with a velocity  $v_B = 4.5$  m/s towards east as shown in the figure. Find the velocity  $v_C$  of the piece C.



**Sol.** Explosion takes negligible duration; therefore, impulse of gravity, which is a finite external force, can be neglected. The pieces fly off acquiring above-mentioned velocities due to internal forces developed due to expanding gases produced during the explosion. The forces applied by the expanding gases are internal forces; hence, momentum of the system of the three pieces remains conserved during the explosion and total momentum before and after the explosion are equal.

Assuming the east as positive  $x$ -direction and the north as positive  $y$ -direction, the momentum vectors  $\vec{p}_A$  and  $\vec{p}_B$  of pieces A and B become

$$\vec{p}_A = m_A v_A \hat{j} = 12 \text{ kg-m/s} \quad \text{and} \quad \vec{p}_B = m_B v_B \hat{i} = 9 \text{ kg-m/s}$$

Before the explosion, momentum of the shell was zero, therefore from the principle of conservation of momentum, the total momentum of the fragments also remains zero.

$$\vec{p}_A + \vec{p}_B + \vec{p}_C = \vec{0} \rightarrow \vec{p}_C = -(9\hat{i} + 12\hat{j})$$

From the above equation, velocity of the piece C is

$$\vec{v}_C = \frac{\vec{p}_C}{m_C} = -(3\hat{i} + 4\hat{j}) = 5 \text{ m/s, } 53^\circ \text{ south of west.}$$



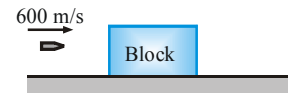
**Ex.** In free space, three identical particles moving with velocities  $v_o \hat{i}$ ,  $-3v_o \hat{j}$  and  $5v_o \hat{k}$  collide successively with each other to form a single particle. Find velocity vector of the particle formed.

**Sol.** Let  $m$  be the mass of a single particle before any of the collisions. The mass of particle formed after collisions must be  $3m$ . In free space, no external forces act on any of the particles, their total momentum remains conserved. Applying principle of conservation of momentum, we have

$$\sum \vec{p}_{initial} = \sum \vec{p}_{final} \rightarrow m v_o \hat{i} - 3m v_o \hat{j} + 5m v_o \hat{k} = 3m \vec{v}$$

$$\vec{v} = \frac{1}{3} v_o (\hat{i} - 3\hat{j} + 5\hat{k}) \text{ m/s}$$

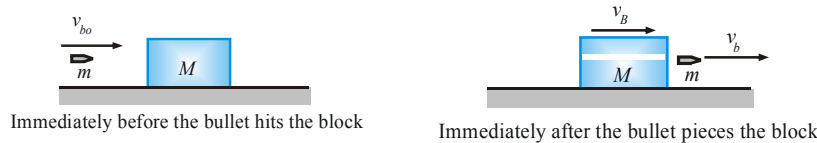
**Ex.** A bullet of mass 50 g moving with velocity 600 m/s hits a block of mass 1.0 kg placed on a rough horizontal ground and comes out of the block with a velocity of 400 m/s. The coefficient of friction between the block and the ground is 0.25. Neglect loss of mass of the block as the bullet pierces through it.



- (a) In spite of the fact that friction acts as an external force, can you apply principle of conservation of momentum during interaction of the bullet with the block?
- (b) Find velocity of the block immediately after the bullet pierces through it.
- (c) Find the distance the block will travel before it stops.

**Sol.** (a) There is no net external force in the vertical direction and in the horizontal direction, only external force friction is non-impulsive, therefore momentum of the bullet-block system during their interaction remains conserved.

(b) Let us denote velocities of the bullet before it hits the block and immediately after it pierces through the block by  $v_{bo}$  and  $v_b$ , velocity of the block immediately after the bullet pierces through it is  $v_B$  and masses of the bullet and the block by  $m$  and  $M$  respectively. These are shown in the adjacent figure.

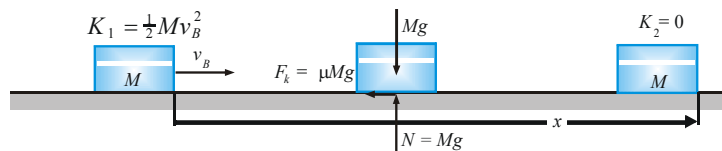


Applying principle of conservation of momentum for horizontal component, we have

$$m v_{bo} = m v_b + M v_B \rightarrow v_B = \frac{m (v_{bo} - v_b)}{M}$$

Substituting the given values, we have  $v_B = 10$  m/s

- (c) To calculate distance traveled by the block before it stops, work kinetic energy theorem has to be applied.



During sliding of the box on the ground only the force of kinetic friction does work.

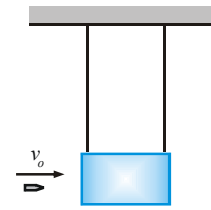
$$W_{1 \rightarrow 2} = K_2 - K_1 \rightarrow -\mu M g x = 0 - \frac{1}{2} M v_B^2 \Rightarrow x = \frac{v_B^2}{2 \mu g}$$

Substituting given values, we have  $x = 20$  m

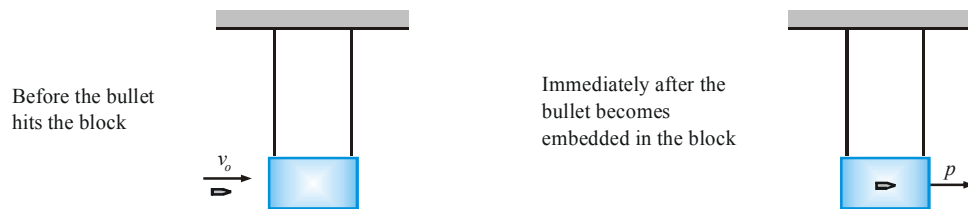
## PHYSICS FOR JEE MAIN & ADVANCED

**Ex. Ballistic Pendulum :** A ballistic pendulum is used to measure speed of bullets. It consists of a wooden block suspended from fixed support.

A wooden block of mass  $M$  is suspended with the help of two threads to prevent rotation while swinging. A bullet of mass  $m$  moving horizontally with velocity  $v_o$  hits the block and becomes embedded in the block. Receiving momentum from the bullet, the bullet-block system swings to a height  $h$ . Find expression for speed of the bullet in terms of given quantities.



**Sol.** When the bullet hits the block, in a negligible time interval, it becomes embedded in the block and the bullet-block system starts moving with horizontally. During this process, net force acting on the bullet-block system in vertical direction is zero and no force acts in the horizontal direction. Therefore, momentum of the bullet-block system remains conserved.



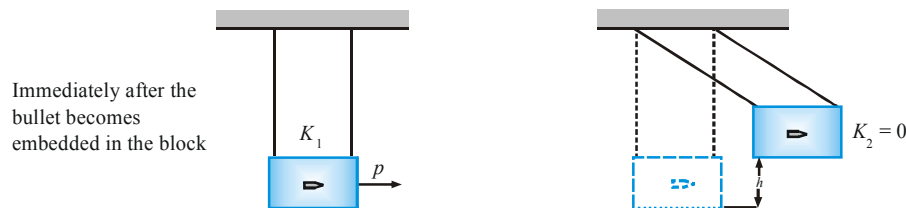
Let us denote momentum of the bullet-block system immediately after the bullet becomes embedded in the block by  $p$  and apply principle of conservation of momentum to the system for horizontal component of momentum.

$$p = mv_o$$

Using equation  $K = p^2 / 2m$ , we can find kinetic energy  $K_1$  of the bullet-block system immediately after the bullet becomes embedded in the block.

$$K_1 = \frac{(mv_o)^2}{2(M+m)}$$

During swing, only gravity does work on the bullet-block system. Applying work-kinetic energy theorem during swing of the bullet-block system, we have



$$W_{1 \rightarrow 2} = K_2 - K_1 \rightarrow \quad -(M+m)gh = 0 - \frac{(mv_o)^2}{2(M+m)}$$

Rearranging terms, we have

$$v_o = \frac{(M+m)}{m} \sqrt{2gh}$$

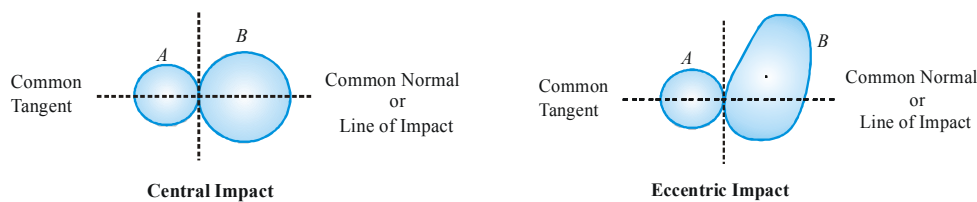
### Impact between two bodies

Impact or collision is interaction of very small duration between two bodies in which the bodies apply relatively large forces on each other.

Interaction forces during an impact are created due to either direct contact or strong repulsive force fields or some connecting links. These forces are so large as compared to other external forces acting on either of the bodies that the effects of later can be neglected. The duration of the interaction is short enough as compared to the time scale of interest as to permit us only to consider the states of motion just before and after the event and not during the impact. Duration of an impact ranges from  $10^{-23}$  s for impacts between elementary particles to millions of years for impacts between galaxies. The impacts we observe in our everyday life like that between two balls last from  $10^{-3}$  s to few seconds.

### Central and Eccentric Impact

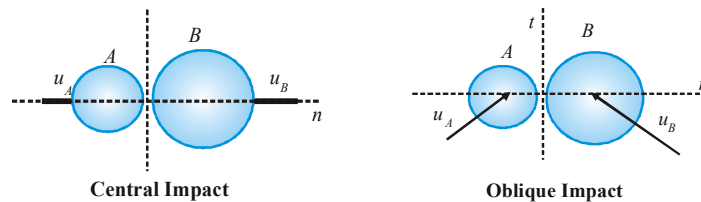
The common normal at the point of contact between the bodies is known as line of impact. If mass centers of the both the colliding bodies are located on the line of impact, the impact is called central impact and if mass centers of both or any one of the colliding bodies are not on the line of impact, the impact is called eccentric impact.



Central impact does not produce any rotation in either of the bodies whereas eccentric impact causes the body whose mass center is not on the line of impact to rotate. Therefore, at present we will discuss only central impact and postpone analysis of eccentric impact to cover after studying rotation motion.

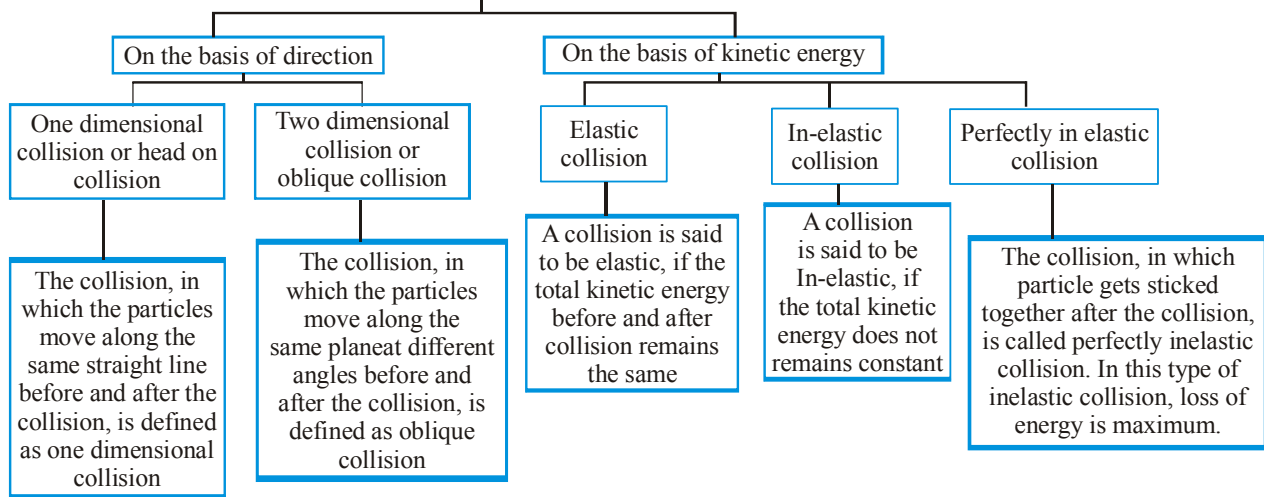
### Head-on (Direct) and Oblique Central Impact

If velocities vectors of the colliding bodies are directed along the line of impact, the impact is called a direct or head-on impact; and if velocity vectors of both or of any one of the bodies are not along the line of impact, the impact is called an oblique impact.



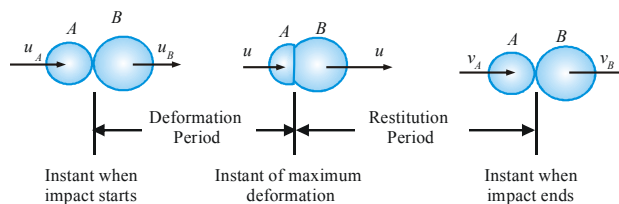
In this chapter, we discuss only central impact, therefore the term central we usually not use and to these impacts, we call simply head-on and oblique impacts. Furthermore, use of the line of impact and the common tangent is so frequent in analysis of these impacts that we call them simply  $t$ -axis and  $n$ -axis.

## TYPES OF COLLISIONS



### Head-on (Direct) Central Impact

To understand what happens in a head-on impact let us consider two balls  $A$  and  $B$  of masses  $m_A$  and  $m_B$  moving with velocities  $u_A$  and  $u_B$  in the same direction as shown. Velocity  $u_A$  is larger than  $u_B$  so the ball  $A$  hits the ball  $B$ . During impact, both the bodies push each other and first they get deformed till the deformation reaches a maximum value and then they try to regain their original shape due to elastic behaviors of the materials forming the balls.



The time interval when deformation takes place is called the deformation period and the time interval in which the balls try to regain their original shapes is called the restitution period. Due to push applied by the balls on each other during the period of deformation the speed of ball  $A$  decreases and that of ball  $B$  increases and at the end of the deformation period, when the deformation is maximum, both balls move with the same velocity, say it is  $u$ . Thereafter, during the period of restitution, due to push applied by the balls on each other, the speed of ball  $A$  decreases further and that of ball  $B$  increases further till they separate from each other. Let us denote the velocities of balls  $A$  and  $B$  after the impact by  $v_A$  and  $v_B$  respectively.

### Equation of Impulse and Momentum during impact

Impulse momentum principle describes the motion of ball  $A$  during the deformation period.

$$\begin{array}{c} \bullet \xrightarrow{m_A u_A} \end{array} + \begin{array}{c} \bullet \xleftarrow{\int D dt} \end{array} = \begin{array}{c} \bullet \xrightarrow{m_A u} \end{array} \quad m_A u_A - \int D dt = m_A u \quad \dots(i)$$

Impulse momentum principle describes the motion of ball  $B$  during the deformation period.

$$\begin{array}{c} \bullet \xrightarrow{m_B u_B} \end{array} + \begin{array}{c} \int D dt \rightarrow \bullet \end{array} = \begin{array}{c} \bullet \xrightarrow{m_B u} \end{array} \quad m_B u_B + \int D dt = m_B u \quad \dots(ii)$$

Impulse momentum principle describes motion of ball A during restitution period.

$$\begin{array}{c}
 \text{Ball A} \xrightarrow{m_A u} + \int R dt = \text{Ball A} \xrightarrow{m_A v_A} \quad m_A u - \int R dt = m_A v_A \quad \dots\text{(iii)}
 \end{array}$$

Impulse momentum principle describes motion of ball B during restitution period.

$$\begin{array}{c}
 \text{Ball B} \xrightarrow{m_B u} + \int R dt = \text{Ball B} \xrightarrow{m_B v_B} \quad m_B u + \int R dt = m_B v_B \quad \dots\text{(iv)}
 \end{array}$$

**Conservation of Momentum during impact**

From equations, (i) and (ii) we have  $m_A u_A + m_B u_B = (m_A + m_B)u \quad \dots\text{(v)}$

From equations, (iii) and (iv) we have  $(m_A + m_B)u = m_A v_A + m_B v_B \quad \dots\text{(vi)}$

From equations, (v) and (vi) we obtain the following equation.

$$m_A v_A + m_B v_B = m_A u_A + m_B u_B \quad \dots\text{(vii)}$$

The above equation elucidates the principle of conservation of momentum.

**Coefficient of Restitution**

Usually the force  $D$  applied by the bodies  $A$  and  $B$  on each other during period differs from the force  $R$  applied by the bodies on each other during period of restitution. Therefore, it is not necessary that magnitude of impulse

$\int D dt$  of deformation equals to the magnitude of impulse  $\int R dt$  restitution.

The ratio of magnitudes of impulse of restitution to that of deformation is called the coefficient of restitution and is denoted by  $e$ .

$$e = \frac{\int R dt}{\int D dt} \quad \dots\text{(viii)}$$

Now from equations (i), (ii), (iii) and (iv), we have

$$e = \frac{v_B - v_A}{u_A - u_B} \quad \dots\text{(ix)}$$

Coefficient of restitution depend on various factors as elastic properties of materials forming the bodies, velocities of the contact points before impact, state of rotation of the bodies and temperature of the bodies. In general, its value ranges from zero to one but in collision where kinetic energy is generated its value may exceed one.

Depending on values of coefficient of restitution, two particular cases are of special interest.

**Perfectly Plastic or Inelastic Impact**

For these impacts  $e = 0$ , and bodies undergoing impact stick to each other after the impact.

**Perfectly Elastic Impact**

For these impacts  $e = 1$ .

**Strategy to solve problems of head-on impact**

Write momentum conservation equation

$$m_A v_A + m_B v_B = m_A u_A + m_B u_B \quad \dots\text{(A)}$$

Write rearranging terms of equation of coefficient of restitution

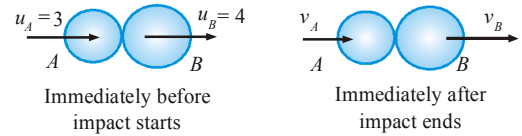
$$v_B - v_A = e(u_A - u_B) \quad \dots\text{(B)}$$

Use the above equations A and B.

## PHYSICS FOR JEE MAIN & ADVANCED

**Ex.** A ball of mass 2 kg moving with speed 5 m/s collides directly with another of mass 3 kg moving in the same direction with speed 4 m/s. The coefficient of restitution is  $\frac{2}{3}$ . Find the velocities after collision.

**Sol.** Denoting the first ball by  $A$  and the second ball by  $B$  velocities immediately before and after the impact are shown in the figure.



Applying principle of conservation of momentum, we have

$$m_B v_B + m_A v_A = m_A u_A + m_B u_B \rightarrow 3v_B + 2v_A = 2 \times 5 + 3 \times 4$$

$$3v_B + 2v_A = 22 \quad \dots\text{(i)}$$

Applying equation of coefficient of restitution, we have

$$v_B - v_A = e(u_A - u_B) \rightarrow v_B - v_A = \frac{2}{3}(5 - 4)$$

$$3v_B - 3v_A = 2 \quad \dots\text{(ii)}$$

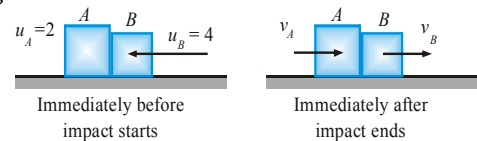
From equation (i) and (ii), we have  $v_A = 4$  m/s and  $v_B = 4.67$  m/s                      **Ans.**

**Ex.** A block of mass 5 kg moves from left to right with a velocity of 2 m/s and collides with another block of mass 3 kg moving along the same line in the opposite direction with velocity 4 m/s.

(a) If the collision is perfectly elastic, determine velocities of both the blocks after their collision.

(b) If coefficient of restitution is 0.6, determine velocities of both the blocks after their collision.

**Sol.** Denoting the first block by  $A$  and the second block by  $B$  velocities immediately before and after the impact are shown in the figure.



Applying principle of conservation of momentum, we have

$$m_B v_B + m_A v_A = m_A u_A + m_B u_B \rightarrow 3v_B + 5v_A = 5 \times 2 + 3 \times (-4)$$

$$3v_B + 5v_A = -2 \quad \dots\text{(i)}$$

Applying equation of coefficient of restitution, we have

$$v_B - v_A = e(u_A - u_B) \rightarrow v_B - v_A = e\{2 - (-4)\}$$

$$v_B - v_A = 6e \quad \dots\text{(ii)}$$

(a) For perfectly elastic impact  $e = 1$ . Using this value in equation (ii), we have

$$v_B - v_A = 6 \quad \dots\text{(i)}$$

Now from equation (i) and (iia), we obtain

$$v_A = -2.5 \text{ m/s and } v_B = 3.5 \text{ m/s}$$

(b) For value  $e = 0.6$ , equation 2 is modified as

$$v_B - v_A = 3.6 \quad \dots\text{(ii)}$$

Now from equation (i) and (iib), we obtain

$$v_A = -1.6 \text{ m/s and } v_B = 2.0 \text{ m/s}$$

Block  $A$  reverse back with speed 1.6 m/s and  $B$  also move in opposite direction to its original direction with speed 2.0 m/s.

**Ex.** Two identical balls  $A$  and  $B$  moving with velocities  $u_A$  and  $u_B$  in the same direction collide. Coefficient of restitution is  $e$ .

**(a)** Deduce expression for velocities of the balls after the collision.

**(b)** If collision is perfectly elastic, what do you observe?

**Sol.** Equation expressing momentum conservation is

$$v_A + v_B = u_A + u_B \quad \dots(\mathbf{A})$$

Equation of coefficient of restitution is

$$v_B - v_A = eu_A - eu_B \quad \dots(\mathbf{B})$$

**(a)** From the above two equations, velocities  $v_A$  and  $v_B$  are

$$v_A = \left(\frac{1-e}{2}\right)u_A + \left(\frac{1+e}{2}\right)u_B \quad \dots(\mathbf{i})$$

$$v_B = \left(\frac{1+e}{2}\right)u_A + \left(\frac{1-e}{2}\right)u_B \quad \dots(\mathbf{ii})$$

**(b)** For perfectly elastic impact  $e = 1$ , velocities  $v_A$  and  $v_B$  are

$$v_A = u_B \quad \dots(\mathbf{iii})$$

$$v_B = u_A \quad \dots(\mathbf{iv})$$

Identical bodies exchange their velocities after perfectly elastic impact.

### Conservation of kinetic energy in perfectly elastic impact

For perfectly elastic impact equation for conservation of momentum and coefficient of restitution are

$$m_A v_A + m_B v_B = m_A u_A + m_B u_B \quad \dots(\mathbf{A})$$

$$v_B - v_A = u_A - u_B \quad \dots(\mathbf{B})$$

Rearranging the terms of the above equations, we have

$$m_A (v_A - u_A) = m_B (u_B - v_B)$$

$$u_A + v_A = v_B + u_B$$

Multiplying LHS of both the equations and RHS of both the equations, we have

$$m_A (v_A^2 - u_A^2) = m_B (u_B^2 - v_B^2)$$

Multiplying by  $\frac{1}{2}$  and rearranging term of the above equation, we have

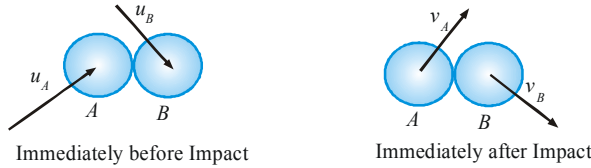
$$\frac{1}{2} m_A u_A^2 + \frac{1}{2} m_B u_B^2 = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2$$

In perfectly elastic impact total kinetic energy of the colliding body before and after the impact are equal.

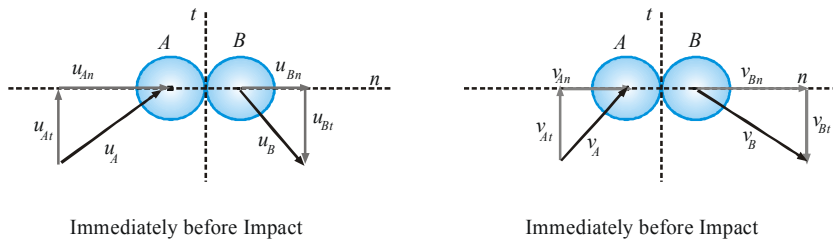
In inelastic impacts, there is always loss of kinetic energy.

**Oblique Central Impact**

In oblique central impact, velocity vectors of both or of any one of the bodies are not along the line of impact and mass center of bodies are on the line of impact. Due to impact speeds and direction of motion of both the balls change. In the given figure is shown two balls A and B of masses  $m_A$  and  $m_B$  moving with velocities  $u_A$  and  $u_B$  collide obliquely. After the collision let they move with velocities  $v_A$  and  $v_B$  as shown in the next figure.



To analyze the impact, we show components of velocities before and after the impact along the common tangent and the line of impact. These components are shown in the following figure.



**Component along the t-axis**

If surfaces of the bodies undergoing impact are smooth, they cannot apply any force on each other along the  $t$ -axis and component of momentum along the  $t$ -axis of each bodies, considered separately, is conserved. Hence,  $t$ -component of velocities of each of the bodies remains unchanged.

$$v_{At} = u_{At} \text{ and } v_{Bt} = u_{Bt} \quad \dots(\text{A})$$

**Component along the n-axis**

For components of velocities along the  $n$ -axis, the impact can be treated same as head-on central impact. The component along the  $n$ -axis of the total momentum of the two bodies is conserved

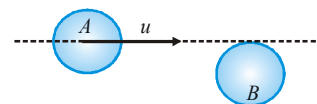
$$m_B v_{Bn} + m_A v_{An} = m_B u_{Bn} + m_A u_{An} \quad \dots(\text{B})$$

Concept of coefficient of restitution  $e$  is applicable only for the  $n$ -component velocities.

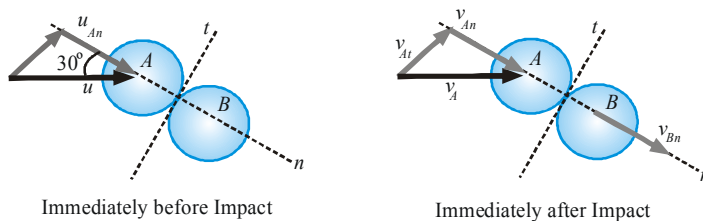
$$v_{Bn} - v_{An} = e(u_{An} - u_{Bn}) \quad \dots(\text{C})$$

The above four independent equation can be used to analyze oblique central impact of two freely moving bodies.

**Ex.** A disk sliding with velocity  $u$  on a smooth horizontal plane strikes another identical disk kept at rest as shown in the figure. If the impact between the disks is perfectly elastic impact, find velocities of the disks after the impact.



**Sol.** (a) We first show velocity components along the  $t$  and the  $n$ -axis immediately before and after the impact. angle that the line of impact makes with velocity  $u$  is  $30^\circ$ .





**Component along t-axis**

Components of momentum along the  $t$ -axis of each disk, considered separately, is conserved. Hence,  $t$ -component of velocities of each of the bodies remains unchanged.

$$v_{At} = u_{At} = \frac{u}{2} \text{ and } v_{Bt} = u_{Bt} = 0 \quad \dots\text{(i)}$$

**Component along n-axis**

The component along the  $n$ -axis of the total momentum of the two bodies is conserved

$$m_B v_{Bn} + m_A v_{An} = m_B u_{Bn} + m_A u_{An} \rightarrow m v_{Bn} + m v_{An} = m \times 0 + m \frac{u\sqrt{3}}{2}$$

$$v_{Bn} + v_{An} = \frac{u\sqrt{3}}{2} \quad \dots\text{(ii)}$$

Concept of coefficient of restitution  $e$  is applicable only for the  $n$ -component velocities.

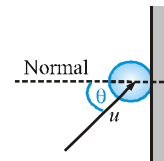
$$v_{Bn} - v_{An} = e(u_{An} - u_{Bn}) \rightarrow v_{Bn} - v_{An} = \frac{u\sqrt{3}}{2} \quad \dots\text{(iii)}$$

From equations (ii) and (iii), we have  $v_{An} = 0$  and  $v_{Bn} = \frac{u\sqrt{3}}{2}$  ...**(iv)**

From equations (i) and (iv) we can write velocities of both the disks.

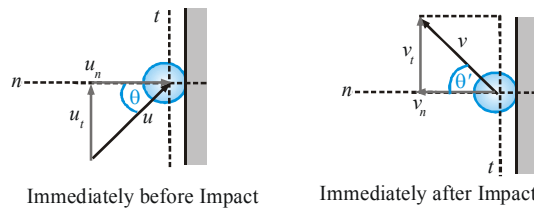
**Ex.** A ball collides with a frictionless wall with velocity  $u$  as shown in the figure. Coefficient of restitution for the impact is  $e$ .

- (a)** Find expression for the velocity of the ball immediately after the impact.
- (b)** If impact is perfectly elastic what do you observe?



**Sol.** **(a)** Let us consider the ball as the body  $A$  and the wall as the body  $B$ . Since the wall has infinitely large inertia (mass) as compared to the ball, the state of motion of the wall, remains unaltered during the impact i.e. the wall remain stationary.

Now we show velocities of the ball and its  $t$  and  $n$ -components immediately before and after the impact. For the purpose we have assumed velocity of the ball after the impact  $v$ .



**Component along t-axis**

Components of momentum along the  $t$ -axis of the ball is conserved. Hence,  $t$ -component of velocities of each of the bodies remains unchanged.

$$v_t = u_t = u \sin \theta \quad \dots\text{(i)}$$

**Component along n-axis**

Concept of coefficient of restitution  $e$  is applicable only for the  $n$ -component velocities.

$$v_{Bn} - v_{An} = e(u_{An} - u_{Bn}) \rightarrow -v_n = eu_n$$

$$v_n = -eu \cos \theta \quad \dots(\text{ii})$$

From equations (i) and (ii), the  $t$  and  $n$ -components of velocity of the ball after the impact are

$$v_t = u \sin \theta \quad \text{and} \quad v_n = eu \sin \theta$$

- (b) If the impact is perfectly elastic, we have  $v_t = u \sin \theta$ ,  $v_n = u \sin \theta$  and  $\theta' = \theta$

The ball will rebound with the same speed making the same angle with the vertical at which it has collided. In other words, a perfectly elastic collision of a ball with a wall follows the same laws as light follows in reflection at a plane mirror.

**Oblique Central Impact when one or both the colliding bodies are constrained in motion**

In oblique collision, we have discussed how to analyze impact of bodies that were free to move before as well as after the impact. Now we will see what happens if one or both the bodies undergoing oblique impact are constrained in motion.

**Component along the t-axis**

If surfaces of the bodies undergoing impact are smooth, the  $t$ -component of the momentum of the body that is free to move before and after the impact remain conserved.

If both the bodies are constrained, the  $t$ -component of neither one remains conserved.

**Momentum Conservation**

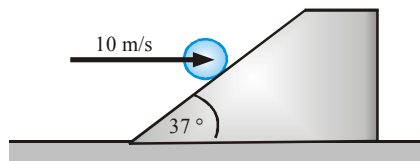
We may find a direction in which no external force acts on both the bodies. The component of total momentum of both the bodies along this direction remains conserved.

**Coefficient of restitution**

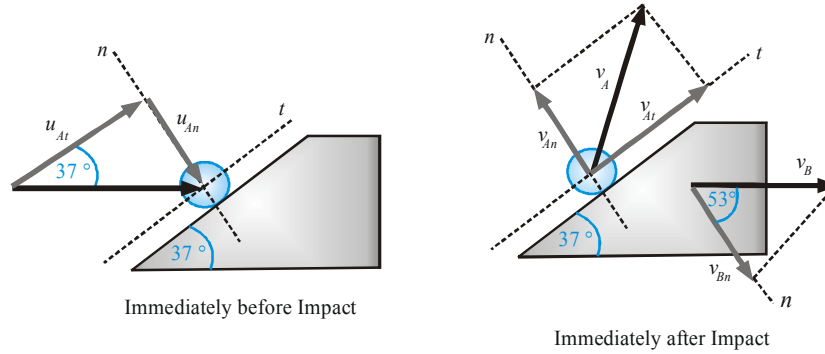
Concept of coefficient of restitution  $e$  is applicable only for the  $n$ -component velocities.

$$v_{Bn} - v_{An} = e(u_{An} - u_{Bn})$$

- Ex.** A 250 g ball moving horizontally with velocity 10.0 m/s strikes inclined surface of a 720 g smooth wedge as shown in the figure. The wedge is placed at rest on a frictionless horizontal ground. If the coefficient of restitution is 0.8, calculate the velocity of the wedge after the impact.



**Sol.** Let us consider the ball as the body  $A$  and the wedge as the body  $B$ . After the impact, the ball bounces with velocity  $v_A$  and the wedge advances in horizontal direction with velocity  $v_B$ . These velocities and their  $t$  and  $n$ -components are immediately before and after the impact are shown in the following figures.



**Component along t-axis**

The ball is free to move before and after the impact, therefore its  $t$ -component of momentum conserved. Hence,  $t$ -component of velocities of the ball remains unchanged.

$$v_{At} = u_{At} = 10 \cos 37^\circ = 8 \text{ m/s} \quad \dots(\text{i})$$

**Momentum Conservation**

In the horizontal direction, there is no external force on both the bodies. Therefore horizontal component of total momentum of both the bodies remain conserved.

$$m_A u_A = m_A (v_{At} \cos 37^\circ - v_{An} \cos 53^\circ) + m_B v_B$$

$$0.25 \times 10 = 0.25 \left( 8 \times \frac{4}{5} - \frac{3v_{An}}{5} \right) + 0.72v_B \quad \dots(\text{ii})$$

**Coefficient of restitution**

Concept of coefficient of restitution  $e$  is applicable only for the  $n$ -component velocities.

$$v_{Bn} - v_{An} = e(u_{An} - u_{Bn}) \rightarrow \frac{3v_B}{5} - v_{An} = 0.8(6 - 0) \quad \dots(\text{iii})$$

From equations (i), (ii) and (iii), we obtain  $v_B = 2.0 \text{ m/s}$

## ● SYSTEM OF PARTICLES ●

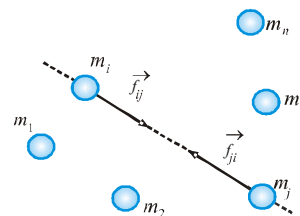
Study of kinematics enables us to explore nature of translation motion without any consideration to forces and energy responsible for the motion. Study of kinetics enables us to explore effects of forces and energy on motion. It includes Newton's laws of motion, methods of work and energy and methods of impulse and momentum. The methods of work and energy and methods of impulse and momentum are developed using equation  $\vec{F} = m\vec{a}$  together with the *methods of kinematics*. The advantage of these methods lie in the fact that they make determination of acceleration unnecessary. Methods of work and energy directly relate force, mass, velocity and displacement and enable us to explore motion between two points of space i.e. in a space interval whereas methods of impulse and momentum enable us to explore motion in a time interval. Moreover methods of impulse and momentum provides only way to analyze impulsive motion.

The work energy theorem and impulse momentum principle are developed from Newton's second law, and we have seen how to apply them to analyze motion of single particle i.e. translation motion of rigid body. Now we will further inquire into possibilities of applying these principles to a system of large number of particles or rigid bodies in translation motion.

### SYSTEM OF PARTICLES

By the term system of particles, we mean a well defined collection of several or large number of particles, which may or may not interact or be connected to each other.

As a schematic representation, consider a system of  $n$  particles of masses  $m_1, m_2, \dots, m_i, \dots, m_j, \dots$  and  $m_n$  respectively. They may be actual particles or rigid bodies in translation motion. Some of them may interact with each other and some of them may not. The particles, which interact with each other, apply forces on each other. The forces of interaction  $\vec{f}_{ij}$  and  $\vec{f}_{ji}$  between a pair of  $i^{\text{th}}$  and  $j^{\text{th}}$  particles are shown in the figure.



System of  $n$  interacting particles.

Similar to these other particles may also interact with each other. These forces of mutual interaction between the particles of the system are *internal forces* of the system.

These internal forces always exist in pairs of forces of equal magnitudes and opposite directions. It is not necessary that all the particles interact with each other; some of them, which do not interact with each other, do not apply mutual forces on each other. Other than internal forces, external forces may also act on all or some of the particles. Here by the term *external force* we mean a force that is applied on any one of the particle included in the system by some other body out-side the system.

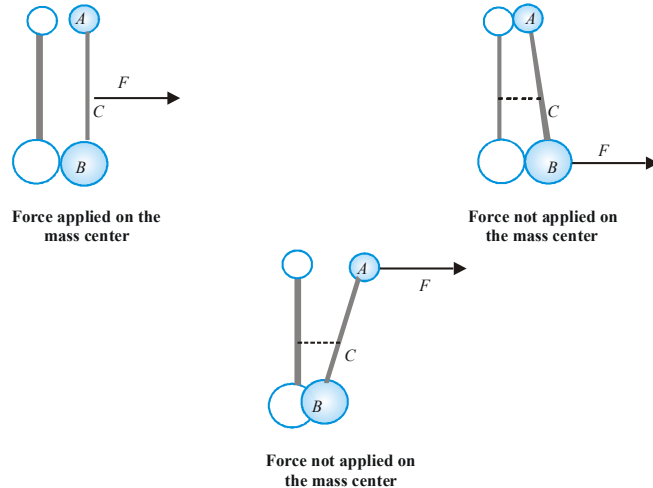
In practice we usually deal with extended bodies, which may be deformable or rigid. An extended body is also a system of infinitely large number of particles having infinitely small separations between them. When a body undergoes deformation, separations between its particles and their relative locations change. A rigid body is an extended body in which separations and relative locations of all of its particles remain unchanged under all circumstances.

### System of Particles and Mass Center

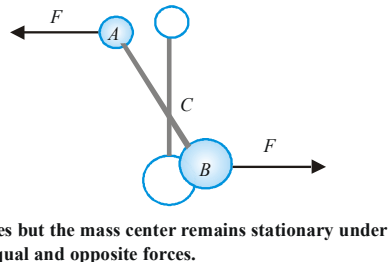
Until now we have deal with translation motion of rigid bodies, where a rigid body can be treated as a particle. When a rigid body undergoes rotation, all of its particles do not move in identical fashion, still we must treat it a system of particles in which all the particles are rigidly connected to each other. On the other hand we may have

particles or bodies not connected rigidly to each other but may be interacting with each other through internal forces. Despite the complex motion of which a system of particles is capable, there is a single point, known as **center of mass** or **mass center** (CM), whose translation motion is characteristic of the system.

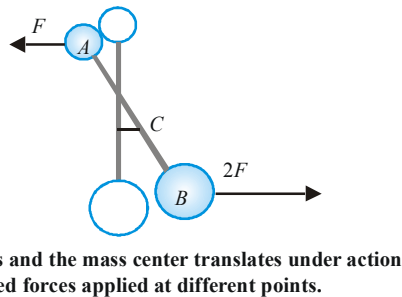
The existence of this special point can be demonstrated in the following examples dealing with a rigid body. Consider two disks  $A$  and  $B$  of unequal masses connected by a very light rigid rod. Place it on a very smooth table. Now pull it horizontally applying a force at different points. You will find a point nearer to the heavier disk, on which if the force is applied the whole assembly undergoes translation motion. Furthermore you cannot find any other point having this property. This point is the mass center of this system. We can assume that all the mass were concentrated at this point. In every rigid body we can find such a point. If you apply the force on any other point, the system moves forward and rotates but the mass center always translates in the direction of the force.



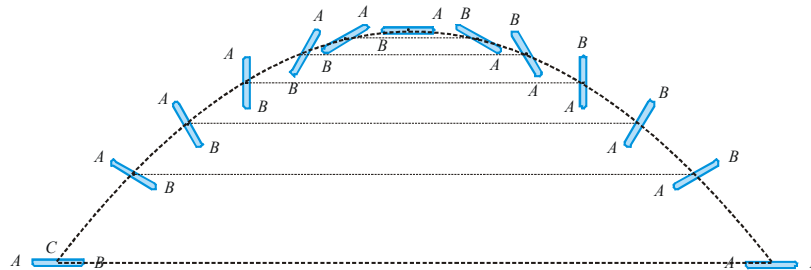
In another experiment, if two forces of equal magnitudes are applied on the disks in opposite directions, the system will rotate, but the mass center  $C$  remains stationary as shown in the following figure.



If the above experiment is repeated with both disks  $A$  and  $B$  of identical masses, the mass center will be the mid point. And if the experiment is repeated with a uniform rod, the mass center again is the mid point.



As another example let us throw a uniform rod in air holding it from one of its ends so that it rotates also. Snapshots taken after regular intervals of time are shown in the figure. The rod rotates through  $360^\circ$ . As the rod moves all of its particles move in a complex manner except the mass center  $C$ , which follows a parabolic trajectory as if it were a particle of mass equal to that of the rod and force of gravity were acting on it.



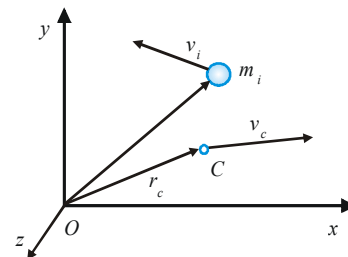
Thus mass center of a rigid body or system of particles is a point, whose translation motion under action of unbalanced forces is same as that of a particle of mass equal to that of the body or system under action the same unbalanced forces. And if different forces having a net resultant are applied at different particles, the system rotates but the mass center translates as if it were a particle of the mass same as that of the system and the net resultant were applied on it.

Concept of mass center provides us a way to look into motion of the system as a whole as superposition of translation of the mass center and motion of all the particles relative to the mass center. In case of rigid bodies all of its particles relative to the mass center can move only on circular paths because they cannot changes their separations.

The concept of mass center is used to represent gross translation of the system. Therefore total linear momentum of the whole system must be equal to the linear momentum of the system due to translation of its mass center.

### Center of Mass of System of Discrete Particles

A system of several particles or several bodies having finite separations between them is known as system of discrete particles. Let at an instant particles of such a system  $m_1, m_2, \dots, m_p, \dots$  and  $m_n$  are moving with velocities  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_i, \dots$  and  $\vec{v}_n$  at locations  $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_i, \dots$  and  $\vec{r}_n$  respectively. For the sake of simplicity only  $i^{\text{th}}$  particle and the mass center  $C$  are shown in the figure. The mass center  $C$  located at  $\vec{r}_c$  is moving with velocity  $\vec{v}_c$  at this instant.



As the mass center represents gross translation motion of the whole system, the total linear i.e. sum of linear momenta of all the particles must be equal to linear momentum of the whole mass due to translation of the mass center.

$$m_1\vec{v}_1 + m_2\vec{v}_2 + \dots + m_i\vec{v}_i + \dots + m_n\vec{v}_n = M\vec{v}_c$$

We can write the following equation in terms of masses and position vectors as an analogue to the above equation. This equation on differentiating with respect to time yields the above equation therefore can be thought as solution of the above equation.

$$m_1\vec{r}_1 + m_2\vec{r}_2 + \dots + m_i\vec{r}_i + \dots + m_n\vec{r}_n = M\vec{r}_c$$

If  $M = \sum m_i$  denotes total mass of the system, the above two equations can be written in short as

$$\sum m_i\vec{v}_i = M\vec{v}_c \quad \dots\dots(1)$$

$$\sum m_i\vec{r}_i = M\vec{r}_c \quad \dots\dots(2)$$

The above equation suggests location of mass center of a system of discrete particles.

$$\vec{r}_c = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \dots + m_i\vec{r}_i + \dots + m_n\vec{r}_n}{M} = \frac{\sum m_i\vec{r}_i}{M} \quad \dots\dots(3)$$

Cartesian coordinate  $(x_c, y_c, z_c)$  of the mass center are components of the position vector  $\vec{r}_c$  of the mass center.

$$x_c = \frac{\sum m_i x_i}{M}; y_c = \frac{\sum m_i y_i}{M}; z_c = \frac{\sum m_i z_i}{M} \quad \dots\dots(4)$$

**1. Center of Mass of Two Particle System**

- Ex.**
- (a) Find expression of position vector of mass center of a system of two particles of masses  $m_1$  and  $m_2$  located at position vectors  $\vec{r}_1$  and  $\vec{r}_2$ .
  - (b) Express Cartesian coordinates of mass center, if particle  $m_1$  at point  $(x_1, y_1)$  and particle  $m_2$  at point  $(x_2, y_2)$ .
  - (c) If you assume origin of your coordinate system at the mass center, what you conclude regarding location of the mass center relative to particles.
  - (d) Now find location of mass center of a system of two particles masses  $m_1$  and  $m_2$  separated by distance  $r$ .

- Sol.**
- (a) Consider two particles of masses  $m_1$  and  $m_2$  located at position vectors  $\vec{r}_1$  and  $\vec{r}_2$ . Let their mass center  $C$  at position vector  $\vec{r}_c$ .

From eq. , we have

$$\vec{r}_c = \frac{\sum m_i\vec{r}_i}{M} \rightarrow \vec{r}_c = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

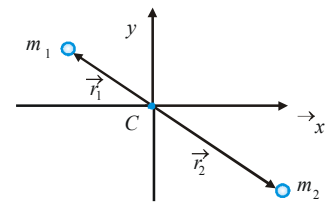
- (b) From result obtained in part (a), we have

$$x_c = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \text{ and } y_c = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}$$

- (c) If we assume origin at the mass center vector  $\vec{r}_c$  vanishes and we have

$$m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{0}$$

Since either of the masses  $m_1$  and  $m_2$  cannot be negative, to satisfy the above equation, vectors  $\vec{r}_1$  and  $\vec{r}_2$  must have opposite signs. It is geometrically possible only when mass center  $C$  lies between the two particles on the line joining them as shown in the figure.

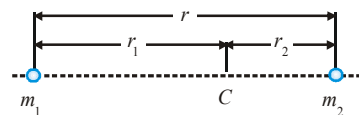


If we substitute magnitudes  $r_1$  and  $r_2$  of vectors  $\vec{r}_1$  and  $\vec{r}_2$  in the above equation, we have  $m_1r_1 = m_2r_2$ , which suggest

$$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

**Now** we conclude that mass center of two particle system lies between the two particles on the line joining them and divide the distance between them in inverse ratio of masses of the particles.

- (d) Consider two particles masses  $m_1$  and  $m_2$  at distance  $r$  from each other. Their mass center  $C$  must lie in between them on the line joining them. Let distances of these particles from the mass center are  $r_1$  and  $r_2$ . Since mass center of two particle system lies between the two particles on the line joining them and divide the distance between them in inverse ratio of masses of the particles, we can write



$$r_1 = \frac{m_2 r}{m_1 + m_2} \text{ and } r_2 = \frac{m_1 r}{m_1 + m_2}$$

## 2. Mass centre of several particles

**Ex.** Find position vectors of mass center of a system of three particle of masses 1 kg, 2 kg and 3 kg located at position vectors  $\vec{r}_1 = (4\hat{i} + 2\hat{j} - 3\hat{k})$  m,  $\vec{r}_2 = (\hat{i} - 4\hat{j} + 2\hat{k})$  m and  $\vec{r}_3 = (2\hat{i} - 2\hat{j} + \hat{k})$  m respectively.

**Sol.** From eq. , we have

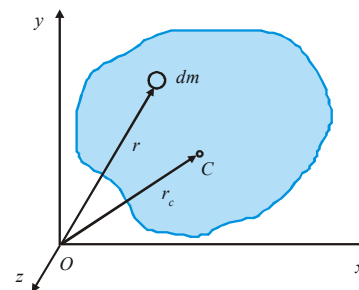
$$\vec{r}_c = \frac{\sum m_i \vec{r}_i}{M} \rightarrow \vec{r}_c = \frac{1(4\hat{i} + 2\hat{j} - 3\hat{k}) + 2(\hat{i} - 4\hat{j} + 2\hat{k}) + 3(2\hat{i} - 2\hat{j} + \hat{k})}{1 + 2 + 3} = 2\hat{i} - 2\hat{j} + \frac{2}{3}\hat{k}$$

## Center of Mass of an Extended Body or Continuous Distribution of Mass

An extended body is collection of infinitely large number of particles so closely located that we neglect separation between them and assume the body as a continuous distribution of mass. A rigid body is an extended body in which relative locations of all the particles remain unchanged under all circumstances. Therefore a rigid body does not get deformed under any circumstances.

Let an extended body is shown as a continuous distribution of mass by the shaded object in the figure. Consider an infinitely small portion of mass  $dm$  of this body. It is called a mass element and is shown at position given by position vector  $\vec{r}$ . Total mass  $M$  of the body is

$M = \int dm$ . The mass center  $C$  is assumed at position given by position vector  $\vec{r}_c$ . Position vector of centre of mass of such a body is given by the following equation.



$$\vec{r}_c = \frac{\int \vec{r} dm}{M} \quad \dots\dots(5)$$

Cartesian coordinate  $(x_c, y_c, z_c)$  of the mass center are components of the position vector  $\vec{r}_c$  of the mass center.

$$x_c = \frac{\int x dm}{M}; y_c = \frac{\int y dm}{M}; z_c = \frac{\int z dm}{M} \quad \dots\dots(6)$$

## 3. Mass centre of uniform symmetrical bodies.

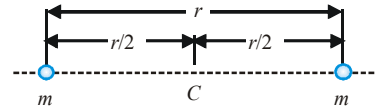
**EX.** Show that mass center of uniform and symmetric mass distributions lies on axis of symmetry.

**Sol.** For simplicity first consider a system of two identical particles and then extend the idea obtained to a straight uniform rod, uniform symmetric plates and uniform symmetric solid objects.



4. **Mass Center of a system of two identical particles**

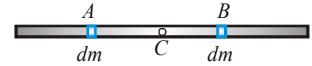
Mass center of a system of two identical particles lies at the midpoint between them on the line joining them.



5. **Mass Center of a system of a straight uniform rod**

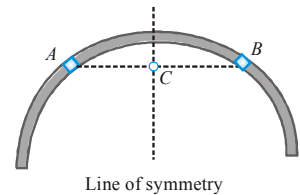
Consider two identical particles A and B at equal distances

from the center C of the rod. Mass center of system these two particles is at C. The whole rod can be assumed to be made of large number of such systems each having its mass center at the midpoint C of the rod. Therefore mass center of the whole rod must be at its midpoint.



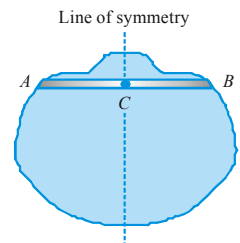
6. **Mass Center of a system of a uniform symmetric curved rod**

Consider two identical particles A and B located at equal distances from the line of symmetry. Mass center of system these two particles is at C. The whole rod can be assumed to be made of large number of such systems each having its mass center at the midpoint C of the joining them. Therefore mass center of the whole rod must be on the axis of symmetry.



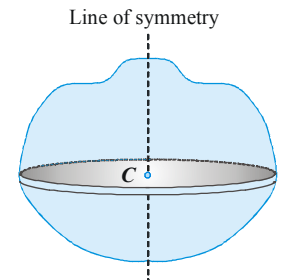
7. **Mass Center of a uniform plate (lamina)**

Consider a symmetric uniform plate. It can be assumed composed of several thin uniform parallel rods like rod AB shown in the figure. All of these rods have mass center on the line of symmetry, therefore the whole lamina has its mass center on the line of symmetry.



8. **Mass Center of a uniform symmetric solid object**

A uniform symmetric solid object occupies a volume that is made by rotating a symmetric area about its line of symmetry through  $180^\circ$ . Consider a uniform symmetric solid object shown in the figure. It can be assumed composed of several thin uniform parallel disks shown in the figure. All of these disks have mass center on the line of symmetry, therefore the whole solid object has its mass center on the line of symmetry.



9. **Mass Center of uniform bodies**

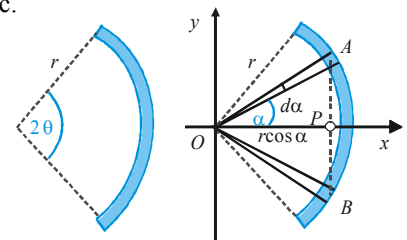
Following the similar reasoning, it can be shown that mass center of uniform bodies lies on their geometric centers.

10. **Mass Center of a system of a segment of a uniform circular rod (arc)**

**Ex.** Find location of mass center of a thin uniform rod bent into shape of an arc.

**Sol.** Consider a thin rod of uniform line mass density  $\lambda$  (mass per unit length) and radius  $r$  subtending angle  $2\theta$  on its center  $O$ .

The angle bisector  $OP$  is the line of symmetry, and mass center lies on it. Therefore if we assume the angle bisector as one of the coordinate axes say  $x$ -axis,  $y$ -coordinate of mass center becomes zero.



## PHYSICS FOR JEE MAIN & ADVANCED

Let two very small segments  $A$  and  $B$  located symmetric to the line of symmetry ( $x$ -axis). Mass center of these two segments is on  $P$  at a distance  $x = r \cos \alpha$  from center  $O$ . Total mass of these two elements is  $dm = 2\lambda r d\alpha$ . Now using eq. , we have

$$x_c = \frac{\int x dm}{M} \rightarrow x_c = \frac{\int_{-\theta}^{\theta} (r \cos \alpha)(2\lambda r d\alpha)}{\lambda r \theta} = \frac{r \sin \theta}{\theta}$$

Mass center of a thin uniform arc shaped rod of radius  $r$  subtending angle  $2\theta$  at the center lies on its angle bisector at distance  $OC$  from the center.

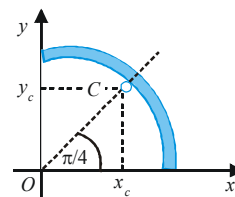
$$OC = \frac{r \sin \theta}{\theta}$$

**Ex.** Find coordinates of mass center of a quarter ring of radius  $r$  placed in the first quadrant of a Cartesian coordinate system, with centre at origin.

**Sol.** Making use of the result obtained in the previous example, distance

$$OC \text{ of the mass center from the center is } OC = \frac{r \sin(\pi/4)}{\pi/4} = \frac{2\sqrt{2}r}{\pi}$$

Coordinates of the mass center  $(x_c, y_c)$  are  $\left(\frac{2r}{\pi}, \frac{2r}{\pi}\right)$

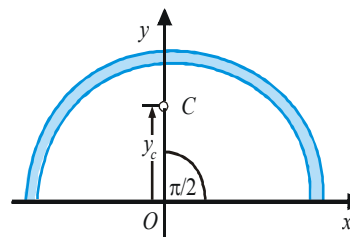


**Ex.** Find coordinates of mass center of a semicircular ring of radius  $r$  placed symmetric to the  $y$ -axis of a Cartesian coordinate system.

**Sol.** The  $y$ -axis is the line of symmetry, therefore mass center of the ring lies on it making  $x$ -coordinate zero.

Distance  $OC$  of mass center from center is given by the result obtained in example 4. Making use of this result, we have

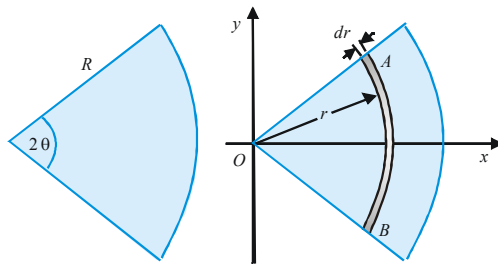
$$OC = \frac{r \sin \theta}{\theta} \rightarrow y_c = \frac{r \sin(\pi/2)}{\pi/2} = \frac{2r}{\pi}$$



### 11. Mass Center of a sector of a uniform circular plate

**Ex.** Find location of mass center of a sector of a thin uniform plate.

**Sol.** Consider a sector of a thin uniform plate of surface mass density  $\sigma$  (mass per unit area) and radius  $r$  subtending angle  $2\theta$  on its center.



Let a thin arc of radius  $r$  and width  $dr$  be an infinitely small part of the sector. Mass  $dm$  of the arc  $AB$  equals to product of mass per unit area and area of the arc.

$$dm = \sigma(2r\theta dr) = 2\sigma r\theta dr$$

Due to symmetry mass center of this arc must be on the angle bisector i.e. on  $x$ -axis at distance  $x = \frac{r \sin \theta}{\theta}$ .

Using above two information in eq. , we obtain the mass center of the sector.

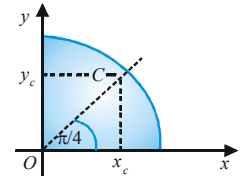
$$x_c = \frac{\int x dm}{M} \rightarrow x_c = \frac{\int_0^R \left(\frac{r \sin \theta}{\theta}\right) (\sigma r \theta dr)}{\sigma(\text{Area of the sector})} = \frac{\int_0^R \left(\frac{r \sin \theta}{\theta}\right) (2\sigma r \theta dr)}{\sigma R^2} = \frac{2r \sin \theta}{3\theta}$$

**Ex.** Find coordinates of mass center of a quarter sector of a uniform disk of radius  $r$  placed in the first quadrant of a Cartesian coordinate system with centre at origin.

**Sol.** Making use of the result obtained in the previous example, distance  $OC$  of the mass center from the center is

$$OC = \frac{2r \sin(\pi/4)}{3\pi/4} = \frac{4\sqrt{2}r}{3\pi}$$

Coordinates of the mass center  $(x_c, y_c)$  are  $\left(\frac{4r}{3\pi}, \frac{4r}{3\pi}\right)$

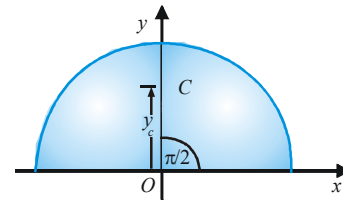


**Ex.** Find coordinates of mass center of a uniform semicircular plate of radius  $r$  placed symmetric to the  $y$ -axis of a Cartesian coordinate system, with centre at origin.

**Sol.** The  $y$ -axis is the line of symmetry, therefore mass center of the plate lies on it making  $x$ -coordinate zero.

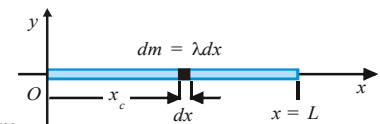
Distance  $OC$  of mass center from center is given by the result obtained in above example. Making use of this result, we have

$$OC = \frac{2r \sin \theta}{3\theta} \rightarrow y_c = \frac{2r \sin(\pi/2)}{3\pi/2} = \frac{4r}{3\pi}$$



**Ex.** Find coordinates of mass center of a nonuniform rod of length  $L$  whose linear mass density  $\lambda$  varies as  $\lambda=a+bx$ , where  $x$  is the distance from the lighter end.

**Sol.** Assume the rod lies along the  $x$ -axis with its lighter end on the origin to make mass distribution equation consistent with coordinate system.



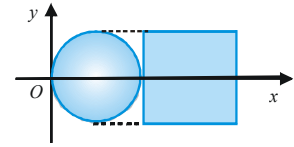
Making use of eq. , we have 
$$x_c = \frac{\int x dm}{M} \rightarrow x_c = \frac{\int_0^L x \lambda dx}{\int_0^L \lambda dx} = \frac{\int_0^L x (ax + b) dx}{\int_0^L (ax + b) dx} = \frac{(2aL + 3b)L}{3(aL + 2b)}$$

## PHYSICS FOR JEE MAIN & ADVANCED

### 12. Mass Center of composite bodies

**Ex.** A composite body is made of joining two or more bodies. Find mass center of the following composite body made by joining a uniform disk of radius  $r$  and a uniform square plate of the same mass per unit area.

**Sol.** To find mass center the component bodies are assumed particle of masses equal to corresponding bodies located on their respective mass centers. Then we use equation to find coordinates of the mass center of the composite body.



To find mass center of the composite body, we first have to calculate masses of the bodies, because their mass distribution is given.

If we denote surface mass density (mass per unit area) by  $\sigma$ , masses of the bodies are

$$\text{Mass of the disk } m_d = \text{Mass per unit area} \times \text{Area} = \sigma(\pi r^2) = \sigma\pi r^2$$

$$\text{Mass of the square plate } m_p = \text{Mass per unit area} \times \text{Area} = \sigma(r^2) = \sigma r^2$$

$$\text{Location of mass center of the disk } x_d = \text{Center of the disk} = r \text{ and } y_d = 0$$

$$\text{Location of mass center of the square plate } x_p = \text{Center of the surface plate} = 3r \text{ and } y_d = 0$$

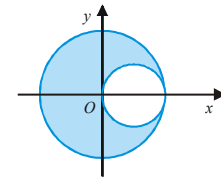
Using eq. , we obtain coordinates  $(x_c, y_c)$  of the composite body.

$$x_c = \frac{m_d x_d + m_s x_s}{m_d + m_s} = \frac{r(\pi + 3)}{(\pi + 1)} \quad \text{and} \quad y_c = \frac{m_d x_d + m_s x_s}{m_d + m_s} = 0$$

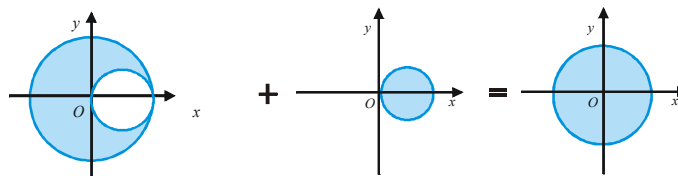
Coordinates of the mass center are  $\left( \frac{r(\pi + 3)}{(\pi + 1)}, 0 \right)$

### 13. Mass Center of truncated bodies

**Ex.** A truncated body is made by removing a portion of a body. Find mass center of the following truncated disk made by removing disk of radius equal to half of the original disk as shown in the figure. Radius of the original uniform disk is  $r$ .



**Sol.** To find mass center of truncated bodies we can make use of superposition principle that is, if we add the removed portion in the same place we obtain the original body. The idea is illustrated in the following figure.



The removed portion is added to the truncated body keeping their location unchanged relative to the coordinate frame.

Denoting masses of the truncated body, removed portion and original body by  $m_{tb}$ ,  $m_{rp}$  and  $m_{ob}$  and location of their mass centers by  $x_{tb}$ ,  $x_{rp}$  and  $x_{ob}$ , we can write  $m_{tb}x_{tb} + m_{rp}x_{rp} = m_{ob}x_{ob}$

From the above equation we obtain position co-ordinate  $x_{tb}$  of the mass center of the truncated body.

$$x_{tb} = \frac{m_{ob}x_{ob} - m_{rp}x_{rp}}{m_{tb}} \quad \dots\dots\dots(1)$$

Denoting mass per unit area by  $\sigma$ , we can express the masses  $m_{tb}$ ,  $m_{rp}$  and  $m_{ob}$ .

Mass of truncated body  $m_{tb} = \sigma \left\{ \pi \left( r^2 - \frac{r^2}{4} \right) \right\} = \frac{3\sigma\pi r^2}{4}$

Mass of the removed portion  $m_{rp} = \frac{\sigma\pi r^2}{4}$

Mass of the original body  $m_{ob} = \sigma\pi r^2$

Mass center of the truncated body  $x_{tb}$

Mass center of the removed portion  $x_{rp} = \frac{r}{2}$

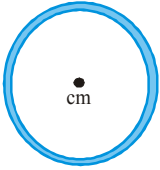
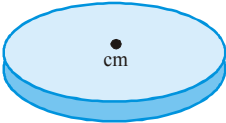
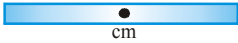
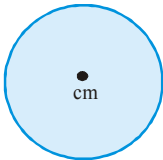
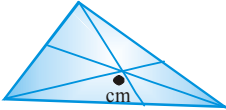

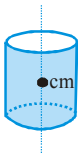
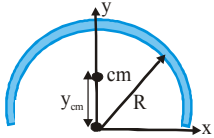
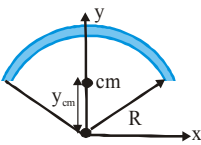
Mass center of the original body  $x_{ob} = 0$

Substituting the above values in equation (1), we obtain the mass the center of the truncated body.

$$x_{tb} = \frac{m_{ob}x_{ob} - m_{rp}x_{rp}}{m_{tb}} = \frac{(\sigma\pi r^2) \times 0 - \left(\frac{\sigma\pi r^2}{4}\right) \left(\frac{r}{2}\right)}{\frac{3\sigma\pi r^2}{4}} = -\frac{r}{6}$$

Mass center of the truncated body is at point  $\left(-\frac{r}{6}, 0\right)$

**PHYSICS FOR JEE MAIN & ADVANCED**

Body	Shape of Body	Position of centre of mass
Uniform Ring		Centre of ring
Uniform Disc		Centre of disc
Uniform Rod		Centre of Rod
Solid sphere/ hollow sphere		Centre of sphere
Triangular plane lamina		Point of intersection of the medians of the triangle i.e. centroid
Plane lamina in the form of a square or rectangle or parallelogram		Point of intersection of diagonals
Hollow/solid cylinder		Middle point of the axis of cylinder
Half ring		$y_{cm} = \frac{2R}{\pi}$
Segment of a ring		$y_{cm} = \frac{R \sin \theta}{\theta}$

Body	Shape of Body	Position of centre of mass
Half disc		$y_{cm} = \frac{4R}{3\pi}$
Sector of a disc (plate)		$y_{cm} = \frac{2R\sin\theta}{3\theta}$
Hollow hemisphere		$y_{cm} = \frac{R}{2}$
Solid hemisphere		$y_{cm} = \frac{3R}{8}$
Hollow cone		$y_{cm} = \frac{h}{3}$
Solid cone		$y_{cm} = \frac{h}{4}$

**Center of Mass Frame of Reference or Centroidal Frame**

Center of mass frame of reference or centroidal frame is reference frame assume attached with the mass center of the system at its origin. It moves together with the mass center.

It is a special frame and presents simple interpretations and solutions to several phenomena. Let us first discuss some of its fundamental properties.

In centroidal frame center of mass is assumed at the origin, therefore position vector, velocity and acceleration of the mass center in centroidal frame all become zero.

- Sum of mass moments in centroidal frame vanishes.

Mass moment of a particle is product of mass of the particle and its position vector.

$$\sum m_i \vec{r}_{i/c} = \vec{0} \text{ or } m_1 \vec{r}_{1/c} + m_2 \vec{r}_{2/c} + \dots + m_i \vec{r}_{i/c} + \dots + m_n \vec{r}_{n/c} = \vec{0} \quad \dots\dots(7)$$

- Total linear momentum of the system in centroidal frame vanishes..

$$\sum m_i \vec{v}_{i/c} = \vec{0} \text{ or } m_1 \vec{v}_{1/c} + m_2 \vec{v}_{2/c} + \dots + m_i \vec{v}_{i/c} + \dots + m_n \vec{v}_{n/c} = \vec{0} \quad \dots\dots(8)$$

## PHYSICS FOR JEE MAIN & ADVANCED

### Ex. Motion of Mass Center in One Dimension

A jeep of mass 2400 kg is moving along a straight stretch of road at 80 km/h. It is followed by a car of mass 1600 kg moving at 60 km/h.

- (a) How fast is the center of mass of the two cars moving?  
 (b) Find velocities of both the vehicles in centroidal frame.

**Sol.** (a) Velocity of the mass center 
$$\vec{v}_c = \frac{m_{jeep} \vec{v}_{jeep} + m_{car} \vec{v}_{car}}{m_{jeep} + m_{car}}$$

Assuming direction of motion in the positive  $x$ -direction, we have

$$\vec{v}_c = \frac{m_{jeep} \vec{v}_{jeep} + m_{car} \vec{v}_{car}}{m_{jeep} + m_{car}} \rightarrow \vec{v}_c = \frac{2400 \times 80 + 1600 \times 60}{2400 + 1600} = 72 \text{ km/h}$$

- (b) Velocity of the jeep in centroidal frame  $v_{jeep/c} = 80 - 72 = 8 \text{ km/h}$  in positive  $x$ -direction.

Velocity of the car in centroidal frame  $v_{car/c} = 60 - 72 = -12 \text{ km/h}$

12 km/h negative  $x$ -direction direction.

### Ex. Motion of Mass Center in Vector Form

A 2.0 kg particle has a velocity of  $\vec{v}_1 = (2.0\hat{i} - 3.0\hat{j}) \text{ m/s}$ , and a 3.0 kg particle has a velocity

$\vec{v}_2 = (1.0\hat{i} + 6.0\hat{j}) \text{ m/s}$ .

- (a) How fast is the center of mass of the particle system moving?  
 (b) Find velocities of both the particles in centroidal frame.

**Sol.** (a) Velocity of the mass center 
$$\vec{v}_c = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

$$\vec{v}_c = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} \rightarrow \vec{v}_c = \frac{2(2.0\hat{i} - 3.0\hat{j}) + 3(1.0\hat{i} + 6.0\hat{j})}{2 + 3} = (1.4\hat{i} + 2.4\hat{j}) \text{ m/s}$$

- (b) Velocity of the first particle in centroidal frame

$$\vec{v}_{1/c} = \vec{v}_1 - \vec{v}_c \rightarrow \vec{v}_{1/c} = (2.0\hat{i} - 3.0\hat{j}) - (1.4\hat{i} + 2.4\hat{j}) = 0.6(\hat{i} + \hat{j}) \text{ m/s}$$

Velocity of the second particle in centroidal frame

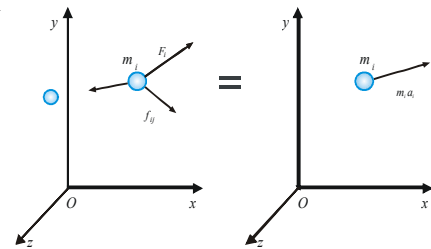
$$\vec{v}_{2/c} = \vec{v}_2 - \vec{v}_c \rightarrow \vec{v}_{2/c} = (1.0\hat{i} + 6.0\hat{j}) - (1.4\hat{i} + 2.4\hat{j}) = -(0.4\hat{i} + 3.6\hat{j}) \text{ m/s}$$

## APPLICATION OF NEWTON'S LAWS OF MOTION TO A SYSTEM OF PARTICLES

In order to write equation of motion for a system of particles, we begin by applying Newton's second law to an individual particle.

Consider  $i^{\text{th}}$  particle of mass  $m_i$ . Internal force applied on it by  $j^{\text{th}}$  particle is shown by  $\vec{f}_{ij}$ . Other particles of the system may also apply internal forces on it. One of them is shown in the figure by an unlabeled vector. In addition to these internal forces, external forces may also be applied on it by bodies outside the system. Resultant of all these

external forces is shown by vector  $\vec{F}_i$ . If under the action of these forces this particles has acceleration  $\vec{a}_i$  relative to an inertial frame  $Oxyz$ , its free body diagram and kinetic diagram can be represented by the following figure and





Newton's second law can be written by the following equation.

$$\vec{F}_i + \sum \vec{f}_{ij} = m_i \vec{a}_i$$

In similar fashion, we can write Newton's second law for all the particles of the system. These equations are

For 1<sup>st</sup> particle  $\vec{F}_1 + \sum \vec{f}_{1j} = m_1 \vec{a}_1$

For 2<sup>nd</sup> particle  $\vec{F}_2 + \sum \vec{f}_{2j} = m_2 \vec{a}_2$

.....

For  $i^{\text{th}}$  particle  $\vec{F}_i + \sum \vec{f}_{ij} = m_i \vec{a}_i$

.....

For  $n^{\text{th}}$  particle  $\vec{F}_n + \sum \vec{f}_{nj} = m_n \vec{a}_n$

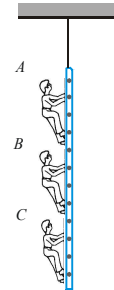
Every internal force  $\vec{f}_{ij}$  on particle  $m_i$  due to particle  $m_j$  and  $\vec{f}_{ji}$  on the particle  $m_j$  due to particle  $m_i$  constituting Newton's third law pair must be equal in magnitude and opposite in direction, therefore the sum all these internal forces for all the particles must be zero. Keeping this fact in mind and denoting the mass of the whole system by  $M$  and acceleration of the mass center  $C$  by  $\vec{a}_c$  relative to the inertial frame, Newton's second law representing translation motion of the system of particles can be represented by the following equation.

$$\sum \vec{F}_i = \sum (m_i \vec{a}_i) = M \vec{a}_c \quad \text{.....(9)}$$

$$\sum \vec{F}_i = M \vec{a}_c = \frac{d\vec{p}_c}{dt} \quad \text{.....(10)}$$

**Ex. Newton' Laws of Motion and System of Particles**

A ladder of mass 20 kg is hanging from ceiling as shown in figure. Three men  $A, B$  and  $C$  of masses 40 kg, 60 kg, and 50 kg are climbing the ladder. Man  $A$  is climbing with upward retardation  $2 \text{ m/s}^2$ ,  $B$  is climbing up with a constant speed of  $0.5 \text{ m/s}$  and  $C$  is climbing with upward acceleration of  $1 \text{ m/s}^2$ . Find the tension in the string supporting the ladder.



**Sol.** External forces acting on the system are weights of the men, weight of the ladder and tension supporting the ladder. Denoting masses of men  $A, B, C$  and ladder by  $m_A, m_B, m_C$  and  $m_L$ , acceleration due to gravity by  $g$ , tension in the string by  $T$  and accelerations of the men  $A, B, C$  and ladder by  $a_A, a_B, a_C$  and  $a_L$  respectively, we can write the following equation according to equation .

$$\Sigma \vec{F}_i = \Sigma (m_i \vec{a}_i) \rightarrow T - m_A g - m_B g - m_C g - m_L g = m_A a_A + m_B a_B + m_C a_C + m_L a_L$$

Substituting given values of masses  $m_A = 40 \text{ kg}$ ,  $m_B = 60 \text{ kg}$ ,  $m_C = 50 \text{ kg}$ ,  $m_L = 20 \text{ kg}$ ,

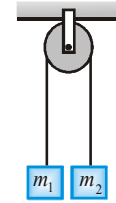
given values of accelerations  $g = 10 \text{ m/s}^2$ ,  $a_A = -2 \text{ m/s}^2$ ,  $a_B = 0 \text{ m/s}^2$ ,  $a_C = 1 \text{ m/s}^2$ , and  $a_L = 0 \text{ m/s}^2$ ,

we obtain  $T - 400 - 600 - 500 - 200 = -80 + 0 + 50 + 0$

$$T = 1670 \text{ N}$$

**Ex.** Simple Atwood Machine as System of Particles

The system shown in the figure is known as simple Atwood machine. Initially the masses are held at rest and then let free. Assuming mass  $m_2$  more than the mass  $m_1$ , find acceleration of mass center and tension in the string supporting the pulley.



**Sol.** We know that accelerations  $a_1$  and  $a_2$  are given by the following equations.

$$a_2 = \frac{m_2 - m_1}{m_2 + m_1} g \downarrow \quad \text{and} \quad a_1 = \frac{m_2 - m_1}{m_2 + m_1} g \uparrow$$

Making use of eq. , we can find acceleration  $a_c$  of the mass center. We denote upward direction positive and downward direction negative signs respectively.

$$M\vec{a}_c = \Sigma(m_i\vec{a}_i) \rightarrow (m_1 + m_2)a_c = m_1a_1 - m_2a_2$$

Substituting values of accelerations  $a_1$  and  $a_2$ , we obtain

$$a_c = \frac{2m_1m_2 - (m_1^2 + m_2^2)}{(m_1 + m_2)^2} g$$

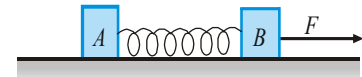
To find tension  $T$  in the string supporting the pulley, we again use eq. (9)

$$\Sigma\vec{F}_i = M\vec{a}_c \rightarrow T - m_1g - m_2g = (m_1 + m_2)a_c$$

Substituting expression obtained for  $a_c$ , we have

$$T = \frac{4m_1m_2}{m_1 + m_2} g$$

**Ex.** Two blocks each of mass  $m$ , connected by an un-stretched spring are kept at rest on a frictionless horizontal surface. A constant force  $F$  is applied on one of the blocks pulling it away from the other as shown in figure.



- (a) Find acceleration of the mass center.
- (b) Find the displacement of the centre of mass as function of time  $t$ .
- (c) If the extension of the spring is  $x_0$  at an instant  $t$ , find the displacements of the two blocks relative to the ground at this instant.

**Sol.** (a) Forces in vertical direction on the system are weights of the blocks and normal reaction from the ground. They balance themselves and have no net resultant. The only external force on the system is the applied force  $F$  in the horizontal direction towards the right.

$$\Sigma\vec{F}_i = M\vec{a}_c \rightarrow F = (m + m)a_c$$

$$a_c = \frac{F}{2m} \text{ towards right}$$

(b) The mass center moves with constant acceleration, therefore its displacement in time  $t$  is given by equation of constant acceleration motion.

$$x = ut + \frac{1}{2}at^2 \rightarrow x_c = \frac{Ft^2}{4m}$$

- (c) Positions  $x_A$  and  $x_B$  of particles  $A$  and  $B$  forming a system and position  $x_c$  mass center are obtained by following eq.

$$M\vec{r}_c = \sum m_i \vec{r}_i$$

Substituting values we obtain  $2mx_c = mx_A + mx_B$        $x_c = \frac{x_A + x_B}{2}$

Now using result obtained in part (b), we have       $x_A + x_B = \frac{Ft^2}{2m}$

Extension in the spring at this instant is  $x_o = x_B - x_A$

From the above two equations, we have       $x_A = \frac{1}{2} \left( \frac{Ft^2}{2m} - x_o \right)$  and  $x_B = \frac{1}{2} \left( \frac{Ft^2}{2m} + x_o \right)$

### Application of Methods of Impulse and Momentum to a System of Particles

In a phenomenon, when a system changes its configuration, some or all of its particles change their respective locations and momenta. Sum of linear momenta of all the particles equals to the linear momentum due to translation of mass center. Principle of impulse and moment suggests net impulse of all the external forces equals to change in momentum of mass center.

$$\sum \int \vec{F}_i dt = \vec{p}_{cf} - \vec{p}_{ci} \quad \dots\dots(11)$$

### Conservation of Linear momentum

The above event suggests that total linear momentum of a system of particle remains conserved in a time interval in which impulse of external forces is zero.

Total momentum of a system of particles cannot change under the action of internal forces and if net impulse of the external forces in a time interval is zero, the total momentum of the system in that time interval will remain conserved.

$$\sum \vec{p}_{initial} = \sum \vec{p}_{final} \text{ or } \vec{p}_{ci} = \vec{p}_{cf} \quad \dots\dots(12)$$

The above statement is known as the *principle of conservation of momentum*.

Since force, impulse and momentum are vectors, component of momentum of a system in a particular direction is conserved, if net impulse of all external forces in that direction vanishes.

During an event the net impulse of external forces in a direction is zero in the following cases.

- When no external force acts in a particular direction on any of the particles or bodies.
- When resultant of all the external forces acting in a particular direction on all the particles or bodies is zero.
- In impulsive motion, where time interval is negligibly small, the direction in which no impulsive forces act.

### No external force: Stationary mass relative to an inertial frame remains at rest

**Ex.** A man of mass  $m$  is standing at on end of a plank of mass  $M$ . The length of the plank is  $L$  and it rests on a frictionless horizontal ground. The man walks to the other end of the plank. Find displacement of the plank and man relative to the ground.



**Sol.** Denoting  $x$ -coordinates of the man, mass center of plank and mass center of the man-plank system by  $x_m$ ,  $x_p$  and  $x_c$ , we can write the following equation.

$$(\sum m_i) \vec{r}_c = \sum m_i \vec{r}_i \rightarrow (m + M) \vec{x}_c = m \vec{x}_m + M \vec{x}_p$$

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Net force on the system relative to the ground is zero. Therefore mass center of the system which is at rest before the man starts walking, remains at rest ( $\Delta\vec{x}_c = 0$ ) after while the man walks on the plank.

$$(\Delta\vec{x}_c = 0) \rightarrow m\Delta\vec{x}_m + M\Delta\vec{x}_p = 0 \quad \dots\dots(1)$$

The man walks displacement ( $\Delta\vec{x}_{m/p} = -L\hat{i}$ ) relative to the plank. Denoting displacements of the man and the plank relative to the ground by  $\Delta\vec{x}_m$  and  $\Delta\vec{x}_p$ , we can write

$$\Delta\vec{x}_{m/p} = \Delta\vec{x}_m - \Delta\vec{x}_p \rightarrow \Delta\vec{x}_m - \Delta\vec{x}_p = -L\hat{i} \quad \dots\dots(2)$$

From the above equations (1) and (2), we have

$$\Delta\vec{x}_m = -\frac{ML\hat{i}}{m+M}$$

The man moves a distance  $\frac{ML}{m+M}$  towards left relative to the ground.

$$\Delta\vec{x}_p = \frac{mL\hat{i}}{m+M}$$

The plank moves a distance  $\frac{mL}{m+M}$  towards right relative to the ground.

### Ex. No external force: Mass center moving relative to an inertial frame moves with constant velocity

Two particles of masses 2 kg and 3 kg are moving under their mutual interaction in free space. At an instant they were observed at points  $(-2\text{ m}, 1\text{ m}, 4\text{ m})$  and  $(2\text{ m}, -3\text{ m}, 6\text{ m})$  with velocities  $(3\hat{i} - 2\hat{j} + \hat{k})\text{ m/s}$  and  $(-\hat{i} + \hat{j} - 2\hat{k})\text{ m/s}$  respectively. If after 10 sec, the first particle passes the point  $(6\text{ m}, 8\text{ m}, -6\text{ m})$ , find coordinate of the point where the second particle passes at this instant?

**Sol.** System of these two particles is in free, therefore no external forces act on them. Their total linear momentum remains conserved and their mass center moves with constant velocity relative to an inertial frame.

Velocity of the mass center

$$\vec{v}_c = \frac{\sum m_i \vec{v}_i}{\sum m_i} = \frac{2(3\hat{i} - 2\hat{j} + \hat{k}) + 3(-\hat{i} + \hat{j} - 2\hat{k})}{2+3} = \frac{3\hat{i} - \hat{j} - 4\hat{k}}{5} \text{ m/s}$$

Location  $\vec{r}_{co}$  of the mass center at the instant  $t = 0\text{ s}$

$$\vec{r}_c = \frac{\sum m_i \vec{r}_i}{\sum m_i} \rightarrow \vec{r}_{co} = \frac{2(-2\hat{i} + \hat{j} + 4\hat{k}) + 3(2\hat{i} - 3\hat{j} + 6\hat{k})}{2+3} = \frac{2\hat{i} - 7\hat{j} + 26\hat{k}}{5}$$

New location  $\vec{r}_c$  of the mass center at the instant  $t = 10\text{ s}$

$$\vec{r}_c = \vec{r}_{co} + \vec{v}_c t \rightarrow \vec{r}_c = \frac{2\hat{i} - 7\hat{j} + 26\hat{k}}{5} + \frac{3\hat{i} - \hat{j} - 4\hat{k}}{5} \times 10 = \frac{32\hat{i} - 17\hat{j} - 14\hat{k}}{5}$$

New location  $(x, y, z)$  of the second particle.

$$\vec{r}_c = \frac{\sum m_i \vec{r}_i}{\sum m_i} \rightarrow \frac{32\hat{i} - 17\hat{j} - 14\hat{k}}{5} = \frac{2(6\hat{i} + 8\hat{j} - 6\hat{k}) + 3(x\hat{i} + y\hat{j} + z\hat{k})}{2+3}$$

Solving the above equation, we obtain the coordinates of the second particle  $(20/3, -11, -2/3)$

**APPLICATION OF METHODS OF WORK AND ENERGY TO A SYSTEM OF PARTICLES**

In a system of particles, all the particles occupy different locations at every instant of time and may change their locations with time. At an instant of time set of locations of all the particles of a system is known as *configuration* of the system. We say something has happened with the system only when some or all of its particles change their locations. It means that in every event or phenomena the system changes its configuration.

Methods of work and energy equips us to analyze what happens when a particle moves form one point of space to other. Now we will apply these methods to analyze a phenomenon in which a system of particle changes its configuration.

**Kinetic Energy of a System of Particle**

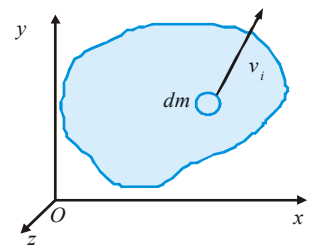
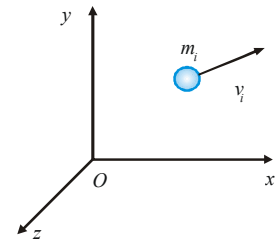
Kinetic energy of a system of particles is defined as sum of kinetic energies of all the particles of the system.

If at an instant particles of masses  $m_1, m_2, \dots, m_i, \dots, m_j, \dots, m_n$  are observed moving with velocities  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_i, \dots, \vec{v}_n$  respectively relative to a reference frame, the kinetic energy of the whole system relative to the reference frame is given by the following equation.

$$K = \frac{1}{2} \sum m_i v_i^2 \quad \dots\dots(13)$$

If the system consists of continuous distribution of mass, instead of discrete particles, expression of kinetic energy becomes

$$K = \frac{1}{2} \int v^2 dm \quad \dots\dots(14)$$



**Kinetic Energy of a System of Particle using Centroidal Frame**

Centroidal frame of reference or center of mass frame is reference frame attached with the mass center of the system.

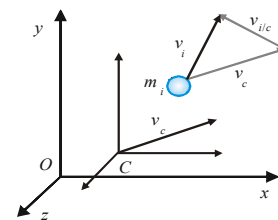
Let velocity of  $i^{\text{th}}$  particle of mass  $m_i$  is moving with velocity  $\vec{v}_i$

relative to frame  $Oxyz$ . Mass center  $C$  and hence the centroidal frame  $Cxyz$  is moving with velocity  $\vec{v}_c$ . Therefore velocity of  $i^{\text{th}}$  particle relative to the centroidal frame is  $\vec{v}_{i/c}$ .

Kinetic energy of the whole system is given by the following equation.

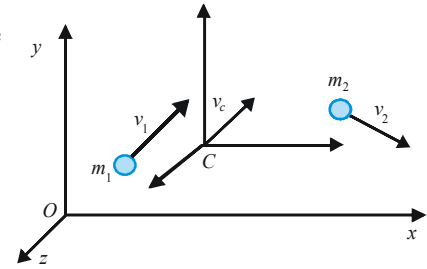
$$K = \frac{1}{2} \sum m_i v_i^2 = \frac{1}{2} \sum m_i v_c^2 + \frac{1}{2} \sum m_i v_{i/c}^2 \quad \dots\dots(15)$$

Here the first term on the right hand side is kinetic energy due to translation of the mass center and the second term is kinetic energy of the system relative to the centroidal frame.



**Kinetic Energy of a Two Particle System using Centroidal Frame**

A two particle system consists of only two particles. Let a two particle system consists of particles of masses  $m_1$  and  $m_2$  moving with velocities  $\vec{v}_1$  and  $\vec{v}_2$  relative to a frame  $Oxyz$ . Their mass center  $C$  lies on the line joining them and divides separation between them in reciprocal ratio of masses  $m_1$  and  $m_2$ . The mass center and hence the centroidal frame is moving with velocity  $\vec{v}_c$ .



Kinetic energy of this two particle system relative to a frame  $Oxyz$  is given by the following equation.

$$K = \frac{1}{2}(m_1 + m_2)v_c^2 + \frac{1}{2}\mu v_{rel}^2 \quad \text{.....(16)}$$

The first term on the right hand side is kinetic energy due to translation of the mass center and the second term is kinetic energy of the system relative to the centroidal frame.

Here symbol  $\mu$  is known as reduced mass of the two particle system and symbol  $v_{rel}$  is magnitude of velocity of either of the particles relative to the other.

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{and} \quad v_{rel} = |\vec{v}_1 - \vec{v}_2| = |\vec{v}_2 - \vec{v}_1|$$

**Work Energy Theorem for a System of Particles**

The work energy theorem can be applied to each particle of the system. For  $i^{\text{th}}$  particle of the system, we can write

$$K_{i,i} + W_{i,i \rightarrow f} = K_{i,f}$$

Here  $W_{i,i \rightarrow f}$  is total work done by all the internal forces  $\vec{f}_{ij}$  and resultant external force  $\vec{F}_i$  on the  $i^{\text{th}}$  particle, when the system goes from one configuration to other.

Adding kinetic energies of all particles, we can write kinetic energies  $K_i$  and  $K_f$  of the whole system in the initial as well as the final configuration. Adding work done  $W_{i,i \rightarrow f}$  by internal as well as external forces on every particle we find total work done  $W_{i \rightarrow f}$  by all the internal as well as external forces on the system. Now we can write work energy theorem.

$$K_i + W_{i \rightarrow f} = K_f \quad \text{.....(17)}$$

While applying the above equation to a system, care must be taken in calculating  $W_{i \rightarrow f}$ . In spite of the fact that the internal forces  $\vec{f}_{ij}$  and  $\vec{f}_{ji}$  being equal in magnitude and opposite in direction, the work done by them on the  $i^{\text{th}}$  and the  $j^{\text{th}}$  particles will not, in general, cancel out, since  $i^{\text{th}}$  and the  $j^{\text{th}}$  particles may undergo different amount of displacements.

The above description at first presents calculating of  $W_{i \rightarrow f}$  as a cumbersome task. However for systems, which we usually encounter are not as complex as a general system of large number of particles may be. Systems which we usually face to analyze have limited number of particles or bodies interacting. For these systems we can simplify the task by calculating work of conservative internal forces as decrease in potential energy of the system. Total work of internal forces other than internal conservative forces vanishes, if these forces are due connecting inextensible links or links of constant length. These forces include string tension and normal reaction at direct contacts between the bodies included I the system. Work of internal forces of the kind other than these and work of external forces, can be calculated by definition of work.

**Conservation of Mechanical Energy**

If total work of internal forces other than conservative is zero and no external forces act on a system, total mechanical energy remains conserved.

$$K_i + U_i = K_f + U_f \quad \dots\dots(18)$$

Since external forces are capable of changing mechanical energy of the system, under their presence total mechanical energy changes by amount equal to work  $W_{ext, i \rightarrow f}$  done by all the external forces.

$$W_{ext, i \rightarrow f} = E_f - E_i = (K_f + U_f) - (K_i + U_i) \quad \dots\dots(19)$$

**Total work of pseudo forces in centroidal frame.**

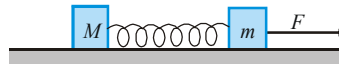
**Ex.** Show that total work done in centroidal frame on all the particles of a system by pseudo forces due to acceleration of mass center is zero.

**Sol.** Let acceleration of mass centre relative to an inertial frame is  $\vec{a}_c$ . Pseudo force on  $i^{th}$  particle of mass  $m_i$  in centroidal frame is  $(-m_i \vec{a}_c)$ . Let displacement of  $i^{th}$  particle in a time interval is  $\Delta \vec{r}_i$  relative to the centroidal frame.

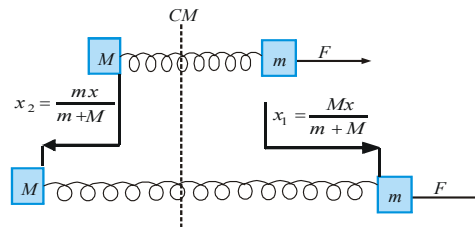
Total work of pseudo forces on all the particles in centroidal frame can now be expressed by the expression

$$\Sigma (-m_i \vec{a}_c) \cdot \Delta \vec{r}_i = -\vec{a}_c \cdot \Sigma (m_i \Delta \vec{r}_i) = -\vec{a}_c \cdot \vec{0} = 0$$

**Ex.** Two blocks of masses  $m$  and  $M$  connected by a spring are placed on frictionless horizontal ground. When the spring is relaxed, a constant force  $F$  is applied as shown. Find maximum extension of the spring during subsequent motion.



**Sol.** If we use ground as inertial frame as we usually do, solution of the problem becomes quite involved. Therefore, we prefer to use the centroidal frame, in which mass center remains at rest.



In the adjacent figure is shown horizontal position of mass center (CM) by dashed line. It remains unchanged in centroidal frame.

Mass center of two particle system divides separation between them in reciprocal ratio of the masses; therefore displacements  $x_1$  and  $x_2$  of the blocks must also be in reciprocal ratio of their masses. The extension  $x$  is sum of displacements  $x_1$  and  $x_2$  of the blocks as shown in the figure.

When extension of the spring achieves its maximum value, both the block must stop receding away from the mass center, therefore, velocities of both the blocks in centroidal frame must be zero.

During the process when spring is being extended, total work done by pseudo forces in centroidal frame become zero, negative work done by spring forces becomes equal to increase in potential energy and work done by the applied force evidently becomes  $Fx_1$ .

Using above fact in applying work energy theorem on the system relative to the centroidal frame, we obtain

$$K_i + W_{i \rightarrow f} = K_f \rightarrow 0 + W_{i \rightarrow f, springforce} + W_{i \rightarrow f, F} = 0$$

$$-\left(0 - \frac{1}{2} kx^2\right) + \frac{FMx}{m + m} = 0 \Rightarrow x = \frac{2FM}{k(m + m)}$$

**Tips & Formulas •**

**1. Impulse-Momentum theorem**

Impulse of a force is equal to the change of momentum  
force time graph area gives change in momentum.

$$\int_{t_1}^{t_2} \vec{F} dt = \Delta \vec{p}$$

**2. Collision of bodies**

The event of the process, in which two bodies either coming in contact with each other or due to mutual interaction at distance apart, affect each others motion (velocity, momentum, energy or direction of motion) is defined as a collision.

**3. In collision**

The particles come closer before collision and after collision they either stick together or move away from each other.

The particles need not come in contact with each other for a collision.

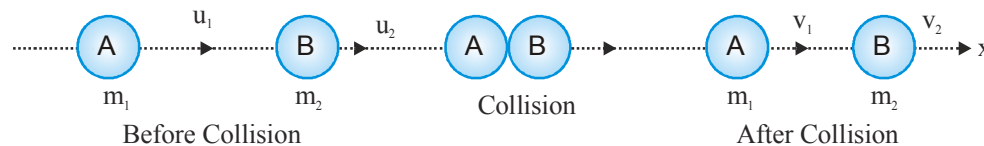
The law of conservation of linear momentum is necessarily applicable in a collision, whereas the law of conservation of mechanical energy is not.

**4. Coefficient of restitution (Newton's law)**

$$e = \frac{\text{velocity of separation along line of impact}}{\text{velocity of approach along line of impact}} = \frac{v_2 - v_1}{u_1 - u_2}$$

Value of e is 1 for elastic collision, 0 for perfectly inelastic collision and  $0 < e < 1$  for inelastic collision.

**5. Head on collision**



**6. Head on inelastic collision of two particles**

Let the coefficient of restitution for collision is e

(a) Momentum is conserved  $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$  ...**(i)**

(b) Kinetic energy is not conserved

(c) According to Newton's law  $e = \frac{v_2 - v_1}{u_1 - u_2}$  ....**(ii)**

By solving eq. **(i)** and **(ii)** :

$$v_1 = \left( \frac{m_1 - em_2}{m_1 + m_2} \right) u_1 + \left( \frac{(1+e)m_2}{m_1 + m_2} \right) u_2 = \frac{m_1 u_1 + m_2 u_2 - m_2 e(u_1 - u_2)}{m_1 + m_2}$$

$$v_2 = \left( \frac{m_2 - em_1}{m_1 + m_2} \right) u_2 + \left( \frac{(1+e)m_1}{m_1 + m_2} \right) u_1 = \frac{m_1 u_1 + m_2 u_2 - m_1 e(u_2 - u_1)}{m_1 + m_2}$$

**7. Elastic Collision (e = 1)**

**If the two bodies are of equal masses :**  $m_1 = m_2 = m$ ,  $v_1 = u_2$  and  $v_2 = u_1$

Thus, if two bodies of equal masses undergo elastic collision in one dimension, then after the collision, the bodies will exchange their velocities.



**If the mass of a body is negligible as compared to other.**

If  $m_1 \gg m_2$  and  $u_2 = 0$  then  $v_1 = u_1, v_2 = 2u_1$  when a heavy body A collides against a light body B at rest, the body A should keep on moving with same velocity and the body B will move with velocity double that of A. If  $m_2 \gg m_1$  and  $u_2 = 0$  then  $v_2 = 0, v_1 = -u_1$

When light body A collides against a heavy body B at rest, the body A should starts moving with same velocity just in opposite direction while the body B should practically remains at rest.

**8. Loss in kinetic energy in inelastic collision**

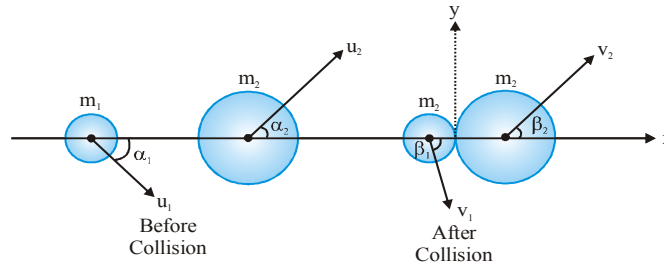
$$\Delta K = \frac{m_1 m_2}{2(m_1 + m_2)} (1 - e^2) (u_1 - u_2)^2$$

**9. Oblique Collision**

Conserving the momentum of system in directions along normal (x-axis in our case) and tangential (y axis our case)

$$m_1 u_1 \cos \alpha_1 + m_2 u_2 \cos \alpha_2 = m_1 v_1 \cos \beta_2 - m_1 v_1 \sin \beta_1$$

$$m_2 u_2 \sin \alpha_2 - m_1 u_1 \sin \alpha_1 = m_2 v_2 \sin \beta_2 - m_1 v_1 \sin \beta_1$$



Since no force is acting on  $m_1$  and  $m_2$  along the tangent (i.e. y-axis) the individual momentum of  $m_1$  and  $m_2$  remains conserved.

$$m_1 u_1 \sin \alpha_1 = m_1 v_1 \sin \beta_1 \quad \& \quad m_2 u_2 \sin \alpha_2 = m_2 v_2 \sin \beta_2$$

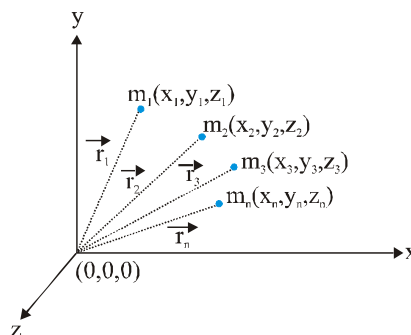
By using Newton's experimental law along the line of impact

$$e = \frac{v_2 \cos \beta_2 - v_1 \cos \beta_1}{u_1 \cos \alpha_1 - u_2 \cos \alpha_2}$$

**10. Centre of mass**

For a system of particles centre of mass is that point at which its total mass is supposed to be concentrated.

**11. Centre of mass of system of discrete particles**



Total mass of the body :  $M = m_1 + m_2 + \dots + m_n$  then

$$\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \dots}{m_1 + m_2 + m_3 + \dots} = \frac{1}{M} \sum m_i \vec{r}_i$$

co-ordinates of centre of mass :

$$x_{cm} = \frac{1}{M} \sum m_i x_i, y_{cm} = \frac{1}{M} \sum m_i y_i \text{ and } z_{cm} = \frac{1}{M} \sum m_i z_i$$

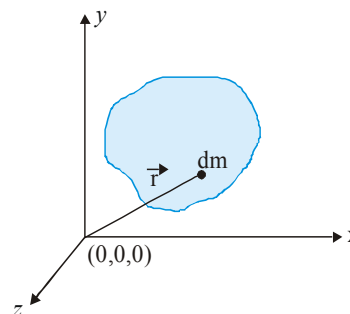
**12. Centre of mass of continuous distribution of particles**

$$\vec{R}_{cm} = \frac{1}{M} \int \vec{r} dm$$

$$x_{cm} = \frac{1}{M} \int x dm, y_{cm} = \frac{1}{M} \int y dm$$

and  $z_{cm} = \frac{1}{M} \int z dm$

x, y, z are the co-ordinate of the COM of the dm mass.



**13. The centre of mass after removal of a part of a body**

Original mass (M) – mass of the removed part (m)

$$= \{ \text{original mass}(M) \} + \{ - \text{mass of the removed part}(m) \}$$

The formula changes to :

$$x_{cm} = \frac{Mx - mx'}{M - m}; y_{cm} = \frac{My - my'}{M - m}; z_{cm} = \frac{Mz - mz'}{M - m}$$

**14. MOTION OF CENTRE OF MASS**

For a system of particles,

velocity of centre of mass  $\vec{v}_{CM} = \frac{d\vec{R}_{CM}}{dt} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \dots}{m_1 + m_2 + \dots}$

Similarly acceleration  $\vec{a}_{CM} = \frac{d}{dt} (\vec{v}_{CM}) = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \dots}{m_1 + m_2 + \dots}$

**15. Law of conservation of linear momentum**

Linear momentum of a system of particles is equal to the product of mass of the system with velocity of its centre of mass.

From Newton's second law  $\vec{F}_{ext.} = \frac{d(M\vec{v}_{CM})}{dt}$

If  $\vec{F}_{ext.} = \vec{0}$  then  $M\vec{v}_{CM} = \text{constant}$

If no external force acts on a system the velocity of its centre of mass remains constant, i.e., velocity of centre of mass is unaffected by internal forces.

**16.** Sum of mass moments about centre of mass is zero. i.e.  $\sum m_i \vec{r}_{i/cm} = \vec{0}$

**17.** A quick collision between two bodies is more violent than slow collision, even when initial and final velocities are equal because the rate of change of momentum determines that the impulsive force small or large.

**18.** Heavy water is used as moderator in nuclear reactors as energy transfer is maximum if  $m_1 = m_2$ .

**19.** Impulse momentum theorem is equivalent to Newton's second law of motion

**20.** For a system, conservation of linear momentum is equivalent to Newton's third law of motion.