HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

1. Let any point on the curve is $\left(\frac{c}{t}, ct^2\right)$

$$y=\frac{c^3}{x^2} \Rightarrow \frac{dy}{dx}=-\frac{2c^3}{x^3}=-\frac{2c^3}{c^3/t^3}$$

$$\frac{dy}{dx}=-2t^3$$

Equation of tangent is

$$y-ct^2=-2t^3\left(x-\frac{c}{t}\right)$$

For x intercept

$$0-ct^2=-2t^3\left(a-\frac{c}{t}\right) \Rightarrow \frac{c}{2t}=a-\frac{c}{t}$$

$$a=\frac{3}{2}\frac{c}{t}$$

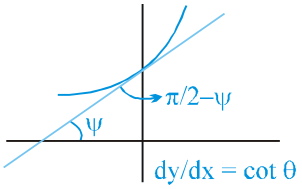
For y intercept

$$b-ct^2=-2t^3\left(0-\frac{c}{t}\right)$$

$$\Rightarrow b-ct^2=2t^2c \quad \Rightarrow b=3ct^2$$

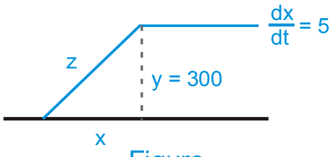
Now $\quad a^2b=\frac{9}{4}\cdot\frac{c^2}{t^2}\times 3ct^2=\frac{27c^3}{4}$

4. $\left.\frac{dy}{dx}\right]_{x=0}=k^2 \Rightarrow \tan\Psi=k^2 \Rightarrow \cot\left(\frac{\pi}{2}-\psi\right)=k^2$



$$\Rightarrow \left(\frac{\pi}{2}-\psi\right)=\cot^{-1}k^2=\sin^{-1}\frac{1}{\sqrt{1+k^4}} \Rightarrow B$$

6.



Figure

From figure $z^2=x^2+y^2$

$$z\frac{dz}{dt}=x\frac{dx}{dt}$$

If $z=500$ then $x=400 \Rightarrow 500\frac{dz}{dt}=400(5)\Rightarrow\frac{dz}{dt}=4$

10. point (2, 0) or (3, 0); $f'(2)=-1$ of $f'(3)=1$

$$\Rightarrow \theta=\frac{\pi}{2}$$

14. $\left|\frac{dx}{dt}\right|<\left|\frac{dy}{dt}\right|$

$$\left|\frac{dy}{dx}\right|>1$$

$$3y^2\frac{dy}{dx}=27 \Rightarrow \frac{dy}{dx}=\frac{9}{y^2}$$

$$\frac{9}{y^2}>1 \quad \Rightarrow \quad y^2<9$$

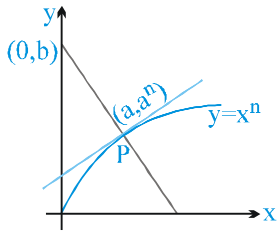
$$\Rightarrow -3<y<3 \qquad \Rightarrow -27<y^3<27$$

$$\Rightarrow -27<27x<27 \qquad \Rightarrow -1<x<1$$

21. $y=x^n$

$$\frac{dy}{dx}=n\,x^{n-1}=na^{n-1}$$

slope of normal $=-\frac{1}{na^{n-1}}$



equation of normal $y-a^n=-\frac{1}{na^{n-1}}(x-a)$

put $x=0$ to get y-intercept

$$y=a^n+\frac{1}{na^{n-2}};$$

Hence $b=a^n+\frac{1}{na^{n-2}}$

$$\lim_{a\to 0} b=\left[\begin{array}{ll} 0 & \text{if } n<2 \\ \frac{1}{2} & \text{if } n=2 \\ \infty & \text{if } n>2 \end{array}\right. \qquad \Rightarrow \text{(C)}1$$

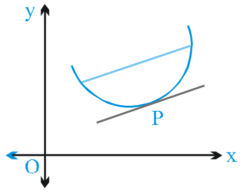
23. Length of subnormal $= y\dfrac{dy}{dx}$

$$y^2 = 8ax \quad \Rightarrow \quad 2y\frac{dy}{dx} = 8a$$

$$\therefore \quad y\frac{dy}{dx} = 4a$$

28. $f(x)=\displaystyle\int_1^x\left(t+\frac{1}{t}\right)dt \Rightarrow f'(x)=x+\frac{1}{x}$

$$\therefore \quad g(x)=x+\frac{1}{x} \text{ for } x\in\left[\frac{1}{2},3\right]$$



$$g\left(\frac{1}{2}\right)=2+\frac{1}{2}=\frac{5}{2},\quad g(3)=3+\frac{1}{3}=\frac{10}{3}$$

Let $P\equiv(c, g(c)),\ c\in\left[\frac{1}{2},3\right]$

By LMVT, $\quad g'(c)=\dfrac{g(3)-g\left(\frac{1}{2}\right)}{3-\frac{1}{2}}$

$$\therefore \quad 1-\frac{1}{c^2}=\frac{\frac{10}{3}-\frac{5}{2}}{3-\frac{1}{2}}$$

$$\Rightarrow \quad c^2=\frac{3}{2} \quad \Rightarrow c=\sqrt{\frac{3}{2}}$$

$$\therefore \quad g(c)=\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{\frac{3}{2}}}=\frac{5}{\sqrt{6}}$$

$$\therefore \quad P\equiv\left(\sqrt{\frac{3}{2}},\frac{5}{\sqrt{6}}\right)$$

31. $f(x)=\sin x-\cos x-ax+b$

$f'(x)=\cos x+\sin x-a\le 0\ \forall\ x\in R$

$\Rightarrow \quad a\ge \cos x+\sin x\ \forall\ x\in R$

$\Rightarrow \quad a\ge\sqrt{2}$

35. $f'(x)=\left(\dfrac{\sqrt{p+4}}{1-p}-1\right)5x^4-3$

It is sufficient to solve for p, the condition

f ɜ(x) $\le 0\ \forall\ x\in R$

$$\left(\frac{\sqrt{p+4}}{1-p}-1\right)5x^4-3\le 0\ \ \forall\ x\in R$$

Case - I $1-p<0 \qquad p>1$

Inequality holds true.

Case - II $1-p>0 \qquad p<1$

$$\text{Inequality holds if } \frac{\sqrt{p+4}}{1-p}-1\le 0$$

$\Rightarrow \quad p\ge-4,\ \ p+4\le(1-p)^2$

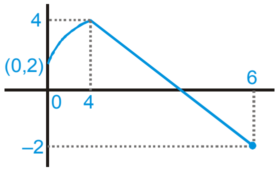
$\Rightarrow \quad p\ge-4,\ \ p^2-3p-3\ge 0$

$\Rightarrow \quad -4\le p\le\dfrac{3-\sqrt{21}}{2}$

Hence $p\in\left[-4,\dfrac{3-\sqrt{21}}{2}\right]\cup(1,\infty)$

36. $f(x)=\begin{cases}\dfrac{(2-\sqrt{x})(2+\sqrt{x})}{(2-\sqrt{x})}, & 0<x<4\\ 4, & x=4\\ 16-3x, & 4<x<6\end{cases}$

$$\Rightarrow \quad f(x)=\begin{cases}2+\sqrt{x}, & 0<x<4\\ 4, & x=4\\ 16-3x, & 4<x<6\end{cases}$$



So f(x) is continuous only

37. Using LMVT in [2, 4]

$$f'(c)=\frac{f(4)-f(2)}{4-2}=\frac{f(4)+4}{2}$$

$$f'(x)\ge 6 \ \Rightarrow \ \frac{f(4)+4}{2}\ge 6 \Rightarrow \ f(4)\ge 8$$

39. $f(-2) = f(3) = 0$

f(x) is continuous in [–2, 3] & derivable in (–2, 3) so Rolle's theorem is applicable.

so $\exists\ c \in (-2, 3)$ such that $f'(c) = 0$

$$\Rightarrow \frac{2c^3 - 5c^2 + 4c - 1}{(c-1)^2} = 0 \quad \Rightarrow \quad c = 1/2$$

44. Using LMVT for f in [1, 2]

$$\forall\ c \in (1, 2) \quad \frac{f(2) - f(1)}{2 - 1} = f'(c) \le 2$$

$$f(2) - f(1) \le 2 \quad \Rightarrow \quad f(2) \le 4 \quad ....\textbf{(1)}$$

again using LMVT in [2, 4]

$$\forall\ d \in (2, 4) \quad \frac{f(4) - f(2)}{4 - 2} = f'(d) \le 2$$

$$\therefore \quad f(4) - f(2) \le 4$$

$$8 - f(2) \le 4$$

$$4 \le f(2) \quad \Rightarrow \quad f(2) \ge 4 \quad ....\textbf{(2)}$$

from **(1)** and **(2)** $\quad f(2) = 4$

47. $f(x) = 3\tan x + x^3 - 2,\ f'(x) = 3(\sec^2 x + x^2) > 0$

$\Rightarrow$ f(x) is increasing in $\forall\ x \in (0, \pi/4)$

$$f(0) < 0\ \&\ f\left(\frac{\pi}{4}\right) > 0$$

$\Rightarrow$ f(x) = 0 has exactly one root in $\left(0, \frac{\pi}{4}\right)$.

49. For $x \in (0, 2)$

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \qquad (\text{Here } c \in (0, x))$$

$\Rightarrow f(x) = 2.f'(x)$

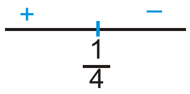
$f(x) \le 1$

52. $f(x) = x^{25}(1-x)^{75}$

$f'(x) = 25.x^{24}(1-x)^{75} - 75.(1-x)^{74}.x^{25}$

$= 25.x^{24}(1-x)^{74}\{1-x-3x\}$

$= 25x^{24}(1-x)^{74}(1-4x)$



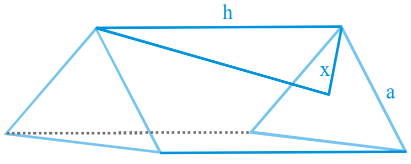
55. $\ell^2 = h^2 + x^2$

Area of base (triangle) is $\frac{\sqrt{3}}{4}a^2$

$$3x = \frac{\sqrt{3}}{2}a$$

$$\text{Volume } V = h\frac{\sqrt{3}}{4}a^2$$

$$= h\frac{\sqrt{3}}{4}.4.3.x^2$$



Figure

$$= 3\sqrt{3}\,h(\ell^2 - h^2)$$

$$\frac{dV}{dh} = 3\sqrt{3}\,(\ell^2 - 3h^2)$$

V is maximum when $h = \frac{\ell}{\sqrt{3}}$.

57. for $x \ge d,\ f(x) = 4x - (a + b + c + d)$

$\Rightarrow f'(x) = 4 \quad \Rightarrow \quad f(x)$ is $\uparrow$

for $c \le x < d,\ f(x) = 2x - a - b - c + d$

$\Rightarrow f'(x) = 2 \quad \Rightarrow \quad f(x)$ is $\uparrow$

for $b \le x < c$,

$f(x) = x - a + x - b + c - x + d - x = c + d - a - b$

$\Rightarrow f'(x) = 0$

$\Rightarrow f(x) = \text{constant}$

for $a \le x < b$,

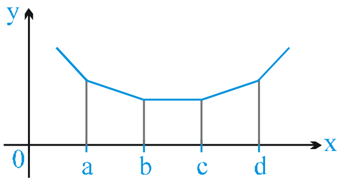
$f(x) = x - a + b - x + c - x + d - x = b + c + d - a - 2x$

$\Rightarrow f'(x) = -2$

$\Rightarrow f(x)$ is $\downarrow$

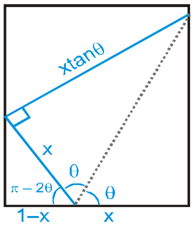
for $x < a$ f(x) is again decreasing. Hence f(x) is least when $b \le x \le c$.

As shown :



59. $f'(x) = x(2^2 + 4^2.x^2 + 6^2.x^4 + ...+ 100^2.x^{98}\}$

60.



Figure

$$\cos(\pi - 2\theta) = \frac{1-x}{x}$$

.Area $= \frac{1}{2}\, x.x\tan\theta = \frac{1}{8\sin^3\theta.\cos\theta}$

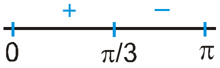
Minimum area when $\tan\theta = \sqrt{\frac{3}{1}} \Rightarrow x = \frac{2}{3}$

61. $f(x) = \frac{1}{2}\{1 + \cos x\} \sin x$

$$= \frac{\sin x}{2} + \frac{\sin 2x}{4}$$

$$f'(x) = \frac{\cos x}{2} + \frac{\cos 2x}{2}$$

$$= \frac{2\cos^2 x + \cos x - 1}{2} = \frac{(2\cos x - 1)(\cos x + 1)}{2}$$



$f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{8} = \frac{3\sqrt{3}}{8}$ = maximum value

$f(0) = f(\pi)$ = minimum value

62. obviously f is increasing and g is decreasing in (x_1, x_2)

hence $f\left(g(\alpha^2 - 2\alpha)\right) > f\left(g(3\alpha - 4)\right)$ as f is increasing

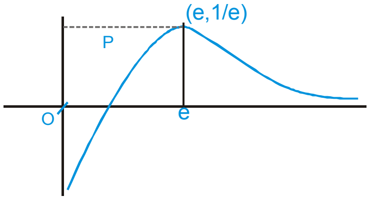
$\Rightarrow$ $g(a^2 - 2\alpha) > g(3\alpha - 4)$

$\therefore$ $\alpha^2 - 2\alpha < 3\alpha - 4$ as g is decreasing

$\alpha^2 - 5\alpha + 4 < 0$

$(\alpha - 1)(\alpha - 4) < 0 \Rightarrow \alpha \in (1, 4)$

63. $A = \frac{\ell n\, x}{x}$



$$A' = \frac{1 - \ell n\, x}{x^2} = 0 \text{ at } x = e$$

$$A'' = \frac{-x - 2x(2 - \ell n x)}{x^2}$$

$A'' < 0$ at $x = e$ $\Rightarrow$ maxima

$$A_{\max|x=e} = \frac{1}{e}$$

67. $f(x) = \frac{t + 3x - x^2}{x - 4}$;

$$f'(x) = \frac{(x-4)(3-2x) - (t + 3x - x^2)}{(x-4)^2}$$

for maximum or minimum, $f'(x) = 0$

$$-2x^2 + 11x - 12 - t - 3x + x^2 = 0$$

$$-x^2 + 8x - (12 + t) = 0$$

for one M and m,

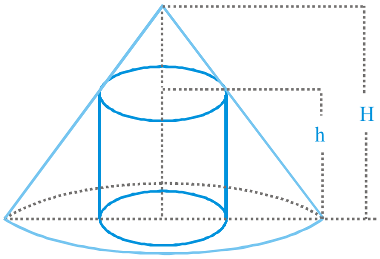
$D > 0$

$64 - 4(12 + t) > 0$

$16 - 12 - t > 0 \Rightarrow 4 > t$ or $t < 4$

68. $\frac{H}{R} = \frac{H - h}{r}$

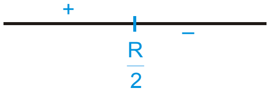
$S = 2\pi rh$



Figure

$$= 2\pi H\left(r - \frac{r^2}{R}\right)$$

$$\frac{dS}{dr} = 2\pi H\left(1 - \frac{2r}{R}\right)$$



sign of $\frac{dS}{dr}$

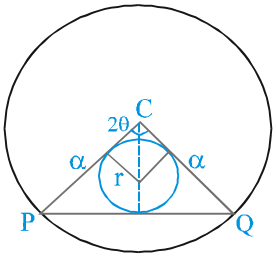
Maximum at $r = \frac{R}{2}$

72. $r=\dfrac{\Delta}{s}$

where Δ = Area of triangle CPQ and s = semiperimeter of ΔCPQ.

$$r=\frac{\alpha^2\sin 2\theta}{2s}=\frac{\alpha^2\sin 2\theta}{2\alpha+2\alpha\sin\theta}=\frac{\alpha}{2}.\frac{\sin 2\theta}{1+\sin\theta}$$

Consider $f(\theta)=\dfrac{\sin 2\theta}{1+\sin\theta}$



$$f'(\theta)=\frac{(1+\sin\theta)\,2\cos 2\theta-\sin 2\theta.\cos\theta}{(1+\sin\theta)^2}=0$$

$2(1+\sin\theta)(1-2\sin^2\theta)-2\sin\theta\,(1-\sin^2\theta)=0$

$2(1-2\sin^2\theta)=2\sin\theta\,(1-\sin\theta)$

$1-2\sin^2\theta=\sin\theta-\sin^2\theta$

$\sin^2\theta+\sin\theta-1=0$

$\sin\theta=\dfrac{-1\pm\sqrt{1+4}}{2}$ $\quad\therefore\quad \sin\theta=\dfrac{\sqrt{5}-1}{2}$

74. Let d be distance between (k, 0) and any point (x, y) on curve.

$d=\sqrt{(k-x)^2+y^2}$

$d=\sqrt{-x^2+2(1-k)x+k^2}$

$(\because\ y^2=2x-2x^2)$.

$\text{Maximum } d=\sqrt{\dfrac{4(-1)k^2-4(1-k)^2}{4(-1)}}$

$\text{Maximum } d=\sqrt{2k^2-2k+1}$

EXERCISE - 2

Part # I : Multiple Choice

1. **(A)** $2y\dfrac{dy}{dx}=4a \Rightarrow \left(\dfrac{dy}{dx}\right)_1=\dfrac{2a}{y_1}=\dfrac{2a}{e^{-x/2a}}=m_1$

For IInd curve $\left(\dfrac{dy}{dx}\right)_2=\dfrac{-1}{2a}e^{\frac{-x}{2a}}=m_2$

$m_1\,m_2=-1$

(B) $2y\left(\dfrac{dy}{dx}\right)_1=4a\ ;\ 2x=4a\left(\dfrac{dy}{dx}\right)_2$

$m_1=\dfrac{2a}{y_1}$ $\qquad m_2=\dfrac{x_1}{2a}$

$y_1^2=4ax_1$ ....(i) $\qquad x_1^2=4ay_1$ ......(ii)

$m_1m_2\neq-1$

(C) $y=\dfrac{a^2}{x}$; $\quad x^2-y^2=b^2$

$m_1=-\dfrac{a^2}{x_1^2}$; $2x_1-2y_1m_2=0\Rightarrow m_2=\dfrac{x_1}{y_1}$

$m_1m_2=\dfrac{-a^2}{x_1y_1}=\dfrac{-a^2}{a^2}=-1$

(D) $m_1=\dfrac{dy}{dx}=a$; $\quad 2x+2ym_2=0$

$m_2=-\dfrac{x}{y}$

$m_1m_2=-\dfrac{ax}{y}=-\dfrac{ax}{ax}=-1$

3. $f(x)=2x^3-3(2+\lambda)x^2+12\lambda x$

$f'(x)=6x^2-6(2+\lambda)x+12\lambda$

$D>0$

$36(2+\lambda)^2-24.12.\lambda>0$

$\Rightarrow\ (\lambda-2)^2>0$

$\Rightarrow\ \lambda\neq 2$

so required set is option (A,C,D)

4. $2y^3 = ax^2 + x^3$

$$6y^2\frac{dy}{dx} = 2ax + 3x^2$$

$$\left.\frac{dy}{dx}\right|_{(a,a)} = \frac{5a^2}{6a^2} = \frac{5}{6}$$

Tangent at (a, a) is $5x - 6y = -a$

$$\alpha = \frac{-a}{5}, \beta = \frac{a}{6}$$

$$\alpha^2 + \beta^2 = 61 \Rightarrow \frac{a^2}{25} + \frac{a^2}{36} = 61$$

$a^2 = 25.36$

$a = \pm 30$

6. $\frac{dy}{dx} = K^2e^{kx}$

$$\left.\frac{dy}{dx}\right|_{x=0} = K^2 = \tan\theta$$

(where θ is angle made by x-axis)

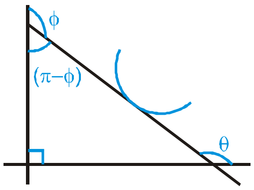
Let ϕ be the angle made by y-axis

$$\tan\theta = \tan\left(\frac{3\pi}{2} - \phi\right) = \cot\phi$$

$\cot\phi = K^2$

$\phi = \cot^{-1}(K^2)$

$$\Rightarrow \phi = \sin^{-1}\left(\frac{1}{\sqrt{1+K^4}}\right)$$



7. $f(0) = 0 \neq f(1)$

there will be no $x \in (0, \infty)$ (∴ Rolle's theorem is not applicable)

for which $f'(x) = 0$ i.e, $\cot^{-1}x = \frac{x}{1+x^2}$

$$f'(x) = \frac{-1}{1+x^2} - \frac{(1+x^2) - 2x^2}{(1+x^2)^2} = \frac{-1}{1+x^2} + \frac{x^2-1}{(x^2+1)^2}$$

$$f''(x) = \frac{-2}{(x^2+1)^2} < 0$$

$f'(x)$ is strictly decreasing

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}\left(\frac{-x}{1+x^2} + \cot^{-1}x\right) = 0$$

$$f(0+) = \lim_{x\to 0^+}\left(\cot^{-1}x - \frac{x}{1+x^2}\right) = \frac{\pi}{2}$$

$$\frac{f\left(x+\frac{2}{\pi}\right) - f(x)}{2/\pi} = f'(c) \qquad c \in \left(0, \frac{\pi}{2}\right)$$

(∴ LMVT is applicable)

$$\therefore \quad f'(c) < \frac{\pi}{2}$$

$$f\left(x+\frac{2}{\pi}\right) - f(x) < \frac{2}{\pi} \times \frac{\pi}{2}$$

$$f\left(x+\frac{2}{\pi}\right) - f(x) < 1$$

$f'(x) \geq 0$; f(x) is increasing

$\therefore \quad f(x) \in [f(0), f(\infty))$

$f(0) = 0$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x\cot^{-1}x = \lim_{x\to 0}\frac{\cot^{-1}x}{1/x}$$

$$= \lim_{x\to\infty}\frac{-1}{1+x^2} \times (-x^2) = 1$$

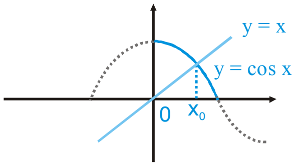
$f(x) \in [0, 1)$

$f(x) = \sec x$ will have no solution

8. $$f'(x) = \frac{\sec^2 x(\cos x + x)(\cos x - x)}{(1 + x\tan x)^2}$$

The only factor in $f'(x)$ which changes sign is $\cos x - x$.

Let us consider graph of $y = \cos x$ and $y = x$



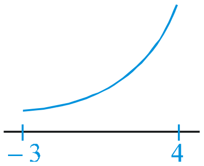
Figure

It is clear from figure that for $x \in (0, x_0)$, $\cos x - x > 0$

and for $x \in \left(x_0, \frac{\pi}{2}\right)$

$\cos x - x < 0$, $\Rightarrow$ f' (x) has maxima at x_0

10. **(A)** f (x) has no relative minimum on (–3, 4)



(B) $f(x)$ is continuous function on $[-3, 4]$

$\Rightarrow$ $f(x)$ has min. and max. on $[-3, 4]$ by IVT

(C) $f''(x) > 0 \Rightarrow$ $f(x)$ is concave upwards on $[-3, 4]$

(D) $f(3) = f(4)$

By Rolle's theorem

$c \in (3, 4)$, where $f'(c) = 0$

$\Rightarrow$ critical point in $[-3, 4]$

11. $y = \dfrac{2(x-2)+3}{x-2}$

$y = 2 + \dfrac{3}{(x-2)}$

$\dfrac{dy}{dx} = \dfrac{-3}{(x-2)^2} < 0$

$\therefore$ y decreases $\forall\ x \in R$

Now, $x = \dfrac{2y-1}{y-2}$

$xy - 2x = 2y - 1$

$y(x-2) = 2x - 1$

$y = \dfrac{2x-1}{x-2} = f^{-1}(x)$ [Also, $y \in R - \{1\}$]

14. Slope of tangent $= 1$

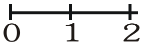
$f'(x) = 1$

$x^2 - 5x + 7 = 1$

$x^2 - 5x + 6 = 0$

$x = 2, 3$

$f(2) = \dfrac{8}{3}$, $f(3) = \dfrac{7}{2}$

19. 

$\because$ $f(0) = f(1)$ & 'f' is continuous in $[0, 1]$ & derivable in $(0, 1)$

$\therefore$ $f'(c_1) = 0$ for atleast one $c_1 \in (0, 1)$

Similarly, $\because$ $f(1) = f(2)$

$\therefore$ $f'(c_2) = 0$ for atleast one $c_2 \in (1, 2)$

$\Rightarrow$ $f'(c_1) = f'(c_2)$

$\Rightarrow$ $f''(c) = 0$ for atleast one $c \in (c_1, c_2)$

20. **(A)** $f(x) = x - \tan^{-1}x$

$f'(x) = 1 - \dfrac{1}{1+x^2} = \dfrac{x^2}{1+x^2} > 0$

$\Rightarrow$ f is increasing in $(0, 1)$

$f(x) > f(0)$ but $f(0) = 0$

$f(x) > 0 \Rightarrow x > \tan^{-1}x$ in $(0, 1)$

(B) $f(x) = \cos x - 1 + \dfrac{x^2}{2}$

$f'(x) = -\sin x + x = x - \sin x > 0$ in $(0, 1)$

$\Rightarrow$ **(B) is not correct**

(C) $f(x) = 1 + x\,ln\left(x + \sqrt{1+x^2}\right) - \sqrt{1+x^2}$

$$f'(x) = x\left(\frac{1 + \frac{1}{2}\cdot\frac{2x}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}}\right) + \ln\left(x + \sqrt{1+x^2}\right) - \frac{x}{\sqrt{1+x^2}}$$

$$= \frac{x}{\sqrt{1+x^2}} + \ln\left(x + \sqrt{1+x^2}\right) - \frac{x}{\sqrt{1+x^2}} > 0\ \forall\ x \in R$$

$\Rightarrow$ **(C) is true**

(D) $f(x) = x - \dfrac{x^2}{2} - \ln(1+x)$

$f'(x) = (1-x) - \dfrac{1}{1+x} = \dfrac{(1-x^2)-1}{1+x} = -\dfrac{x^2}{1+x} < 0$

$\Rightarrow$ **(D) is correct**

hence $f(x)$ is decreasing in $(0, 1)$

23. $\dfrac{dx}{dt} = \dfrac{2\left(-\cos ec^2 t\right)}{\cot t}$

at $t = \dfrac{\pi}{4}, \dfrac{dx}{dt} = -4$

$\dfrac{dy}{dt} = \sec^2 t - \cos ec^2 t$

at $t = \dfrac{\pi}{4}$ $\dfrac{dy}{dt} = 0$

$\dfrac{dy}{dx} = 0$ for tangent & hence it is parallel to x–axis & its normal is parallel to y axis

24. $2y = x^2$

$2y' = 2x$

$y' = h$

Equation of normal at (h, k)

$$(y-k) = -\frac{1}{h}(x-h)$$

As it passes through (0, 3)

So, $(3-k)\,h = -(-h) \quad \Rightarrow \quad (3-k)\,h = h$

or, $h\,(3-k-1) = 0$

or, $h\,(2-k) = 0$

or, $2h - hk = 0$

or, $2h - \frac{h^3}{2} = 0 \qquad \left(\because \quad k = \frac{h^2}{2}\right)$

or, $4h - h^3 = 0$

or, $h = 0, \ \pm\ 2$

$\therefore$ Required points are (2, 2) & (–2, 2)

(Rejecting (0, 0) since, its distance from point (0,3) is 3 which is not shortest.)

26. $f'(x) = 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{x^2+1}} = 1 - \frac{1}{1+x^2} + 1 - \frac{1}{\sqrt{x^2+1}}$

$$= \frac{x^2}{1+x^2} + \left(1 - \frac{1}{\sqrt{x^2+1}}\right) \geq 0$$

27. $\phi(x) = f^3(x) - 3f^2(x) + 4f(x) + 5x + 3\sin x + 4\cos x$

$\phi'(x) = (3f^2(x) - 6f(x) + 4)f'(x) + 5 + 3\cos x - 4\sin x$ .....**(i)**

$3\cos x - 4\sin x \geq -5$

$5 + (3\cos x - 4\sin x) \geq 0$

also $3f^2(x) - 6f(x) + 4 > 0 \quad \because \ D < 0$

$\phi'(x) > 0 \quad \forall\, f'(x) > 0$

Now let $f'(x) = -11$

$\phi'(x) = -11(3f^2(x) - 6f(x) + 4) + 5 + 3\cos x - 4\sin x$

Now $3f^2(x) - 6f(x) + 4 \geq 1$

$\Rightarrow \quad -11\,(3f^2(x) - 6f(x) + 4) \leq -11$ ..... **(ii)**

$3\cos x - 4\sin x \leq 5$

$\Rightarrow \quad 5 + (3\cos x - 4\sin x) \leq 10$ ..... **(iii)**

(ii) + (iii)

$\Rightarrow \quad -11(3f^2(x) - 6f(x) + 4) + 5 + (3\cos x - 4\sin x) \leq -1$

$\Rightarrow \quad \phi'(x) \leq -1$

28. $f(x) = \int_0^{\pi} \cos t \cos(x-t)\,dt$ ....**(1)**

$$= \int_0^{\pi} -\cos t \cdot \cos(x - \pi + t)\,dt \ \text{(using King)}$$

$f(x) = \int_0^{\pi} \cos t \cdot \cos(x+t)\,dt$ ....**(2)**

(1) + (2) gives

$$2\,f(x) = \int_0^{\pi} \cos t\,(2\cos x \cdot \cos t)\,dt$$

$$\therefore \quad f(x) = \cos x \int_0^{\pi} \cos^2 t\,dt = 2\cos x \int_0^{\pi/2} \cos^2 t\,dt$$

$f(x) = \frac{\pi \cos x}{2}$ Now verify.

Only **(A)** & **(B)** are correct.

30. $f(x) = \int_0^{x} \sqrt{1-t^4}\,dt$ a

$f(-x) = \int_0^{-x} \sqrt{1-t^4}\,dt$

$= -\int_0^{x} \sqrt{1-u^4}\,du \quad$ (Put t = –u)

$f(-x) = -f(x) \qquad \Rightarrow$ 'f' is odd function.

Check other options.

31. $f'(x) = \frac{1}{3x^{2/3}}$

$f'(0) \to \infty$ tangent is vertical at x = 0

Equation of tangent at (0, 0) is x = 0

Equation of normal is y = 0

$f(x) = f^{-1}(x)$

$x^{\frac{1}{3}} = x^3 \quad \Rightarrow \quad x^9 = x$

$\Rightarrow \quad x = 0\ ;\ 1 \ ;\ -1$

36. $y = x^{1/3}(x-1)$

$$\frac{dy}{dx} = \frac{4}{3}x^{1/3} - \frac{1}{3}\cdot\frac{1}{x^{2/3}} = \frac{1}{3x^{2/3}}[4x-1]$$

hence f is ↑ for $x > \frac{1}{4}$ and f ↓ for $x < \frac{1}{4}$

$\left.\begin{array}{l} x^{2/3} \text{ is always positive and } x = 1/4 \\ \text{the curves has a local minima} \end{array}\right]$

now $f'(x) = \frac{4}{3}x^{1/3} - \frac{1}{3}\cdot x^{-2/3}$

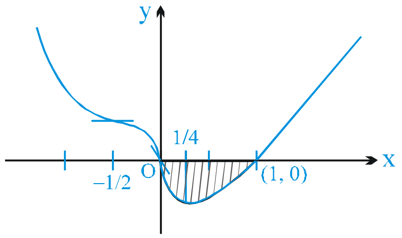
(non existent at x = 0, vertical tangent)

$$f''(x) = \frac{4}{9}\cdot\frac{1}{x^{2/3}} + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{x^{5/3}}$$

$$= \frac{2}{9x^{2/3}}\left[2 + \frac{1}{x}\right] = \frac{2}{9x^{2/3}}\left[\frac{2x+1}{x}\right]$$

$\therefore$ $f''(x) = 0$ at $x = -\frac{1}{2}$ (inflection point)

graph of $f(x)$ is as



$$A = \int_0^1 \left(x^{4/3} - x^{1/3}\right)dx = \left.\frac{3}{7}x^{3/7} - \frac{3}{4}x^{4/3}\right]_0^1$$

$$= \left|\frac{3}{7} - \frac{3}{4}\right| = 3\left|\frac{4-7}{28}\right| = \frac{9}{28} \Rightarrow \textbf{(D)}$$

38. $\phi'(x) = (3(f(x))^2 - 6(f(x)) + 4)f'(x) + 5 + 3\cos x - 4\sin x$

$5 - \sqrt{9+16} \le 5 + 3\cos x - 4\sin x \le 5 + \sqrt{9+16}$

adding $(3(f(x))^2 - 6(f(x)) + 4)f'(x)$

$(3(f(x))^2 - 6(f(x)) + 4)f'(x) \le \phi'(x) \le (3(f(x))^2$

$- 6(f(x)) + 4)f'(x) + 10$

$\because$ $3(f(x))^2 - 6f(x) + 4 = 3(f(x) - 1)^2 + 1 > 0$

$(3(f(x))^2 - 6(f(x)) + 4)f'(x) \ge 0$ when ever f(x) is increasing.

$\Rightarrow$ $\phi'(x) \ge 0$

$\Rightarrow$ $\phi(x)$ is increasing, when ever f(x) is increasing.

If $f'(x) = -11$ then

$(3(f(x))^2 - 6f(x) + 4)f'(x) + 10 = -33(f(x) - 1)^2 - 1 < 0$

$\Rightarrow$ $\phi'(x) < 0$ $\Rightarrow$ $\phi(x)$ is decreasing.

41. $f'(x) = (x-1)^{n-1}(x+1)^{n-1}$

$[2(n+1)x^3 + (2n+1)x^2 + 2(n-1)x - 1]$

At $x = 1$ $2(n+1)x^3 + (2n+1)x^2 + 2(n-1)x - 1 \ne 0$

for $n \in N$

$\therefore$ $n - 1$ must be odd $\Rightarrow$ n is even

44. **(A)** let $f(x) = \sin x - e^{-x}$

then $f'(x) = \cos x + e^{-x}$

Now between 2 roots of $f(x) = 0$ i.e. $e^x \sin x = -1$

there will be one root of $f'(x) = 0$

$\sin x - e^{-x}$

$e^x \cos x = -1$

(B) Let $f(x) = x^{100} + \sin x - 1$

$f'(x) = 100x^{99} + \cos x > 0, x \in [0, 1]$

$\Rightarrow$ f(x) is increasing.

(C) Suppose $f(x) = ax^3 - 2bx^2 + cx$, then clearly $f(0) = 0$

and $f(1) = a - 2b + c = 0$,

$\because$ $f(0) = f(1)$

$\therefore$ By Rolle's theorem $f'(x) = 3ax^2 - 4bx + c = 0$

for atleast one x in (0, 1) which is positive

(D) $y^2 = 4ax \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$ $\Rightarrow y = e^{\frac{-x}{2a}}$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{2a}e^{\frac{-x}{2a}} = \frac{-1}{2a}y$$

Product of slopes $= \left(\frac{2a}{y}\right)\left(\frac{-y}{2a}\right) = -1$

45. $f(x) = \frac{1}{(x+1)^3} - 3x + \sin x$

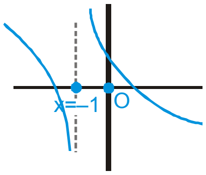
Domain of 'f' is $(-\infty, -1) \cup (-1, \infty)$

$$f'(x) = -3\left(\frac{1}{(x+1)^4} + 1\right) + \cos x.$$

$\Rightarrow$ $f'(x) < 0 \Rightarrow$ f is decreasing

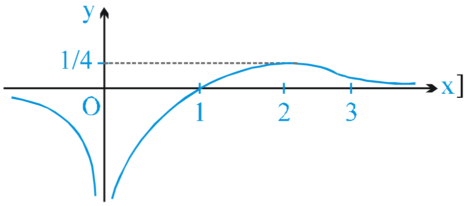
$\lim\limits_{x\to -1^+} f(x) \to \infty$ $\lim\limits_{x\to -1^-} f(x) \to -\infty$

$\lim\limits_{x\to \infty} f(x) \to -\infty$ $\lim\limits_{x\to -\infty} f(x) \to \infty$



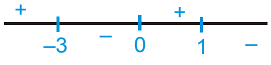
$\Rightarrow$ $f(x) = 0$ has exactly two roots.

46. $f'(x) = \dfrac{2-x}{x^3}$ and $f''(x) = \dfrac{x-3}{x^4}$. Now interpret



48. $f'(x) = \dfrac{-40.12x(x+3)(x-1)}{(3x^4+8x^3-18x^2+60)^2}$

$f'(x) = 0$



signs of f (x)

at $\quad x = 0, \qquad x = -3, x = 1$

so $\quad$ at $x = 0$, $\quad$ f(x) has local minima.

and $\quad$ at $x = -3$, $\quad x = 1$; f(x) has local maxima

$f(1) = \dfrac{40}{53}$, $\quad f(-3) = \dfrac{-40}{75}$. $f(-3) < 0$, $f(1) > 0$ and $f(x) \neq 0$

$\Rightarrow$ f(x) is undefined at point(s) in (–3, 1). Hence f(x) has no absolute maxima.

49. $g(x) = 2f\left(\dfrac{x}{2}\right) + f(1-x)$

and $\quad g'(x) = f'(x/2) - f'(1-x)$

Now $\quad$ g(x) is increasing if $g'(x) \geq 0$

$f'\left(\dfrac{x}{2}\right) \geq f'(1-x)$

$[\because\ f''(x) < 0$ i.e. $f'(x)$ is decreasing]

$\Rightarrow \dfrac{x}{2} \leq 1 - x \qquad \Rightarrow x \leq 2 - 2x$

$\Rightarrow 3x \leq 2 \qquad \Rightarrow x \leq 2/3 \qquad \Rightarrow 0 \leq x \leq \dfrac{2}{3}$

$\Rightarrow$ g(x) increases in $0 \leq x \leq 2/3$

and $g'(x) \leq 0$ for decreasing

$\Rightarrow f'\left(\dfrac{x}{2}\right) \leq f'(1-x) \qquad \Rightarrow \dfrac{x}{2} \geq 1 - x$

$\Rightarrow x \geq 2/3 \qquad \Rightarrow 2/3 \leq x \leq 1$

50. $f'(x) = \dfrac{1}{1+x^2} - \dfrac{1}{2}\dfrac{1}{x}, x > 0 = \dfrac{-(x-1)^2}{2x(1+x^2)} \leq 0 \quad \forall\ x > 0.$

f(x) is decreasing $\forall\ x > 0$.

On $\left[\dfrac{1}{\sqrt{3}}, \sqrt{3}\right]$, greatest value is

$f\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6} - \dfrac{1}{2}\ell n\left(\dfrac{1}{\sqrt{3}}\right)$ and least value is

$f(\sqrt{3}) = \dfrac{\pi}{3} - \dfrac{1}{2}\ell n\sqrt{3}$.

Part # II : Assertion & Reason

3. Statement-II :

$\because$ f(x) is continuous, derivable & $f(1) = f(2) = 0$

$\Rightarrow$ $f'(x) = 0$ has atleast one root in (1, 2).

$\Rightarrow$ $e^{10x}(2x-3) + 10\,e^{10x}(x^2-3x+2) = 0$

has atleast one root in (1, 2).

$\Rightarrow$ $10x^2 - 28x + 17 = 0$ has at least one root in (1, 2).

Statement-I is true & statement-II explains statement-I.

4. $f'(x) = 50x^{49} - 20x^{19}$

$= 10x^{19}(5x^{30} - 2)$

x = 0 is stationary point. Statement-2 is ture.

$f(0) = 0$

$f\left(\left(\dfrac{2}{5}\right)^{1/30}\right) = \left(\dfrac{2}{5}\right)^{5/3} - \left(\dfrac{2}{5}\right)^{2/3} < 0$

$f(1) = 0$

$\therefore$ Global maximum is 0. Statement-1 is true.

5. Consider $f(x) = x^{1/x}$

$f'(x) = x^{1/x}\left(\dfrac{1-\ell n\,x}{x^2}\right) \quad \forall\ x > 0$

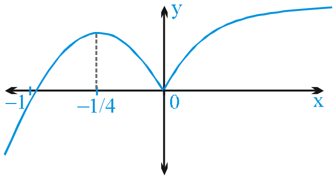
$\therefore$ at x = e, f(x) has absolute maximum value.

$3^{1/3} > 4^{1/4} = 2^{1/2}$.

Hence both statements are true & statement-II explains statements I.

6. $f(x) = \begin{cases} x+\sqrt{x} & \text{if } x \ge 0 \\ & . \\ x+\sqrt{-x} & \text{if } x < 0 \end{cases}$

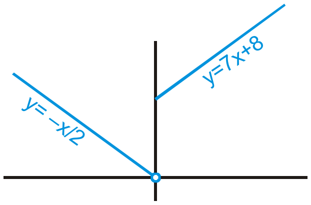
The graph of $f(x)$ is shown with $f'(x) = 0$ as $x = -1/4$. Also derivative fails at $x = 0$. Hence there are two critical points.



7. $\frac{dy}{dx} = 7x^6 + 24x^2 + 2$

which is always positive

8.



From figure st. I is false, because f(0–h)< f(0)

st. II is obviously true.

10. Let $f(x) = 0$ has two roots say $x = r_1$

and $x = r_2$ where $r_1, r_2 \in [a, b]$

$\Rightarrow$ $f(r_1) = f(r_2)$

hence there must exist some $c \in (r_1, r_2)$ where $f'(c) = 0$

but $f'(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$

for $x \ge 1$, $f'(x) = (x^6 - x^5) + (x^4 - x^3) + (x^2 - x) + 1 > 0$

for $x \le 1$, $f'(x) = (1 - x) + (x^2 - x^3) + (x^4 - x^5) + x^6 > 0$

hence $f'(x) > 0$ for all x

$\therefore$ Rolles theorem fails

$\Rightarrow$ $f(x) = 0$ can not have two or more roots.

12. $f'(x) = \frac{x^{1/x}}{x^2}(1 - \ell nx)$

$f'(x) \le 0$, when $x \ge e$

$\therefore$ f(x) is decreasing function, when $x \ge e$

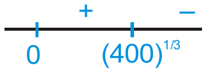
$\pi > e \Rightarrow f(\pi) < f(e)$

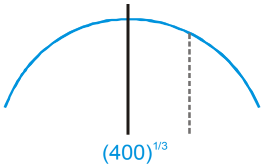
$\pi^{1/\pi} < e^{1/e} \Rightarrow e^\pi > \pi^e$

$\therefore$ Statement-1 is True, Statement-2 is False

13. St. II :- $f(x) = \frac{x^2}{x^3 + 200}$

$$f'(x) = \frac{2x(x^3 + 200) - 3x^4}{(x^3 + 200)^2} = \frac{x(400 - x^3)}{(x^3 + 200)^2}$$





St. II is false.

St. I $\because$ f(x) has maxima at $x = (400)^{1/3}$ & 7 is the closest natural number.

$\therefore$ a_n has greatest value for n = 7.

16. $f(x) = ln(2 + x) - \frac{2x + 2}{x + 3}$ is continuous in $(-2, \infty)$

$$f'(x) = \frac{1}{x+2} - \frac{4}{(x+3)^2} = \frac{(x+3)^2 - 4(x+2)}{(x+2)(x+3)^2}$$

$$= \frac{x^2 + 2x + 1}{(x+2)(x+3)^2} = \frac{(x+1)^2}{(x+2)(x+3)^2} > 0$$

$(f'(x) = 0 \text{ at } x = -1)$

$\Rightarrow$ f is increasing in $(-2, \infty)$

also $\lim_{x \to -2^+} f(x) \to -\infty$ and $\lim_{x \to \infty} f(x) \to \infty$

$\Rightarrow$ unique root

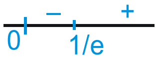
EXERCISE - 3

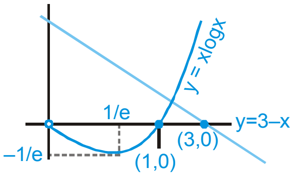
Part # I : Matrix Match Type

3. **(A)** $x\log x = 3 - x$

$y = x\log x$

$y' = 1 + \log x$





$$\lim_{x\to0^+}\frac{\log x}{1/x} = \lim_{x\to0^+}\frac{1/x}{-1/x^2} = 0$$

$$\operatorname*{Lim}_{x\to\infty} x\log x \to \propto$$

There is exactly one root of the equation in (1, 3).

(B) Let $g(x) = \int(4ax^3 + 3bx^2 + 2cx + d)\,dx$

$\Rightarrow$ $g(x) = ax^4 + bx^3 + cx^2 + dx + K$

$\Rightarrow$ $g(0) = g(3) = K$ $\{\because\ 27a + 9b + 3c + d = 0\}$

$\therefore$ By Rolle's Theorem

$g'(x) = 0$ has atleast one root in (0, 3).

(C) Let the required inteval be (a, b).

By LMVT

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\Rightarrow \frac{b + \frac{1}{b} - a - \frac{1}{a}}{b - a} = 1 - \frac{1}{c^2}$$

$$\Rightarrow 1 - \frac{1}{ab} = 1 - \frac{1}{3} \Rightarrow ab = 3$$

(D) $$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\Rightarrow \frac{(2b - b^2) - (2a - a^2)}{b - a} = 2(1 - c)$$

$$\Rightarrow \frac{2(b - a) - (b^2 - a^2)}{b - a} = 1$$

$\Rightarrow$ $2 - (b + a) = 1$ $\Rightarrow$ $(b + a) = 1$

4. **(A)** f(x) is continuous and differentiable $f(0) = f(\pi)$

Hence condition in Rolle's theorem and LMVT are satisfied.

(B) $f(1^-) = -1,\ f(1) = 0,\ f(1^+) = 1$

f(x) is not continuous at x = 1, belonging to $\left[\frac{1}{2}, \frac{3}{2}\right]$

Hence, atleast one condition in LMVT and Rolle's theorem is not satisfied

(C) $f'(x) = \frac{2}{5}(x - 1)^{-3/5},\ x \neq 1$

At $x = 1$, f(x) is not differentiable.

Hence at least one condition in LMVT and Rolle's theorem is not satisfied.

(D) At $x = 0$

$$\text{L.H.D.} = \lim_{x\to0^-} t\frac{x\left(\dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}\right) - 0}{x - 0} = \frac{0 - 1}{0 + 1} = -1$$

R.H.D. $= 1$

At $x = 0$, f(x) is not differentiable

Hence at least one condition in LMVT and Rolle's theorem is not satisfied.

5. $y = ax^2 + bx + c$

$\because$ Points A, B and D lies on the curve.

$\therefore$ $4a - 2b + c = 3$

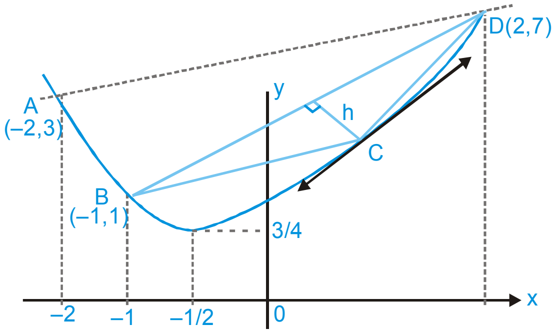
$a - b + c = 1$

$4a + 2b + c = 7$

Solving the equations we get a = b = c = 1.

$\therefore$ $y = x^2 + x + 1$

To maximize area of ☐ABCD, we maximize area (ΔBCD).



To maximize Area(ΔBCD) we have to maximize h (as shown in figure)

for maximum h

$\Rightarrow$ Slope of BD = Slope of tangent at C

$$\frac{7 - 1}{2 + 1} = (2x + 1)$$

$$x = \frac{1}{2}$$

$$y = \frac{1}{4} + \frac{1}{2} + 1 = \frac{7}{4}$$

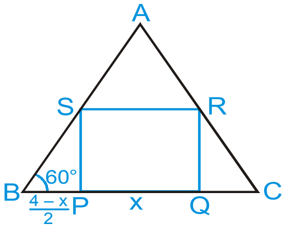
$$\therefore \quad C \equiv \left(\frac{1}{2}, \frac{7}{4}\right)$$

On the basis of this the coloumns can be matched.

6. **(A)** Let $PQ = x$

Then $BP = \frac{4-x}{2}$

$$\therefore PS = \frac{4-x}{2}\tan 60^\circ = \frac{\sqrt{3}\,(4-x)}{2}$$



Figure

$$\therefore \quad \text{area A of rectangle} = \frac{\sqrt{3}}{2}(4-x)\,x$$

$$\frac{dA}{dx} = \frac{\sqrt{3}}{2}(4-2x) = 0 \qquad \Rightarrow \quad x = 2$$

$$\frac{d^2A}{dx^2} = -\sqrt{3} < 0$$

$\therefore$ A is maximum, when $x = 2$.

$$\therefore \quad \text{Maximum area} = \frac{\sqrt{3}}{2}\,2.2 = 2\sqrt{3}\,.$$

Square of maximum area = 12

(B) Dimensions be x, 2x, h

$72 = x\,.\,2x\,.\,h$

$36 = x^2h$ ....(1)

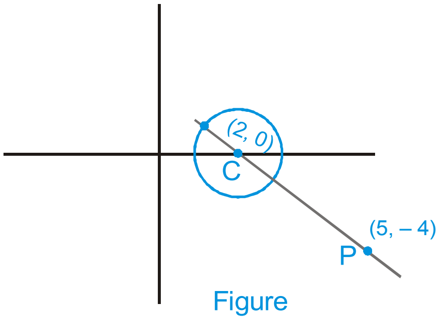
$S = 4x^2 + 6xh$

$$S = 4x^2 + 6\frac{36}{x}$$

$$\frac{dS}{dx} = 8x - \frac{216}{x^2} = \frac{8(x^3 - 3^3)}{x^2}$$

For least S, $x = 3$ and least S is 108.

(C) Let $y = \sqrt{-3 + 4x - x^2}$



Figure

$x^2 + y^2 - 4x + 3 = 0$

$(x-2)^2 + y^2 = 1$, center $C = (2, 0)$

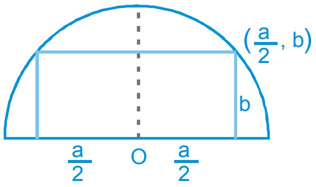
Consider point $P(5, -4)$

$CP = \sqrt{9+16} = 5$

Maximum value of

$$\left(\sqrt{-3+4x-x^2} + 4\right)^2 + (x-5)^2 \text{ is } (5+1)^2 = 36.$$

(D)



Figure

$x^2 + y^2 = 5$

$$\frac{a}{2} = \sqrt{5}\cos\theta,\ b = \sqrt{5}\sin\theta$$

Let $f(\theta)$ be perimeter

$f(\theta) = 2a + 2b$

$= 2\sqrt{5}\,(2\cos\theta + \sin\theta)$

$f'(\theta) = 2\sqrt{5}\,(-2\sin\theta + \cos\theta)$

$f''(\theta) = 2\sqrt{5}\,(-2\cos\theta - \sin\theta)$

$$f'(\theta) = 0 \Rightarrow \tan\theta = \frac{1}{2} \quad \text{and} \quad f''(\theta) < 0$$

$\Rightarrow$ $f(\theta)$ is greatest

$a = 4, b = 1$

$a^3 + b^3 = 65$

7. **(A)** $4y\frac{dy}{dx} = 2ax \Rightarrow -4\frac{dy}{dx} = 2a$

$\Rightarrow \frac{dy}{dx} = \frac{-a}{2} = -1 \Rightarrow a = 2$

$2y^2 = ax^2 + b$

$2 = a + b$

$b = 0$

$a - b = 2 - 0 = 2$

(B) Slope of normal $= -1$

Slope of tangent $= 1 = \frac{dy}{dx}$

$18y\frac{dy}{dx} = 3x^2$

$18b = 3a^2$

$b = \frac{a^2}{6}$ .....**(i)**

$9b^2 = a^3$ .....**(ii)**

$9.\frac{a^4}{36} = a^3 \Rightarrow a = 4 ; b = \frac{16}{6} = \frac{8}{3}$

$a - b = 4 - \frac{8}{3} = \frac{4}{3}$

(C) (1, 2) satisfies $y = ax^2 + bx + \frac{7}{2}$

$\Rightarrow 2 = a + b + \frac{7}{2} \Rightarrow a + b = \frac{-3}{2}$

$\frac{dy}{dx} = 2ax + b = 2a + b$

for IInd curve $\frac{dy}{dx} = 2x + 6 = 2$

Slope of normal $= -\frac{1}{2}$

$2a + b = -\frac{1}{2}$

Solve for a & b

(D) Put, (1, 1) $1 + a + b = 0$ ......**(i)**

$\frac{dy}{dx} = 2$

$y + xy' + a + by' = 0$

$1 + 2 + a + 2b = 0$

$a + 2b = -3$ ......**(ii)**

get the values of a & b

Part # II : Comprehension

Comprehension # 1

$f(x) = x^2f(1) - xf'(2) + f''(3)$

$f(0) = 2 \Rightarrow f''(3) = 2$

$f(x) = x^2f(1) - xf'(2) + 2$

$f'(x) = 2xf(1) - f'(2)$

$f'(2) = 4f(1) - f'(2)$ ......**(i)**

$f''(x) = 2f(1)$

$f''(3) = 2f(1)$

$2 = 2f(1) \Rightarrow f(1) = 1$

$f'(2) = 4(1) - f'(2)$ (from **(i)**)

$f'(2) = 2$

$f(x) = x^2 - 2x + 2$

1. $f'(x) = 2x - 2$

$\Rightarrow f'(1) = 0$

2. $f'(x) = 2x - 2 \Rightarrow f'(3) = 4$

equation of tangent at (3, 5) is

$y - 5 = 4(x - 3)$

$y = 4x - 7$

3. $2e^{2x} = x^2 - 2x + 2$

interseting at (0, 2)

$\left(\frac{dy}{dx}\right)_1 = -2 \; ; \; \left(\frac{dy}{dx}\right)_2 = 4$

angle of intersection $= \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$

$\tan\theta = \left|-\frac{6}{7}\right| \Rightarrow \theta = \tan^{-1}\left(\frac{6}{7}\right)$

Comprehension # 2

1-3

$\frac{da}{dt} = 2 \Rightarrow a = 2t + c$

$\because$ c = 0 $\{\because$ a = 0, when t = 0$\}$

$\therefore$ a = 2t

$\therefore$ the curve $y = x^2 - 2ax + a^2 + a$ becomes

$y = x^2 - 4tx + 4t^2 + 2t$

if $x = 0$, then $y = 4t^2 + 2t$

$$\frac{dy}{dx} = 2x - 4t \quad \therefore \quad \left.\frac{dy}{dx}\right|_{at\ x=0} = -4t$$

$\therefore$ equation of the tangent

$$y - (4t^2 + 2t) = -4t\,(x - 0)$$

i.e. $y = -4t\,x + 4t^2 + 2t$

vertex of $y = x^2 - 4t\,x + 4t^2 + 2t$ is $(2t, 2t)$

$\therefore$ distance of vertex from the origin $= 2\sqrt{2}\,t$

$\therefore$ rate of change of distance of vertex from origin with respect to t $= 2\sqrt{2}$

i.e. $k = 2\sqrt{2}$

$c(t) = 4t^2 + 2t$

$$\therefore \quad \frac{dc}{dt} = 8t + 2 \quad \therefore \quad \left.\frac{dc}{dt}\right|_{at\ t = 2\sqrt{2}} = 16\sqrt{2} + 2 \quad \therefore$$

$\ell = 16\sqrt{2} + 2$

$m(t) = -4t$

$$\therefore \quad \frac{dm}{dt} = -4 \quad \therefore \quad \left.\frac{dm}{dt}\right|_{at\ t = \ell} = -4$$

Comprehension # 3

1. $a = 1$

$f(x) = 8x^3 + 4x^2 + 2bx + 1$

$f'(x) = 24x^2 + 8x + 2b = 2(12x^2 + 4x + b)$

for increasing function, $f'(x) \geq 0 \quad \forall\ x \in R$

$$\therefore \quad D \leq 0 \quad \Rightarrow \quad 16 - 48b \leq 0 \quad \Rightarrow \quad b \geq \frac{1}{3} \quad \Rightarrow \quad (C)$$

2. if $b = 1$

$f(x) = 8x^3 + 4ax^2 + 2x + a$

$f'(x) = 24x^2 + 8ax + 2$ or $2(12x^2 + 4ax + 1)$

for non monotonic $f'(x) = 0$ must have distinct roots

hence $D > 0$

i.e. $16a^2 - 48 > 0 \quad \Rightarrow \quad a^2 > 3$;

$\therefore$ $a > \sqrt{3}$ or $a < -\sqrt{3}$

$\therefore$ $a \in 2, 3, 4, ...$

sum $= 5050 - 1 = 5049$ Ans.

3. If x_1, x_2 and x_3 are the roots then

$\log_2 x_1 + \log_2 x_2 + \log_2 x_3 = 5$

$\log_2(x_1 x_2 x_3) = 5$

$x_1 x_2 x_3 = 32$

$$-\frac{a}{8} = 32 \quad \Rightarrow \quad a = -256 \text{ Ans.}$$

Comprehension # 4

2. At $x = -5$ $f'(x)$ changes from + ve to – ve and $x = 4$, $f'(x)$ change sign for + ve to – ve hence maxima at $x = -5$ and 4. f is continuous and $f'(x)$ is not defined hence $x = 2$ must be geometrical sharp corner

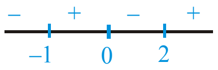
Comprehension # 7

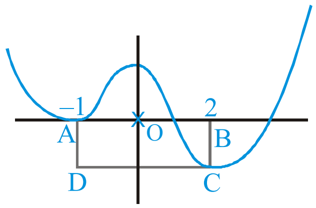
$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$f'(x) = 12x^3 - 12x^2 - 24x$

$= 12x\,(x - 2)\,(x + 1)$

$\therefore$ $a_1 = -1$, $a_2 = 0$ & $a_3 = 2$.





on the basis of above graph, the given questions can be solved.

Comprehension # 8

1. $$\lim_{x \to 0^+} x \ln\left(1 + \frac{1}{x}\right) = \lim_{x \to 0^+} \frac{\ln\left(\dfrac{x+1}{x}\right)}{\dfrac{1}{x}} \qquad \left(\frac{\infty}{\infty}\right)$$

Using L'Hospital's Rule

$$l = \lim_{x \to 0} -\left(\frac{1}{x+1} - \frac{1}{x}\right)x^2 = \lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x+1}\right) \cdot x^2$$

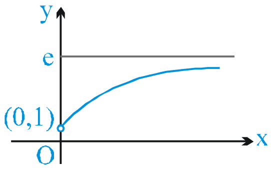
$$= \lim_{x \to 0} \frac{1}{x(x+1)} \cdot x^2 = \lim_{x \to 0} \frac{x}{(x+1)} = 0 \text{ Ans.}$$

2. $\lim\limits_{x\to 0} f(x) = 1$ (can be verified)

$\lim\limits_{x\to\infty} f(x) = e$

Also f is increasing for all $x > 0$ ⇒ **(D)**

(can be verified)



3. $l = \left(\prod\limits_{k=1}^{n}\left(1+\frac{n}{k}\right)^{k/n}\right)^{1/n}$

{given $f(x) = (1 + 1/x)^x$ and $f(k/n) = \left(1+\frac{n}{k}\right)^{k/n}$ }

taking log,

$$\ln l = \lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n}\ln\left(1+\frac{n}{k}\right)^{k/n}$$

$$= \lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n}\frac{k}{n}\ln\left(1+\frac{1}{k/n}\right)dx$$

$$= \int_0^1 \underbrace{x}_{II}\ \underbrace{\ln\left(1+\frac{1}{x}\right)}_{I}dx$$

$$= \ln\left(1+\frac{1}{x}\right)\cdot\frac{x^2}{2}\Bigg]_0^1 + \int_0^1\left(\frac{1}{x}-\frac{1}{x+1}\right)\cdot\frac{x^2}{2}dx$$

$$= \left(\frac{1}{2}\ln 2 - 0\right) + \frac{1}{2}\int_0^1\frac{x+1-1}{x+1}dx$$

$$= \frac{1}{2}\ln 2 + \frac{1}{2}[x - \ln(x+1)]_0^1$$

$$= \frac{1}{2}\ln 2 + \frac{1}{2}[(1-\ln 2) - 0] = \frac{1}{2}$$

$l = \sqrt{e}$ Ans.

Comprehension # 9

1-3

Let $g(x) = \frac{x+\sin x}{2}$, $x \in [0,\pi]$. g(x) is increasing function of x.

$\therefore$ range of g(x) is $\left[0, \frac{\pi}{2}\right]$

$\therefore$ $f(x) = \frac{x+\sin x}{2}$, $x \in [0, \pi]$

Now let $\pi \le t \le 2\pi$,

then $f(t) + f(2\pi - t) = \pi$

i.e $f(t) + \frac{2\pi - t + \sin(2\pi - t)}{2} = \pi$

i.e $f(t) + \pi - \frac{t}{2} - \frac{\sin t}{2} = \pi$

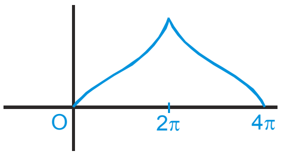
i.e $f(t) = \frac{t+\sin t}{2}$

$\therefore$ $f(x) = \frac{x+\sin x}{2}$ for $\pi \le x \le 2\pi$

Thus $f(x) = \frac{x+\sin x}{2}$ for $0 \le x \le 2\pi$

Also $f(x) = f(4\pi - x)$ for all $x \in [2\pi, 4\pi]$

⇒ f(x) is symmetric about $x = 2\pi$



Figure

$\therefore$ from graph of f(x)

$\therefore$ $\alpha = 2\pi - 0 = 2\pi$

$\therefore$ $\beta = \alpha$

Maximum value is $f(2\pi) = \pi = \frac{\beta}{2}$

Comprehension # 10

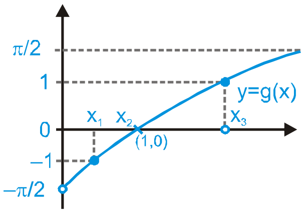
$f(x) = \tan^{-1}(\ell n\, x)$

1. $\because$ $\tan^{-1}(x)$ & $\ell n\, x$ are increasing functions.

⇒ f(x) is also increasing function.

2. $\lim\limits_{x\to 0^+}\tan^{-1}(\ell n\, x) \to -\frac{\pi}{2}$

$\lim\limits_{x\to\infty}\tan^{-1}(\ell n\, x) \to \frac{\pi}{2}$ ⇒ range of 'f' is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.



3. From graph, g(x) is discontinuous at $x = x_1, x_2, x_3$

$\tan^{-1}(\ell n\, x_1) = -1;\ \tan^{-1}(\ell n\, x_2) = 0;\ \tan^{-1}(\ell n\, x_3) = 1$

$$\Rightarrow\ x_1 = \frac{1}{e^{\tan 1}};\quad x_2 = 1;\quad x_3 = e^{\tan 1}$$

$$x_1 + x_2 + x_3 = e^{\tan 1} + \frac{1}{e^{\tan 1}} + 1 > 3.$$

Comprehension # 11

$$\underset{x\to 0}{\text{Lt}}\ \frac{1}{x}\,\ell n \begin{vmatrix} \frac{f(x)}{x} & 1 & 0 \\ 0 & \frac{1}{x} & 1 \\ 1 & 0 & \frac{1}{x} \end{vmatrix} = 2$$

$$\underset{x\to 0}{\text{Lt}}\ \frac{1}{x}\ \ell n\left(\frac{f(x)}{x^3} + 1\right) = 2 \ ....\textbf{(1)}$$

for limit to exist

$$\underset{x\to 0}{\text{Lt}}\ \frac{f(x)}{x^3} = 0$$

$\Rightarrow\ f(x) = a_0x^6 + a_1x^5 + a_2x^4$

Also $\quad f'(0) = f'(2) = f'(1) = 0$

$f'(x) = 6a_0x^5 + 5a_1x^4 + 4a_2x^3$

$= x^3(6a_0x^2 + 5a_1x + 4a_2)$

$f'(2) = 0$

$\Rightarrow\ 24a_0 + 10a_1 + 4a_2 = 0 \quad ......\textbf{(2)}$

$f'(1) = 0$

$6a_0 + 5a_1 + 4a_2 = 0 \quad ....\textbf{(3)}$

Consider eq^n. (1)

$$\ell n\left\{\underset{x\to 0}{\text{Lt}}\left(\frac{f(x)}{x^3} + 1\right)^{\frac{1}{x}}\right\} = 2$$

$$\ell n\ e^{\left(\lim\limits_{x\to 0}\frac{f(x)}{x^4}\right)} = 2$$

$$\Rightarrow\ \underset{x\to 0}{\text{Lt}}\ \frac{a_0x^6 + a_1x^5 + a_2x^4}{x^4} = 2$$

$\Rightarrow\ a_2 = 2$

Putting a_2 in (2) & (3)

$24a_0 + 10a_1 = -8$

$6a_0 + 5a_1 = -8$

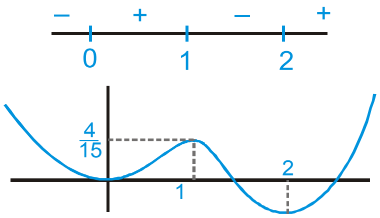
on solving this we get

$$a_1 = -\frac{12}{5},\ a_0 = \frac{2}{3}$$

$$f(x) = \frac{2}{3}x^6 - \frac{12}{5}x^5 + 2x^4$$

$f'(x) = 4x^5 - 12x^4 + 8x^3 = 4x^3(x^2 - 3x + 2)$

$= 4x^3(x-2)(x-1)$



On the above basis the answers can be given.

EXERCISE - 4

Subjective Type

2. $f(x) = \sin 2x - 8(a+1)\sin x + (4a^2 + 8a - 14)x$

$f'(x) = 2\cos 2x - 8(a+1)\cos x + (4a^2 + 8a - 14)$

$f'(x) = 2(2\cos^2 x - 1) - 8(a+1)\cos x + 4a^2 + 8a - 14$

$= 4\{\cos^2 x - 2(a+1)\cos x\} + 4a^2 + 8a - 16$

$= 4\{\cos x - (a+1)\}^2 - 20 > 0$

$= \{\cos x - (a+1)\}^2 - (\sqrt{5})^2 > 0$

$f'(x) = \{\cos x - (a+1) - \sqrt{5}\}\ \{\cos x - (a+1) + \sqrt{5}\} > 0$

$\Rightarrow \cos x > a + 1 + \sqrt{5}$ or $\cos x < (a+1) - \sqrt{5}$

$\forall\ x \in R$

$a + 1 + \sqrt{5} < -1$ or $(a+1) - \sqrt{5} > 1$

$a < -2 - \sqrt{5}$ or $a > \sqrt{5}$

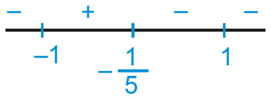
$a \in (-\infty, -2 - \sqrt{5}) \cup (\sqrt{5}, \infty)$

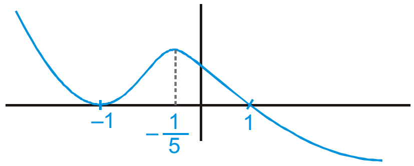
4. **(B)** $f(x) = -(x-1)^3 (x+1)^2$

$f'(x) = -\{3(x-1)^2(x+1)^2 + (x-1)^3 2(x+1)\}$

$= -(x-1)^2 (x+1)\{3x + 3 + 2x - 2\}$

$= -(x-1)^2 (x+1)(5x+1)$

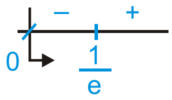




(C) $f(x) = x\,\ell n\,x$

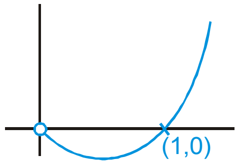
$f'(x) = 1 + \ell n\,x$

$f''(x) = \frac{1}{x} > 0$



$\Rightarrow$ concave up

$\underset{x\to 0^+}{\text{Lt}}\ x\,\ell n\,x = 0$, $\underset{x\to\infty}{\text{Lt}}\ x\,\ell n\,x \to \infty$



5. At t = 0 the point is origin

$$\frac{dx}{dt} = \lim_{t\to 0}\frac{2t + t^2 \sin 1/t - 0}{t} = 2$$

$$\frac{dy}{dt} = \lim_{t\to 0}\frac{\frac{1}{t}\sin t^2}{t} = 1$$

$$\frac{dy}{dx} = \frac{1}{2}$$

equation of tangent is $y - 0 = \frac{1}{2}(x - 0)$

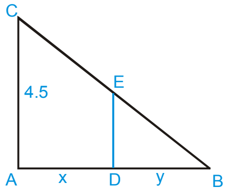
equation of normal is $y - 0 = -2(x - 0)$

7. Let AC be pole, DE be man and B be farther end of shadow as shown in figure

From triangles ABC and DBE

$$\frac{4.5}{x+y} = \frac{1.5}{y}$$

$3y = 1.5\,x$



$$\frac{dy}{dt} = 2,\quad \frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt}$$

$= 4 + 2 = 6$

10. Consider $g(x) = \begin{vmatrix} f(a) & f(b) & f(x) \\ \phi(a) & \phi(b) & \phi(x) \\ \psi(a) & \psi(b) & \psi(x) \end{vmatrix}$

Apply LMVT in g(x) in [a, b]

12. Let the point is (x_1, y_1)

Slope of line joining (0, 0) & (x_1, y_1) is

$$m_1 = \frac{y_1}{x_1}$$

$$\frac{(2x + 2yy')}{(x^2 + y^2)} = \frac{C\left(\frac{y'}{x} - \frac{y}{x^2}\right)}{\left(1 + \frac{y^2}{x^2}\right)}$$

$$\frac{2(x_1 + y_1 y')}{(x_1^2 + y_1^2)} = \frac{C(y' x_1 - y_1)}{(x_1^2 + y_1^2)}$$

$2x_1 + 2y_1 y' = Cx_1 y' - Cy_1$

$2x_1 + Cy_1 = y'(Cx_1 - 2y_1)$

$$y' = \frac{(2x_1 + Cy_1)}{(Cx_1 - 2y_1)} = m_2$$

Calculate $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

14. Let (h, k) be point of inflection h sin h = k ...**(i)**

$y' = \sin x + x \cos x$

$y'' = \cos x + \cos x - x \sin x$

$y'' = 0 \Rightarrow 2 \cos h - h \sinh = 0 \Rightarrow 2 \cos h = k$ ...**(ii)**

$\sin^2 h + \cos^2 h = 1$

$$\frac{k^2}{h^2} + \frac{k^2}{4} = 1 \quad \Rightarrow \quad 4k^2 + h^2k^2 = 4h^2$$

$\therefore$ locus $y^2 (4 + x^2) = 4x^2$

15. $f(x) = \begin{vmatrix} 2ax & 2ax - 1 & 2ax + b + 1 \\ b & b + 1 & -1 \\ 0 & 0 & 1 \end{vmatrix}$

$(R_3 \rightarrow R_3 - (R_1 + 2R_2))$

$f'(x) = 2ax + b \quad \Rightarrow \quad f(x) = ax^2 + bx + c$

f(x) is maximum at $x = \frac{5}{2}$

$f'\left(\frac{5}{2}\right) = 0 \quad \Rightarrow \quad 5a + b = 0$

$f(0) = 2 \Rightarrow c = 2,\ f(1) = 1 \Rightarrow a + b + c = 1$

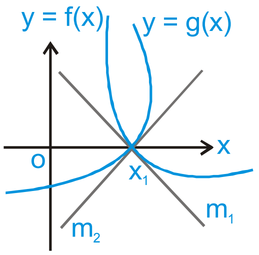
$\therefore \quad a = \frac{1}{4},\ b = -\frac{5}{4},\ c = 2$

$$f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2$$

20. $f(x_1) = g(x_1) = 0$

$m_1 m_2 = -1$ and $|m_1| = |m_2|$

$\Rightarrow m_1 = 1 ; m_2 = -1$ or $m_1 = -1 ; m_2 = 1$

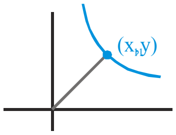


$$\lim_{h\to 0} [(0 + h)(0 - h)] = \lim_{h\to 0} [0 - h^2] = -1$$

22. $ax^2 + 2bxy + ay^2 - c = 0$ ....**(i)**

$$2xa + 2b\left(y + x\frac{dy}{dx}\right) + 2ay\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-(2ax + 2by)}{2bx + 2ay}$$



slope of normal $= \frac{bx + ay}{ax + by}$

slope of line joining origin & point $(x_1, y_1) = \frac{y_1}{x_1}$

minimum distance is along normal.

so $\frac{bx_1 + ay_1}{ax_1 + by_1} = \frac{y_1}{x_1} \quad \Rightarrow \quad x_1^2 = y_1^2$

$\Rightarrow \quad x_1 = y_1$ or $x_1 = -y_1$ ....**(ii)**

from **(i)** & **(ii)** required points are

for $x_1 = y_1; \left(\sqrt{\frac{c}{2(a+b)}}, \sqrt{\frac{c}{2(a+b)}}\right)$ & $\left(-\sqrt{\frac{c}{2(a+b)}}, -\sqrt{\frac{c}{2(a+b)}}\right)$

for $x_1 = -y_1 \left(\pm\sqrt{\frac{c}{2(a-b)}}, \mp\sqrt{\frac{c}{2(a-b)}}\right)$ not possible

since $a - b < 0$

26. $f(x) = \sin^3 x + \lambda \sin^2 x$

$f'(x) = \sin x \cos x(3\sin x + 2\lambda)$

$f''(x) = 6\sin x\cos^2 x - 3\sin^3 x + 2\lambda\cos 2x$

$f'(x) = 0 \Rightarrow \sin x = 0$ or $\cos x = 0$ or $\sin x = \frac{-2\lambda}{3}$

$\cos x \neq 0$ if $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\sin x = 0 \quad \Rightarrow \quad x = 0$

$\sin x = \frac{-2\lambda}{3}$

$-1 < \sin x < 1 \Rightarrow -1 < \frac{-2\lambda}{3} < 1$

$\Rightarrow \quad \frac{-3}{2} < \lambda < \frac{3}{2}$

$\lambda \neq 0$ otherwise there is only one critical point.

If $\lambda > 0$, then $f''(0) > 0$

$\Rightarrow$ x = 0 point of minima & f '(x) changes sign from positive to negative for $x = \sin^{-1}\left(\frac{-2\lambda}{3}\right)$ (point of maxima).

If $\lambda < 0$ then $x = 0$ is a point of maxima while $x = \sin^{-1}\left(\frac{-2\lambda}{3}\right)$ is a point of minima. Thus for $\lambda \in \left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\}$ function has exactly one maxima & exactly one minima.

27. Let No. of children of john & anglina = y

$\therefore\ x + (x+1) + y = 24$

$y = 23 - 2x$

Number of fights

$F = x(x+1) + x(23-2x) + (x+1)(23-2x)$

$F = -3x^2 + 45x + 23$

$\frac{df}{dx} = 0 \Rightarrow -6x + 45 = 0 \Rightarrow x = 7.5$

But 'x' wil be integral.

check x = 6 or x = 7

F = 191

30. Any point on curve $y = x^2$ is $P(t, t^2)$

$\frac{dy}{dx} = 2x$

equation of normal at (t, t^2) is

$y - t^2 = -\frac{1}{2t}(x - t)$

Solving with $y = x^2$ we get

$x^2 - t^2 = \frac{-1}{2t}(x-t) \Rightarrow (x-t)\left(x + t + \frac{1}{2t}\right) = 0$

$\Rightarrow x = -t - \frac{1}{2t}$

So normal cuts the curve again at

$Q\left(-t - \frac{1}{2t}, \left(-t - \frac{1}{2t}\right)^2\right)$

$z = PQ^2 = 4t^2\left(1 + \frac{1}{4t^2}\right)^3$

Now $\frac{dz}{dt} = 0 \Rightarrow t = \pm\frac{1}{\sqrt{2}}, 0$

$\frac{dz}{dt}$ changes sign from negative to positive about $t = \frac{1}{\sqrt{2}}$ as well as $t = -\frac{1}{\sqrt{2}}$

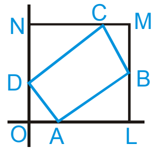
(No chord is formed for t = 0)

z is minimum at $t = \pm\frac{1}{\sqrt{2}}$ & minimum value of $z = PQ^2 = 3$

Shortest normal chord has length $\sqrt{3}$ & its equation is $x + \sqrt{2}y - \sqrt{2} = 0$

or $x - \sqrt{2}y + \sqrt{2} = 0$

36. Let the vertices L, M, N of the square S be (1, 0), (1, 1) & (0, 1) respectively & the vertex O be origin. Let the co-ordinate of vertices A, B, C, D of the quadrilateral be (p, 0)(1, q)(r, 1) & (0, s)



Then $a^2 = (1-p)^2 + q^2$

$b^2 = (1-q)^2 + (1-r)^2$

$c^2 = (1-s)^2 + r^2$

$d^2 = p^2 + s^2$

Thus $a^2 + b^2 + c^2 + d^2 = (1-p)^2 + q^2 + (1-q)^2 + (1-r)^2 + (1-s)^2 + r^2 + p^2 + s^2$

Let $f(x) = x^2 + (1-x)^2 \quad 0 \le x \le 1$

$f'(x) = 2x - 2(1-x)$

$f'(x) = 0 \Rightarrow x = 1/2$

$f''(x) = 4$

$\Rightarrow$ f(x) is minimum at x = 1/2 & max. value of f(x) occur at x = 0, x = 1

$\therefore\ 1/2 \le f(x) \le 1$

So $2 \le a^2 + b^2 + c^2 + d^2 \le 4$

37. $A + B + C = \pi \Rightarrow dA + dB = 0 \Rightarrow dA = -dB$

$\frac{c}{\sin C} = 2R = \text{constant}$

$a = 2R\sin A \Rightarrow da = 2R\cos A dA$ .....**(i)**

similarly $db = 2R\cos B dB$ .....**(ii)**

Divide **(i)** by **(ii)**

$\frac{da}{db} = \frac{\cos A(dA)}{\cos B(dB)}$

$\Rightarrow \frac{da}{db} = -\frac{\cos A}{\cos B}$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

2. $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1 \quad a > 0$

$\therefore\ f'(x) = 6x^2 - 18ax + 12a^2$

$\therefore\ f''(x) = 12x - 18a$

for maximum or minimum

$6x^2 - 18ax + 12a^2 = 0$

$x^2 - 3ax + 2a^2 = 0$

$x = a \quad \text{or} \quad x = 2a$

maximum at $x = a$ and minimum at $x = 2a$

$\therefore\ (a > 0)$ given)

$p = a, \quad q = 2a$

$\therefore\ p^2 = q$

$a^2 = 2a$

$a(a - 2) = 0$

$a = 2$

3. $f(x) = x + \dfrac{1}{x} \qquad f'(x) = 1 - \dfrac{1}{x^2}$

$x = \pm 1$

$f''(x) = \dfrac{2}{x^3}$

minimum at $x = 1$

4. $u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$

$$u^2 = a^2 + b^2 + 2\sqrt{(a^2\cos^2\theta + b^2\sin^2\theta)(a^2\sin^2\theta + b^2\cos^2\theta)}$$

$$u^2 = a^2 + b^2 + 2\sqrt{\begin{array}{l} a^4\cos^2\theta\sin^2\theta + a^2b^2\cos^4\theta \\ \quad + a^2b^2\sin^4\theta + b^4\sin^2\theta\cos^2\theta \end{array}}$$

$$u^2 = a^2 + b^2 + 2\sqrt{\begin{array}{l} a^2b^2(1 - 2\sin^2\theta\cos^2\theta) \\ \quad + a^4\cos^2\theta\sin^2\theta + b^4\cos^2\theta\sin^2\theta \end{array}}$$

$$= a^2 + b^2 + 2\sqrt{a^2b^2 + (a^4 - b^4 - 2a^2b^2)\sin^2\theta\cos^2\theta}$$

$$= a^2 + b^2 + 2\sqrt{a^2b^2 + (a^2 - b^2)^2 \times \left(\frac{\sin 2\theta}{2}\right)^2}$$

$$= a^2 + b^2 + \sqrt{4a^2b^2 + (a^2 - b^2)^2\sin^2 2\theta}$$

u^2 is maximum when $\sin^2 2\theta = 1$

u^2 is minimum when $\sin^2 2\theta = 0$

$u^2_{(\max.)} - u^2_{(\min.)}$

$2(a^2 + b^2) - (a + b)^2$

$2a^2 + 2b^2 - a^2 - b^2 - 2ab$

$a^2 + b^2 - 2ab = (a - b)^2$

6. $x = a(1 + \cos\theta),\ y = a\sin\theta$

$\dfrac{dx}{d\theta} = -a\sin\theta \quad ; \quad \dfrac{dy}{d\theta} = a\cos\theta$

$\left(\dfrac{dy}{dx}\right) = -\dfrac{\cos\theta}{\sin\theta} \qquad \text{slope of normal} = -\left(\dfrac{dx}{dy}\right) = \dfrac{\sin\theta}{\cos\theta}$

$y - a\sin\theta = \dfrac{\sin\theta}{\cos\theta}(x - a - a\cos\theta)$

$y\cos\theta - a\sin\theta\cos\theta = x(\sin\theta) - a\sin\theta(1 + \cos\theta)$

$x\sin\theta - y\cos\theta = a\sin\theta(1 + \cos\theta - \cos\theta)$

clearly passes through (a, 0)

7. Check the option one by one

third option $f(x) = 3x^2 - 2x + 1$

$f'(x) = 6x - 2 \geq 0 \qquad x \geq 1/3$ it is incorrect

8. $x = a(\cos\theta + \theta\sin\theta)$ & $y = a(\sin\theta - \theta\cos\theta)$

$\dfrac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta),$

$\dfrac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta)$

$\therefore\ \dfrac{dy}{dx} = \dfrac{\sin\theta}{\cos\theta}$

slope of normal $= -\dfrac{\cos\theta}{\sin\theta} = -\cot\theta$

it makes angle $\left(\dfrac{\pi}{2} + \theta\right)$ with the x-axis

eq of normal $y - a\sin\theta + a\theta\cos\theta = -\dfrac{\cos\theta}{\sin\theta}$

$(x - a\cos\theta - a\theta\sin\theta)$

$\Rightarrow\ x\cos\theta + y\sin\theta = a.$

Hence it is at a constant distance 'a' from the origin.

9. Angle betweeen the tangents $\frac{dy}{dx} = 2x - 5$

$$\left(\frac{dy}{dx}\right)_{(2,0)} = -1 \qquad \left(\frac{dy}{dx}\right)_{(3,0)} = 1 \Rightarrow \text{Angle} = \frac{\pi}{2}$$

10. $f(x) = \frac{x}{2} + \frac{2}{x}$

$\Rightarrow \quad f'(x) = \frac{1}{2} - \frac{2}{x^2}$

For maximum or minimum, $f'(x) = 0$

$\frac{1}{2} - \frac{2}{x^2} = 0$

$\Rightarrow \quad x^2 = 4 \qquad \Rightarrow \quad x = \pm 2$

Now, $\quad f''(x) = \frac{4}{x^3}$

at $\quad x = 2, \qquad f''(x) > 0$

and $x = -2 \qquad f''(x) < 0$

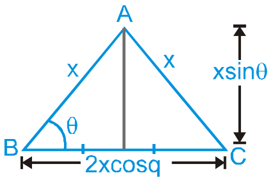
So, there exists a local minimum at $x = 2$.

11. A triangular park

$\Delta = \frac{1}{2}(2x\cos\theta)(x\sin\theta)$

$= \frac{1}{2}x^2\sin 2\theta$

$$\boxed{\Delta_{\max.} = \frac{x^2}{2}}$$



12. $f(x) = \tan^{-1}(\sin x + \cos x)$

$f'(x) = \frac{1 \times (\cos x - \sin x)}{1 + (\sin x + \cos x)^2} > 0$

$\cos x - \sin x > 0 \qquad \cos x > \sin x$

$\sin x < \cos x \qquad \tan x < 1$

$x < \frac{\pi}{4} \qquad \boxed{x \in \left(-\frac{\pi}{2}, \frac{\pi}{4}\right)}$

13. Using A.M. ≥ G.M.

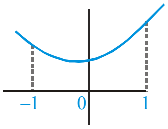
$\frac{p^2 + q^2}{2} \geq p.q$

$\Rightarrow \quad pq \leq \frac{1}{2}$

$\Rightarrow \quad (p + q)^2 = p^2 + q^2 + 2pq$

$\Rightarrow \quad (p + q) \leq \sqrt{2}$

15. Graph of P(x) under given conditions. It is clear that P(x) has max. at 1 but not minimum at –1.



16. Point (t^2, t) is on the parabola $x = y^2$

Its distance from $y - x = 1$

$d(t) = \frac{t^2 - t + 1}{\sqrt{2}}$

$d'(t) = \frac{1}{\sqrt{2}}[2t - 1] = 0$

$t = \frac{1}{2}$

$d''(t) = \frac{2}{\sqrt{2}} > 0$

d(t) is min at $t = \frac{1}{2}$

Its value

$d\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}\left(\frac{1}{4} - \frac{1}{2} + 1\right)$

$d\left(\frac{1}{2}\right) = \frac{3\sqrt{2}}{8}$

17. $f(x) = \frac{1}{e^x + 2e^{-x}}$

$y = \frac{1}{e^x + 2e^{-x}} \qquad$ Let $e^x = t \in (0, \infty)$

$y = \frac{1}{t + \frac{2}{t}} \quad \Rightarrow \quad y = \frac{t}{t^2 + 2} \quad \Rightarrow \quad t^2y - t + 2y = 0$

$D \geq 0$

$1 - 8y^2 \geq 0$

$\Rightarrow \quad 8y^2 - 1 \leq 0 \quad \Rightarrow \quad y \in \left[\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right]$

but $y > 0$

$\therefore \; y \in \left(0, \frac{1}{2\sqrt{2}}\right]$

$\therefore \; f(0) = \frac{1}{3}$

$\therefore \; f(c) = \frac{1}{3} \quad (c \in R)$

So Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

18. $y = x + \frac{4}{x^2}$

$\frac{dy}{dx} = 1 - \frac{8}{x^3}$

Equation of tangent is parallel to x-axis

$\therefore \frac{dy}{dx} = 0$

$\Rightarrow 1 - \frac{8}{x^3} = 0 \Rightarrow x^3 = 8 \Rightarrow x = 2$

At, $x = 2, y = 2 + \frac{4}{4} = 3 \Rightarrow y_1 = 3$

$\therefore$ point is (2, 3) equation of tangent is :

$y - y_1 = 0(x - x_1)$

$y = 3$

19. f has a local minimum at $x = -1$

$\therefore \lim_{x \to -1} f(x) \geq f(-1)$

$k + 2 \leq 1$

$k \leq -1$

$\therefore k = -1$

20. $f'(x) = \sqrt{x} \sin x$

$f'(\pi)$ & $f'(2\pi)$ are 0.

$f'(x)$

$\Rightarrow$ local maximum at $x = \pi$ and local minimum at $x = 2\pi$

22. At $x = 0$ $f(x) = 1$

and for $x = h$ and $x = -h$ $(h \to 0; h > 0)$

$\frac{\tan x}{x} > 1$

$\therefore$ Function has a minima at $x = 0$

$\therefore$ Statement–1 is true.

Now $f(x) = \begin{cases} \frac{\tan x}{x} & ; x \neq 0 \\ 1 & ; x = 0 \end{cases}$

$f'(x) = \begin{cases} \frac{x \sec^2 x - \tan x}{x^2} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

$f'(0) = 0$

$\therefore$ Statement-2 is also true.

23. $V = \frac{4}{3}\pi r^3$

Initially $r = 4500\,\pi$, $r = r_0$

$4500\,\pi = \frac{4}{3}\pi\, r_0^3 \Rightarrow \boxed{r_0 = 15\text{m}}$

Now $\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$

$-72\,\pi = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{-18}{r^2}$ ......**(i)**

$\int r^2 dr = -\int 18\, dt \Rightarrow \frac{r^3}{3} = -18\,t + C$

At $t = 0$, $r = 15$ m

So, $\frac{(15)^3}{3} = -18(0) + C \Rightarrow C = 1125$

$\Rightarrow r^3 = -54t + 3375$ ......**(ii)**

At time $t = 49$ min $r = 9$ m

from eq. **(i)**

$\left(\frac{dr}{dt}\right)_{t=49} = \frac{-18}{(9)^2} = -2/9$

(Negative sign shows decrement in radii)

24. $f'(x) = \frac{1}{x} + 2bx + a$

$f'(-1) = -1 - 2b + a = 0$ ......**(1)**

$f'(2) = \frac{1}{2} + 4b + a = 0$ ...... **(2)**

solve (1) & (2) $\Rightarrow a = \frac{1}{2}$, $b = -\frac{1}{4}$

$\therefore$ st : 2 is true

$f''(x) = -\frac{1}{x^2} - \frac{1}{2} = -\left(\frac{1}{x^2} + \frac{1}{2}\right)$ (always –ive)

$f''(-1) = -\frac{3}{2} < 0$

$f''(2) = -\frac{3}{4} < 0$

$\therefore$ Local maximum at $x = -1$ & 2

25. $y = \int_0^x |t|\, dt$

$\frac{dy}{dx} = |x| = 2 \Rightarrow x = \pm 2$

If $x=2, \quad y = \int_0^2 t\,dt = 2$

If $x=-2, \quad y = \int_0^{-2} -t\,dt = -2$

Tangents are $(y-2) = 2(x-2)$

or $(y+2) = 2(x+2)$

x intercepts $= \pm 1$.

26. $f(x) = 2x^3 + 3x + k$

$f'(x) = 6x^2 + 3 > 0$

$\Rightarrow$ f is increasing function

$\Rightarrow$ $f(x) = 0$ has exactly one real root

(as it is an odd degree polynomial)

28. $4x + 2\pi r = 2 \Rightarrow \quad 2x + \pi r = 1$

$$\Rightarrow \quad r = \frac{1-2x}{\pi}$$

$f(x) = x^2 + \pi r^2$

$$= x^2 + \pi \times \frac{[1-2x]^2}{\pi^2}$$

$$f(x) = x^2 + \frac{(1-2x)^2}{\pi}$$

$$f'(x) = 2x - \frac{2(1-2x)\times(2)}{\pi} = 0$$

$$\Rightarrow \quad x = \frac{2(1-2x)}{\pi}$$

$\Rightarrow \quad \pi x = 2 - 4x$

$$\Rightarrow \quad \pi x = 2 - 4\left[\frac{1-\pi r}{2}\right]$$

$\pi x = 2 - 2(1 - \pi r)$

$\pi x = 2 - 2 + 2\pi r$

$\pi x = 2\pi r.$

Part # II : IIT-JEE ADVANCED

1. Slope of normal

$$= -\frac{1}{dy/dx} = \tan\frac{3\pi}{4} \Rightarrow \frac{dy}{dx} = 1$$

$\therefore f'(3) = 1$

6. $f(x) = 4x^3 - 3x - p$

$$f\left(\frac{1}{2}\right) = -(p+1)$$

$f(1) = (1-p)$

$$f(1)\,.\,f\left(\frac{1}{2}\right) = -(1-p^2) \le 0 \quad \because p \in [-1, 1]$$

$$\therefore \quad f(x) = 0 \text{ has atleast one root in } \left[\frac{1}{2}, 1\right]$$

$f'(x) = 3\,(2x-1)\,(2x+1)$

$$\Rightarrow \quad f'(x) > 0 \quad \forall\, x > \frac{1}{2}$$

$$\Rightarrow \quad f(x) = 0 \text{ has exactly one root in } \left[\frac{1}{2}, 1\right]$$

Let the root be $x = \cos\theta$

$\therefore \quad 4\cos^3\theta - 3\cos\theta = p$

$\cos 3\theta = p$

$$\Rightarrow \quad \theta = \frac{1}{3}\cos^{-1}(p) \quad \Rightarrow x = \cos\left(\frac{1}{3}\cos^{-1}(p)\right)$$

8. $3y^2y' + 6x = 12y'$

$2x = y'(4 - y^2)$

$$y' = \frac{2x}{(4-y^2)}$$

For vertical tangent $y = \underline{+}2$

At $y = 2$

$$8 + 3x^2 = 24 \Rightarrow 3x^2 = 16 \Rightarrow x = \underline{+}\frac{4}{\sqrt{3}}$$

At $y = -2$

$-8 + 3x^2 = -24$

x^2 = negative

Not possible

9. **(A)** $\cos x - 1 > -\frac{x^2}{2}$ (given) ....**(i)**

consider $f(x) = \sin(\tan x) - x$

$f'(x) = \cos(\tan x)\,(1 + \tan^2 x) - 1$

$= (\tan^2 x)\cos(\tan x) + \cos(\tan x) - 1$

$\cos(\tan x) - 1 > -\frac{\tan^2 x}{2}$ from **(i)**

$(\tan^2 x)\cos(\tan x) + \cos(\tan x) - 1 > \tan^2 x \{\cos(\tan x) - \frac{1}{2}\}$

$\Rightarrow \quad f'(x) > \tan^2 x \{\cos(\tan x) - \frac{1}{2}\}$

$0 \le \tan x \le 1 \quad \{\because 0 \le x \le \frac{\pi}{4}\}$

$\Rightarrow \quad \cos(\tan x) > \frac{1}{2}$

$\Rightarrow \quad f'(x) > 0$

$\Rightarrow \quad f(x) \ge f(0) \quad \Rightarrow \quad f(x) \ge 0$

(B) Consider $g(x) = \int_0^{x^2} f(t)dt$

$g(1) - g(0)$

$= g'(\alpha), \alpha \in (0, 1)$ {by LMVT in [0,1]} ....**(i)**

$g(2) - g(1)$

$= g'(\beta), \beta \in (1, 2)$ {by LMVT in [1,2]} ....**(ii)**

(i) + **(ii)** $\Rightarrow$ $g(2) - g(0) = g'(\alpha) + g'(\beta)$

$\Rightarrow \quad \int_0^4 f(t)dt = 2\{\alpha f(\alpha^2) + \beta f(\beta^2)\}$

14. Let $g(x) = \int p(x)\,dx + K$

$g(x) = \frac{x^{102}}{2} - 23x^{101} - \frac{45x^2}{2} + 1035x + K$

$= \frac{x^{102} - 46x^{101} - 45x^2 + 2070x}{2} + K$

$= \frac{x(x^{100} - 45)(x - 46)}{2} + K$

$g(45^{1/100}) = g(46)$

$\Rightarrow$ $g'(x) = 0$ has exactly one root in $(45^{1/100}, 46)$

15. Let $f(x) = \sin x + 2x$ & $g(x) = \frac{3x^2 + 3x}{\pi}$

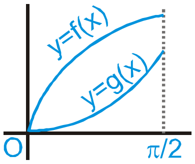
$f'(x) = \cos x + 2 \quad g'(x) = \frac{6x + 3}{\pi}$

$f''(x) = -\sin x \quad g''(x) = \frac{6}{\pi}$

$\Rightarrow$ 'f' is increasing & concave down in $\left[0, \frac{\pi}{2}\right]$

and 'g' is increasing & concave up in $\left[0, \frac{\pi}{2}\right]$

& $f\left(\frac{\pi}{2}\right) > g\left(\frac{\pi}{2}\right)$.



from the graph $f(x) \ge g(x)\ \forall\ x \in \left[0, \frac{\pi}{2}\right]$

17. Consider $g(x) = x^2 - f(x)$

'g' is continuous-derivable

$\therefore$ By Rolle's theorem

$g(1) = g(2) \Rightarrow g'(c_1) = 0$ for atleast one $c_1 \in (1, 2)$

$g(2) = g(3) \Rightarrow g'(c_2) = 0$ for atleast one $c_2 \in (2, 3)$

$g'(c_1) = g'(c_2)$

$\Rightarrow \quad g''(c) = 0$ for atleast one $c \in (c_1, c_2)$.

$\Rightarrow \quad 2 - f''(c) = 0$

$\Rightarrow \quad f''(c) = 2$

19. Put $x_1 = x + h$ & $x_2 = x$

$|f(x + h) - f(x)| \le h^2$

$$\lim_{h \to 0} \left|\frac{f(x + h) - f(x)}{h}\right| \le \lim_{h \to 0} h$$

$|f'(x)| \le 0$

Possible only if $f'(x) = 0$

$f(x) = c$

at point (1, 2) $f(x) = 2$

$y = 2$

20. Let $p(x) = ax^3 + bx^2 + cx + d$

$p(-1) = 10$

$\Rightarrow \quad -a + b - c + d = 10$ ......**(i)**

$p(1) = -6$

$\Rightarrow \quad a + b + c + d = -6$ ......**(ii)**

$p(x)$ has maxima at $x = -1$

$\therefore \quad p'(-1) = 0$

$\Rightarrow$ $3a - 2b + c = 0$ .....**(iii)**

p'(x) has min. at x = 1

$\therefore$ $p''(1) = 0$

$\Rightarrow$ $6a + 2b = 0$ .....**(iv)**

Solving **(i)**, **(ii)**, **(iii)** and **(iv)** we get

From **(iv)** $b = -3a$

From **(iii)** $3a + 6a + c = 0 \Rightarrow c = -9a$

From **(ii)** $a - 3a - 9a + d = -6 \Rightarrow d = 11a - 6$

From **(i)** $-a - 3a + 9a + 11a - 6 = 10$

$\Rightarrow$ $16a = 16 \Rightarrow a = 1$

$\Rightarrow$ $b = -3,\ c = -9,\ d = 5$

$\therefore$ $p(x) = x^3 - 3x^2 - 9x + 5$

$\Rightarrow$ $p'(x) = 3x^2 - 6x - 9 = 0$

$\Rightarrow$ $3(x+1)(x-3) = 0$

$\Rightarrow$ $x = -1$ is a pt. of max (given) and $x = 3$ is at pt. of min.

[$\because$ max and min occur alternatively]

$\therefore$ pt. of local max is $(-1, 10)$ and pt. of local min is $(3, -22)$

And distance between them is

$$= \sqrt{[3-(-1)]^2 + (-22-10)^2} = \sqrt{16 + 1024}$$

$$= \sqrt{1040} = 4\sqrt{65}$$

22. (a,b) $\because$ $g(x) = \int_0^x f(t)\,dt$

$$\Rightarrow g'(x) = f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - e^{x-1} & 1 < x \le 2 \\ x - e & 2 < x \le 3 \end{cases}$$

$\therefore$ $g'(x) = 0$

at $x = 1 + \ell n2$

$x = 0$ & $x = e$

$$g''(x) = \begin{cases} 1 & 0 \le x \le 1 \\ -e^{x-1} & 1 < x \le 2 \\ 1 & 2 < x \le 3 \end{cases}$$

$\therefore$ $g''(1 + \ln 2) = -2$ and $g''(e) = 1$

$\Rightarrow$ g(x) has local max. at $x = 1 + \ln 2$ and local min. at $x = e$.

32. **(A)** $y = \dfrac{x^2 + 2x + 4}{x + 2}$

$\Rightarrow$ $x^2 + (2 - y)x + 4 - 2y = 0$ x is real ; so $D \ge 0$

$y^2 + 4y - 12 \ge 0$

$y \le -6,\ y \ge 2$

so minimum value = 2

(B) $(A + B)(A - B) = (A - B)(A + B)$

$\Rightarrow$ $AB = BA$

as A is symmetric & B is skew symmetric

$\Rightarrow$ $(AB)^t = -AB$

$\Rightarrow$ $k = 1, 3$

(C) $a = \log_3 \log_3 2 \Rightarrow 3^{-a} = \log_2 3$

Now $1 < 2^{(-k+3^{-a})} < 2$

$\Rightarrow$ $1 < 2^{(-k+\log_2 3)} < 2$ $\Rightarrow$ $1 < 3.2^{-k} < 2$

$\Rightarrow$ $\dfrac{1}{3} < 2^{-k} < \dfrac{2}{3}$ $\Rightarrow$ $\dfrac{3}{2} < 2^{k} < 3$

so k = 1 is possible

(D) $\sin\theta = \cos\phi$

$\Rightarrow$ $\cos\left(\dfrac{\pi}{2} - \theta\right) = \cos\phi$

$\dfrac{\pi}{2} - \theta = 2n\pi \pm \phi$

$\dfrac{\pi}{2} - \theta = 2n\pi \pm \phi$

$\Rightarrow$ $\theta \pm \phi - \dfrac{\pi}{2} = -2n\pi$

$\Rightarrow$ $\dfrac{1}{\pi}(\theta \pm \phi - \dfrac{\pi}{2})$ = even integer

33. $f(x) = \dfrac{x^2 - ax + 1}{x^2 + ax + 1}$

$f'(x) = \dfrac{2a(x^2 - 1)}{(x^2 + ax + 1)^2}$ and $f''(x) = \dfrac{4a(-x^3 + 3x + a)}{(x^2 + ax + 1)^3}$

$f''(1) = \dfrac{4a}{(a+2)^2}$ and $f''(-1) = \dfrac{-4a}{(a-2)^2}$

$\therefore$ $(a + 2)^2 f''(1) + (2 - a)^2 f''(-1) = 0$

34. As when $x \in (-1, 1)$, $f'(x) < 0$

so f(x) is decreasing on (–1, 1) at x = 1

$f''(1) = \frac{4a}{(a+2)^2} > 0$ so local minima at x = 1.

35. $g(x) = \int_0^{e^x} \frac{f'(t)}{1+t^2}dt$

$$g'(x) = \frac{f'(e^x)}{1+e^{2x}}e^x = \frac{2a(e^{2x}-1)e^x}{(e^{2x}+ae^x+1)^2(1+e^{2x})}$$

$g'(x) > 0$ when $x > 0$

$g'(x) < 0$ when $x < 0$

36. $f(x) = 2x^3 - 15x^2 + 36x - 48$

Set $A = \{x \mid x^2 + 20 \le 9x\}$

$\because x^2 - 9x + 20 \le 0$

$(x-5)(x-4) \le 0$

$\Rightarrow x \in [4, 5]$

Now, $f'(x) = 6x^2 - 30x + 36 = 0$

$\Rightarrow x^2 - 5x + 6 = 0$

$x = 2, 3$ and $f(x) \uparrow$ in $x \in (-\infty, 2) \cup (3, \infty)$

$\Rightarrow$ In the set A, f(x) is increasing

$\Rightarrow f(x)_{max} = f(5)$

$= 2.125 - 15.25 + 36.5 - 48$

$= 7$

37. $\underset{x\to0}{Lt}\left(1 + \frac{p(x)}{x^2}\right) = 2$

$\Rightarrow \underset{x\to0}{Lt} \frac{p(x)}{x^2} = 1$

Let $p(x) = ax^4 + bx^3 + cx^2$

$\because \underset{x\to0}{Lt} \frac{p(x)}{x^2} = 1 \Rightarrow c = 1$

$p(x) = ax^4 + bx^3 + x^2$

Now, $p'(x) = 4ax^3 + 3bx^2 + 2x$

$\because p'(1) = 0, p'(2) = 0$

$\Rightarrow 4a + 3b + 2 = 0$

$32a + 12b + 4 = 0 \Rightarrow a = \frac{1}{4}, b = -1$

$\Rightarrow p(x) = \frac{1}{4}x^4 - x^3 + x^2 \Rightarrow p(2) = 4 - 8 + 4 = 0$

39. $f(x) = (1 + b^2)x^2 + 2bx + 1$

It is a quadratic expression with coeff. of

$x^2 = 1 + b^2 > 0$.

$\therefore$ f (x) represents an upward parabola whose

min value is $\frac{-D}{4a}$, D being the discriminant.

$\therefore m(b) = -\frac{4b^2 - 4(1+b^2)}{4(1+b^2)} \Rightarrow m(b) = \frac{1}{1+b^2}$

For range of m(b) :

$\frac{1}{1+b^2} > 0$ also $b^2 \ge 0 \Rightarrow 1 + b^2 \ge 1$

$\Rightarrow \frac{1}{1+b^2} \le 1$

Thus m(b) = (0, 1]

41. $f(x) = \ell nx + \int_0^x \sqrt{1+\sin t}\,dt$

$f'(x) = \frac{1}{x} + \sqrt{1+\sin x}$

$f'(x) = \frac{1}{x} + \sqrt{2}\left|\cos\left(\frac{x}{2} - \frac{\pi}{4}\right)\right|$

$\because \left|\cos\left(\frac{x}{2} - \frac{\pi}{4}\right)\right|$ is non-derivable

$\therefore$ f '(x) is non-derivable but continuous.

hence option (A) is incorrect & option (B) is correct.

For option C

$f(x) = (\ell nx) + \int_0^x \left(\sqrt{1+\sin x}\right)dx$

since f(x) is positive increasing function for all x > 1

$\Rightarrow |f(x)| = f(x)$ & $|f'(x)| = f'(x)$

Let f(x) = y

$$f'(x) - f(x) = \frac{1}{x} - \ell nx + \sqrt{1+\sin x} - \int_0^x \sqrt{1+\sin t}\,dt$$

$$f'(x)-f(x)=\frac{1}{x}-\ell nx+\sqrt{1+\sin x}-\sqrt{2}\int_0^x\left|\cos\left(\frac{t}{2}-\frac{\pi}{4}\right)\right|dt$$

$\frac{1}{x}-\ell nx<0$; when α > e

$0\le\sqrt{1+\sin x}\le\sqrt{2}\cdot$

$\int_0^x\left|\cos\left(\frac{t}{2}-\frac{\pi}{4}\right)\right|dt>\sqrt{2}\ \forall\alpha>\frac{3\pi}{2}$

$\Rightarrow$ f'(x) – f(x) < 0 ∀ $\alpha>\frac{3\pi}{2}>1$

Hence option **(C)** is correct.

For option **(D)** |f(x)| + |f'(x)| → ∞

when x → ∞.

Therefore option **(D)** is incorrect.

Alternate :

$f(x)=\ell nx+\int_0^x\sqrt{1+\sin t}dt$

$f'(x)=\frac{1}{x}+\sqrt{1+\sin x}$ ....**(i)**

for x > 1

$\frac{1}{x}+\sqrt{1+\sin x}<1+\sqrt{2}$

but $\ell nx+\int_0^x\sqrt{1+\sin t}\,dt$ will always be more than $1+\sqrt{2}$ for some α > 1

$\because\int_0^x\sqrt{1+\sin t}>0$ & ℓnx is increasing in (1, ∞)

$\Rightarrow$ f(x) > f'(x) ∀ α > 1

∴ **(C)** is correct

$$f''(x)=-\frac{1}{x^2}+\frac{\cos x}{2\sqrt{1+\sin x}}$$

$\Rightarrow$ f ' is not derivable on (0, ∞)

at $\frac{3\pi}{2},\frac{7\pi}{2}$

∴ **(B)** is also correct

f(x) is unbounded near x = 0 in (0, 1) hence |f(x)| can never be made less than a finite number hence |f(x)|+|f '(x)| can never be less than β.

42. If x ∈ [0, 1]

then $x^2\le x\le 1$

$x^2e^{x^2}\le xe^{x^2}\le e^{x^2}$

Add e^{-x^2} to all sides

$x^2e^{x^2}+e^{-x^2}\le xe^{x^2}+e^{-x^2}\le e^{x^2}+e^{-x^2}$

$\Rightarrow$ $h(x)\le g(x)\le f(x)$ ......**(i)**

where, $f(x)=e^{x^2}+e^{-x^2}$

$f'(x)=2x(e^{x^2}-e^{-x^2})>0$

$\Rightarrow$ f(x) has a maxima at x = 1

$\Rightarrow$ $a=e+\frac{1}{e}$

$h(x)=x^2e^{x^2}+e^{-x^2}$

$h'(x)=2x^3e^{x^2}+2xe^{x^2}-2xe^{-x^2}$

$=2x^3e^{x^2}+2x(e^{x^2}-e^{-x^2})>0$

$\Rightarrow$ h(x) has a maxima at x = 1

$\Rightarrow$ $c=e+\frac{1}{e}$

$\because$ $h(x)\le g(x)\le f(x)$

$\Rightarrow$ g(x) also has a maximum value at x = 1

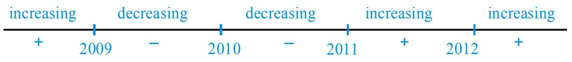
$\Rightarrow$ a = b = c

43. $f'(x)=2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4\ \forall x\in R$

f(x) = ℓn(g(x)) ∀ x ∈ R

$g(x)=e^{f(x)}$

$g'(x)=0\Rightarrow e^{f(x)}.f'(x)=0\Rightarrow f'(x)=0$



local maximum at x = 2009, hence only 1 point.

44. Ans. (A)

$f : (0,1) \to R$

$$f(x) = \frac{b-x}{1-bx} \qquad b \in (0,1)$$

$$\Rightarrow \quad f'(x) = \frac{b^2-1}{(bx-1)^2}$$

$\Rightarrow \quad f'(x) < 0 \ \forall \ x \in (0, 1)$

hence $f(x)$ is decreasing function

hence its range $(-1, b)$

$\Rightarrow$ co-domain $\neq$ range

$\Rightarrow \quad f(x)$ is non-invertible function

45. Ans. 2

Let $f(x) = x^4 - 4x^3 + 12x^2 + x - 1$

$f'(x) = 4x^3 - 12x^2 + 24x$

$f''(x) = 12x^2 - 24x + 24$

$= 12(x^2 - 2x + 2) > 0$

$\Rightarrow \quad f'(x)$ is strictly increasing function

$\because \quad f'(x)$ is cubic polynomial

hence number of roots of $f'(x) = 0$ is 1

$\Rightarrow$ Number of maximum roots of $f(x) = 0$ are 2

Now $f(0) = -1$, $f(1) = 9$, $f(-1) = 15$

$\Rightarrow \quad f(x)$ has exactly 2 distinct real roots.

46. $f(x) = (1-x)^2 \sin^2 x + x^2$

P : $f(x) + 2x = 2(1 + x^2)$

$\Rightarrow \quad (1-x)^2 \sin^2 x + x^2 + 2x = 2 + 2x^2$

$\Rightarrow \quad (1-x)^2 \sin^2 x - x^2 + 2x - 2 = 0$

$(1-x)^2 \cos^2 x + 1 = 0$

which is not possible.

$\therefore$ P is false.

Q : $2f(x) + 1 = 2x(1 + x)$

$2x^2 + 2(1-x)^2 \sin^2 x + 1 = 2x^2 + 2x$

$2(1-x)^2 \sin^2 x - 2x + 1 = 0.$

Let $h(x) = 2(1-x)^2 \sin^2 x - 2x + 1,$

clearly $h(1) = -1$

and $h(x) = 2(x^2 - 2x + 1)\sin^2 x - 2x + 1$

$$= x^2\left[2\left(1 - \frac{2}{x} + \frac{1}{x^2}\right).\sin^2 x - \frac{2}{x} + \frac{1}{x^2}\right]$$

$\therefore \quad h(x) \to \infty$ as $x \to \infty$.

$\therefore$ By intermediate value theorem

$h(x) = 0$ has a root which is greater than 1.

Hence Q is true.

47. $$g(x) = \int_1^x \left(\frac{2(t-1)}{(t+1)} - \ell nt\right) f(t)dt$$

$$g'(x) = \left(\frac{2(x-1)}{x+1} - \ell nx\right) f(x)$$

$f(x) > 0 \quad \forall x \in R$

Suppose.

$$h(x) = \frac{2(x-1)}{x+1} - \ell nx$$

$$h(x) = 2 - \left(\frac{4}{x+1} + \ell nx\right)$$

$$h'(x) = \frac{4}{(x+1)^2} - \frac{1}{x}$$

$$h'(x) = -\frac{(x-1)^2}{x(x+1)^2}$$

$h'(x) < 0$

So $h(x)$ is decreasing

so $h(x) < h(1). \qquad \forall x > 1$

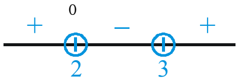
$h(x) < 0 \qquad \forall x > 1$

So $g'(x) = h(x) f(x)$

$g'(x) < 0 \qquad \forall x > 1$

$g(x)$ is decreasing in $(1, \infty)$.

48. $$f(x) = \int_0^x e^{t^2} (t-2)(t-3)dt$$



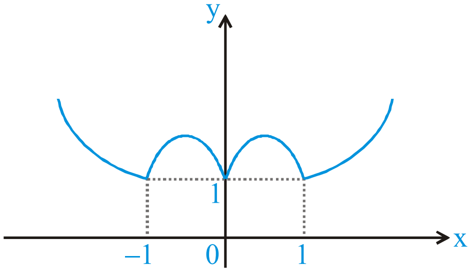
$\Rightarrow \quad f'(x) = e^{x^2} (x-2)(x-3)$

$\therefore \quad f'(2) = f'(3) = 0$

$\Rightarrow \quad f''(c) = 0$ for same $c \in (2,3)$ (by Rolle's theorem)

49. $f(x) = |x| + |(x + 1)(x - 1)|$

$$\Rightarrow \quad f(x) = \begin{cases} x^2 - x - 1 & x < -1 \\ -x^2 - x + 1 & -1 \quad x < 0 \\ -x^2 + x + 1 & 0 \quad x < 1 \\ x^2 + x - 1 & x \quad 1 \end{cases}$$



$\therefore$ f has 5 points where it attains either a local maximum or local minimum.

50. Let $P'(x) = k(x - 1)(x - 3)$

$= k(x^2 - 4x + 3)$

$\Rightarrow \quad P(x) = k\left(\frac{x^3}{3} - 2x^2 + 3x\right) + c$

$\because \quad P(1) = 6$

$\Rightarrow \quad \frac{4k}{3} + c = 6$ ....**(1)**

$P(3) = 2$

$\Rightarrow \quad c = 2$ ....**(2)**

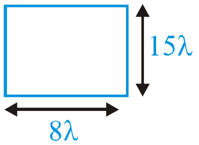
by **(i)** and **(ii)**

$k = 3$

$\therefore P'(x) = 3(x - 1)(x - 3)$

$\Rightarrow \quad P'(0) = 9$

51. Where $P = 8\lambda + 15\lambda + 8\lambda + 15\lambda$ & λ is constant



Let removed length from each sides is x

Removed area is $4x^2 = 100 \quad \Rightarrow \quad x = 5$

$V = (8\lambda - 2x)(15\lambda - 2x)x$

$V = 120l^2x - 46\lambda x^2 + 4x^3$

$$\frac{dv}{dx} = 120\lambda^2 - 92\lambda x + 12x^2 = 0$$

Put $x = 5 \Rightarrow 120\lambda^2 - 460\lambda + 300 = 0$

$12\lambda^2 - 40\lambda + 30 = 0$

$6\lambda^2 - 23\lambda + 15 = 0$

$(\lambda - 3)(6\lambda - 5) = 0$

$\lambda = 3$ & $\lambda = \frac{5}{6}$

$$\frac{d^2v}{dx^2} = -92\lambda + 24x = 120 - 92\lambda$$

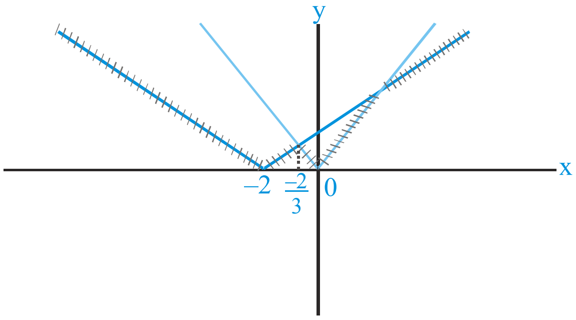
at $\lambda = 3 \Rightarrow \frac{d^2v}{dx^2} < 0$

at $\lambda = \frac{5}{6} \Rightarrow \frac{d^2v}{dx^2} > 0$ (rejected)

52. $f(x) = (a + b) - |b - a|$

$$= \begin{cases} 2a, & a \le b \\ 2b, & a > b \end{cases} = 2\min(a, b)$$

where $a = 2|x|$, $b = |x + 2|$

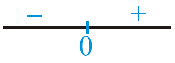


$\therefore$ Local maxima and minima at $x = -2, -\frac{2}{3}$ & 0

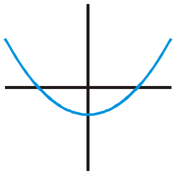
53. $f(x) = x^2 - x\sin x - \cos x$

$f'(x) = 2x - x\cos x - \sin x + \sin x$

$= x(2 - \cos x)$



$\therefore$ graph of $f(x)$ will be



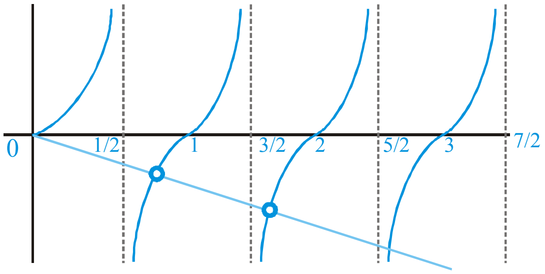
$\therefore$ $f(x)$ is zero for 2 values of x

$\therefore$ **(C)**

54. $f'(x) = \sin\pi x + \pi x \cos\pi x = 0$

$\tan \pi x = -\pi x$

$y = \tan x\pi \;\&\; y = -\pi x$



intersection point lies in

$$\left(\frac{1}{2},1\right)\cup\left(\frac{3}{2},2\right)\cup\left(\frac{5}{2},3\right)...$$

option **(B)** is correct for $\left(n+\frac{1}{2},n\right)$

as well (n, (n + 1)) because root lies in $(0, 1)\cup(1, 2)\cup(2, 3)$

58. $f'(x)+\frac{f(x)}{x}=2$

$\Rightarrow\; xf'(x)+f(x)=2x$

$\Rightarrow\; \int d(x.f(x))=\int 2x dx$

$\Rightarrow\; xf(x)=x^2+c$

$f(x)=x+\frac{c}{x}(c\neq 0 \text{ as } f(1)\neq 1)$

For this function, only **(A)** is correct.

MOCK TEST

2. (A)

Here $f(x) = \frac{a\sin x + b\cos x}{c\sin x + d\cos x}$ is monotonically increasing

then $f'(x) \geq 0$

Now $f'(x)$

$$=\frac{(c\sin x+d\cos x)(a\cos x-b\sin x)-(a\sin x+b\cos x)(c\cos x-d\sin x)}{(c\sin x+d\cos x)^2}$$

$$=\frac{\begin{gathered}ac\sin x\cos x-bc\sin^2 x+ad\cos^2 x-bd\sin x\cos x\\-ac\sin x\cos x+ad\sin^2 x-bc\cos^2 x+bd\sin x\cos x\end{gathered}}{(c\sin x+d\cos x)^2}$$

$$=\frac{ad(\sin^2 x+\cos^2 x)-bc(\sin^2 x+\cos^2 x)}{(c\sin x+d\cos x)^2}$$

$$=\frac{ad-bc}{(c\sin x+d\cos x)^2}$$

Here $f'(x)\geq 0 \;\forall\; x\in R$

$\Rightarrow\; ad - bc \geq 0$

$ad \geq bc$

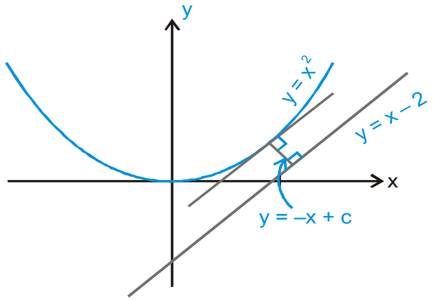
3. $y = x^2$

$\frac{dy}{dx}=2x$

$2x = 1$

$\Rightarrow\; x=\frac{1}{2}$

$\left(\frac{1}{2},\frac{1}{4}\right)$ on parabola



Figure

shortest canal will be along the common normal of $y = x^2$ and $y = x - 2$ which will be,

$y = -x + c$

$\because$ it passes through $\left(\frac{1}{2},\frac{1}{4}\right) \Rightarrow c=\frac{3}{4}$

solving, $y = x - 2$ and $y=-x+\frac{3}{4}$

$y=-\frac{5}{8}$ and $x=\frac{11}{8}$

Hence point on straight line along the shortest canal is $\left(\frac{11}{8},\frac{-5}{8}\right)$

4. (B)

Consider the function $f(x) = \frac{a_0x^{n+1}}{n+1} + \frac{a_1x^n}{n} + \frac{a_2x^{n-1}}{n-1}$

$+ .... + \frac{a_{n-1}x^2}{2} + a_nx.$

Then $f(0) = 0$ and $f(1) = 0$

hence $f'(x) = 0$ has at least one solution in $(0, 1)$

5. $f'(x) = \sqrt{4ax - x^2} + \frac{x(4a - 2x)}{2\sqrt{4ax - x^2}} = \frac{6ax - 2x^2}{\sqrt{4ax - x^2}} < 0,$

$\forall\, x \in (4a, 3a)$

so f(x) is decreasing in [4a, 3a]

6. (D)

$f(x) = 8ax - a \sin 6x - 7x - \sin 5x$

$f'(x) = 8a - 6a \cos 6x - 7 - 5\cos 5x$

$= 8a - 7 - 6a \cos 6x - 5\cos 5x$

f(x) is an increasing function

$f'(x) \geq 0 \quad \therefore \quad 8a - 7 \geq 6a + 5$

$\Rightarrow \quad 2a \geq 12$

$a \geq 6$

$a \in [6, \infty)$

7. $f'(x) = 0 \Rightarrow x = \frac{1}{a}, \frac{-2}{3a}$

since, we have a cubic polynomial with coefficient of x^3 +ve , minima will occur after maxima.

Case - 1 : If $a > 0$

then $\frac{1}{a} = \frac{1}{3} \Rightarrow a = 3$ also $f\left(\frac{1}{3}\right) > 0 \Rightarrow b < -\frac{1}{2}$

Case - 2 : If $a < 0$

then $-\frac{-2}{3a} = \frac{1}{3} \Rightarrow a = -2$

also $f\left(\frac{1}{3}\right) > 0 \Rightarrow \frac{(-2)^2}{3^2} - \frac{(-2)}{2} \cdot \frac{1}{3^2} - 2\left(\frac{1}{3}\right) - b > 0$

$\Rightarrow \quad \frac{4}{27} + \frac{1}{9} - \frac{2}{3} - b > 0 \quad \Rightarrow \quad b < -\frac{11}{27}$

8. (B)

Time taken by the truck $= \frac{300}{x}$ hours

$\therefore$ Fuel consumed $= \left(2 + \frac{x^2}{600}\right)\frac{300}{x}$ litre

$\therefore$ expenses on travelling

$$E = 200 \times \frac{300}{x} + \left(2 + \frac{x^2}{600}\right)\frac{3000}{x}$$

$$= \frac{60000}{x} + \frac{6000}{x} + 5x = \frac{66000}{x} + 5x$$

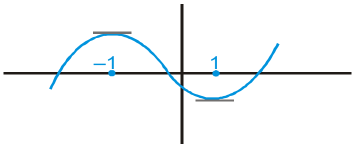
$\therefore \quad \frac{dE}{dx} = -\frac{66000}{x^2} + 5 < 0 \qquad$ for all $x \in [30, 60]$

$\therefore$ most economical speed is 60 kmph.

9. $f(x) = x^3 - 3x + k,\ k = [a]$

$f'(x) = 3(x - 1)(x + 1)$

-1 is maxima is 1 is minima



Figure

for three roots $f(-1)\, f(1) < 0$

$\Rightarrow \quad (k + 2)(k - 2) < 0$

$k \in (-2, 2)$

$\Rightarrow \quad -2 < [a] < 2$

$\Rightarrow \quad -1 \leq a < 2$

10. (A)

S_1 : $f(x) = x\, e^{x(1-x)}$

$f'(x) = e^{x-x^2} + x\, e^{x-x^2}(1 - 2x) = e^{x-x^2}(1 + x - 2x^2)$

$= -e^{x-x^2}(2x + 1)(x - 1) \geq 0$ (for increasing function)

$x \in \left[-\frac{1}{2}, 1\right]$

S_2 : $f(x) = (x - 2)^{2/3}(2x + 1)$

$f'(x) = \frac{2}{3}(x - 2)^{-1/3}(2x + 1) + 2(x - 2)^{2/3} = \frac{2}{3(x - 2)^{1/3}}$

$(2x + 1) + 2(x - 2)^{2/3}$

$$= \frac{2}{(x-2)^{1/3}}\left[\frac{2x+1}{3}+x-2\right]$$

$$= \frac{2}{(x-2)^{1/3}}\ \frac{(5x-5)}{3}$$

$x = 2$ and $x = 1$ are critical points

S_3 : $f'(x) = 2 - \frac{1}{1+x^2} + \frac{1}{\left(\sqrt{1+x^2}-x\right)}\left\{\frac{x}{\sqrt{1+x^2}}-1\right\}$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$$

$$= 2 - \left(\frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2}}\right)$$

$f'(x) > 0\ \forall\ x \in R$

S_4 : $\frac{d}{dx}\ f^2(x) = 2\, f(x)\, f'(x) < 0$

so $f^2(x)$ is decreasing function

11. (B, C)

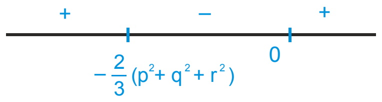
$\because$ g(x) is increasing & f(x) is decreasing.

$\Rightarrow$ $g(x+1) > g(x-1)$ & $f(x+1) < f(x-1)$

$\Rightarrow$ $f\{g(x+1)\} < f\{g(x-1)\}$ & $g\{f(x+1)\} < g\{f(x-1)\}$

12. $f(x) = \begin{vmatrix} x+p^2 & pq & pr \\ pq & x+q^2 & qr \\ pr & qr & x+r^2 \end{vmatrix} = x^3 + (p^2+r^2+q^2)\,x^2$

$f'(x) = 3x^2 + 2x(p^2+q^2+r^2) = x\,\{3x + 2(p^2+q^2+r^2)\}$



Here f(x) is increasing if $x < -\frac{2}{3}(p^2+q^2+r^2)$ and $x > 0$

decreasing is if $-\frac{2}{3}(p^2+q^2+r^2) < x < 0$

13. (B, C)

$f(x) = x^3 - x^2 + 100x + 1001$

$f'(x) = 3x^2 - 2x + 100 > 0\quad \forall\ x \in R$

$\therefore$ f(x) is increasing (strictly)

$\therefore$ $f\left(\frac{1}{1999}\right) > f\left(\frac{1}{2000}\right)$

$f(x+1) > f(x-1)$

14. (A) Let $x \Rightarrow x+h$ and $y \to x$

$|\tan^{-1}x - \tan^{-1}y| \le |x-y|$

$|\tan^{-1}(x+h) - \tan^{-1}x| \le |h|$

$$\left|\frac{d}{dx}(\tan^{-1}x)\right| \le 1$$

$\Rightarrow$ $\left|\frac{1}{1+x^2}\right| \le 1$ hence true

(C) $|\sin x - \sin y| \le |x-y|$

$x \to x+h \qquad y \to x$

$$\left|\frac{\sin(x+h)-\sin x}{h}\right| \le 1$$

$\Rightarrow$ $|\cos x| \le 1$ hence true

Alternative solutions

For x = y, this is true

$\therefore$ Let $x, y \in R$ and $x < y$

consider $f(t) = \tan^{-1}t,\ t \in [x, y]$

Using LMVT, $\frac{\tan^{-1}y - \tan^{-1}x}{y-x} = \frac{1}{1+c^2}$, $c \in (x, y)$

$\Rightarrow$ $\tan^{-1}y - \tan^{-1}x = \frac{y-x}{1+c^2} \le y - x$ ....(i)

similarly $x > y$, $\tan^{-1}x - \tan^{-1}y \le x - y$ ....(ii)

From (i) and (ii) we get $\left|\tan^{-1}x - \tan^{-1}y\right| \le |x-y|$

Similarly considering $g(t) = \sin t$ in $[x, y]$

we get $\frac{\sin y - \sin x}{y-x} = \cos c$

$\Rightarrow$ $\sin y - \sin x = (\cos c)(y-x) \le y - x$ ....(iii)

and $\sin x - \sin y \le x - y$ ....(iv)

(iii), (iv) $\Rightarrow$ $|\sin x - \sin y| \le |x-y|$

15. (A,C,D)

$f(x) = (x-1)^4(x-2)^n,\ n \in N$ ....(1)

$\therefore$ $f'(x) = 4(x-1)^3(x-2)^n + (x-1)^4\, n\,(x-2)^{n-1}$

$= (x-1)^3(x-2)^{n-1}(4x - 8 + nx - n)$

$= (x-1)^3(x-2)^{n-1}[(n+4)x - (n+8)]$

If n is odd, then $f'(x) > 0$ if $x < 1$ and sufficiently close to 1 and $f'(x) < 0$ if $x > 1$ and sufficiently close to 1

$\therefore$ $x = 1$ is point of local maximum

Similary if n is even, then $x = 1$ is a point of local minimum

Further if n is even, then $f'(x) < 0$ for $x < 2$ and sufficiently close to 2 and $f'(x) > 0$ for $x > 2$ and sufficiently close to 2.

$\therefore$ $x = 2$ is a point of local minimum.

16. (D)

Let the slope of the tangent be denoted by $\tan\psi$

length of tangent = $y \operatorname{cosec}\psi$

length of normal = $y \sec\psi$

$$\therefore \quad \frac{\text{length of tangent}}{\text{length of normal}} = \cot\psi \propto y$$

$\therefore$ Statement-1 is false

$$\text{length of normal} = y\sec\psi = \left| y\sqrt{1+m^2} \right|$$

$$\text{length of tangent} = y\operatorname{cosec}\psi = \left| \frac{y\sqrt{1+m^2}}{m} \right|$$

$\therefore$ Statement-1 is False, Statement-2 is True

17. (A)

Statement-II is obviously true.

Statement-I : Consider the function $f(x) = e^x$. $P(x_1, e^{x_1})$ and $Q(x_2, e^{x_2})$ are two points on the curve $y = f(x)$ and a point R divides the line joining P and Q internally in the ratio 1 : 2, then coordinates of R are

$$\left(\frac{2x_1+x_2}{3}, \frac{2e^{x_1}+e^{x_2}}{3}\right).$$

$$\therefore \quad \frac{2e^{x_1}+e^{x_2}}{3} > e^{\left(\frac{2x_1+x_2}{3}\right)}$$

Statement-I is true.

Statement-II is true and it explains Statement - 1.

18. (D)

Let $g(x)$ be the inverse function of $f(x)$. Then $f(g(x)) = x$.

$\therefore$ $f'(g(x)).g'(x) = 1$

i.e. $g'(x) = \dfrac{1}{f'(g(x))}$

$$\therefore \quad g''(x) = -\frac{1}{(f'(g(x))^2}.f''(g(x)).g'(x)$$

In statement-1 $f''(g(x)) > 0$ and $g'(x) > 0$

$\Rightarrow$ $g''(x) < 0$

$\Rightarrow$ concavity of $f^{-1}(x)$ is downwards

$\therefore$ statement is false

In statement-2 $f''(g(x)) > 0$ and $g'(x) < 0$

$\Rightarrow$ $g''(x) > 0$

$\Rightarrow$ concavity of $f^{-1}(x)$ is upwards

$\therefore$ statement is true

19. (D)

Statement-I : $5 - 4(x-2)^{2/3}$ attains greatest value at $x = 2$

Statement-II : obviously true

20. (D)

Statement-2 $f(x) = \dfrac{x^2}{x^3+200}$

$$f'(x) = \frac{(x^3+200)2x - 3x^2x^2}{(x^3+200)^2} = \frac{-x^4+400x}{(x^3+200)^2}$$

As $x \to 0^+$, $f(x) \to 0^+$

At $x = 400^{1/3}$, $f(x) = \dfrac{400^{2/3}}{600}$

As $x \to \infty$, $f(x) \to 0$

So statement : 2 is true

but statement : 1 is false as $400^{1/3} \notin N$

21. (A)→(r), (B)→(q), (C)→(t), (D)→(s)

(A) $\dfrac{dy}{dx} = \dfrac{4t}{3}$

Tangent is $y - at^4 = \dfrac{4t}{3}(x - at^3)$

x-intercept $= \dfrac{at^3}{4}$ y-intercept $= -\dfrac{at^4}{3}$

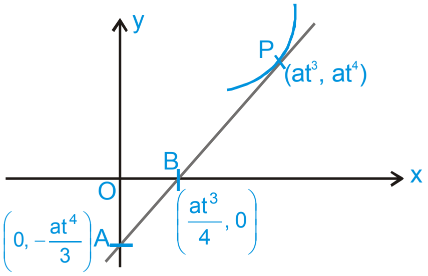
If P divides AB in ratio $\lambda : 1$

$$\Rightarrow \quad at^3 = \frac{\lambda.0 + \frac{at^3}{4}}{\lambda+1}$$

$$\Rightarrow \quad \lambda = \frac{-3}{4}$$

$\therefore \quad \frac{m}{n} = -\frac{3}{4}$

$\therefore \quad m = 3, n = 4$



$\therefore \quad m + n = 7$

(B) $\frac{dx}{dy} = e^{\sin y} \cos y$: slope of normal $= -1$

equation of normal is $x + y = 1$

Area $= \frac{1}{2}$

(C) $y = \frac{1}{x^2}$: $\frac{dy}{dx} = -\frac{2}{x^3}$: slope of tangent $= -2$

$y = e^{2-2x}$: $\frac{dy}{dx} = e^{2-2x} . (-2)$: slope of tangent $= -2$

$\therefore \quad \tan\theta = 0$

(D) Length of subtangent $= \left|\frac{y}{y'}\right| = \left|\frac{be^{x/3}}{b\frac{1}{3}e^{x/3}}\right| = 3$

22. **(A)** → **(q)**, **(B)** → **(r)**, **(C)** → **(r)**, **(D)** → **(t)**

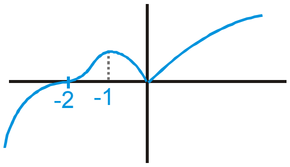
(A) By graph it is clear that at $x = -1$ is local max. and $x = 0$ is local min.

(B) $a + b = 1$

$$\sqrt{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)} = \sqrt{1+\frac{1}{a}+\frac{1}{b}+\frac{1}{ab}} = \sqrt{1+\frac{2}{ab}}$$

$$\sqrt{ab} < \frac{a+b}{2} = \frac{1}{2}$$

$\therefore \quad ab < \frac{1}{4} \qquad \Rightarrow \quad \frac{1}{ab} > 4$



$\therefore \quad \sqrt{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)} \geq \sqrt{1+8} = 3$

(C) $\therefore \quad y = 10 - (10 - x) = x$

$\therefore$ maximum value of $y = 3$

(D) Equation of tangent at P is $ty = x + t^2$

it intersects the line $x = 0$ at Q

$\therefore$ coordinates of Q are $(0, t)$

$\therefore$ area of $\Delta PQS = \frac{1}{2}\begin{vmatrix} 0 & t & 1 \\ 1 & 0 & 1 \\ t^2 & 2t & 1 \end{vmatrix}$

$= \frac{1}{2}[-t(1-t^2) + 2t] = \frac{1}{2}(t + t^3)$

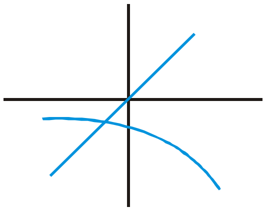
$\frac{dA}{dt} = \frac{1}{2}(3t^2 + 1) > 0 \quad \forall\, t \in [0, 2]$

$\therefore$ Area is maximum for $t = 2$

max area $= \frac{1}{2}[2 + 8] = 5.$

23.

1.



2. (A)

Consider $y = ke^x$ and $y = x$

Let (α, ke^{α}) be a point on $y = ke^x$

if it lies on $y = x$ also then $\alpha = ke^{\alpha}$

$\frac{dy}{dx} = ke^x$

$\therefore \quad \left.\frac{dy}{dx}\right|_{x=\alpha} = ke^{\alpha} = \alpha = 1$

$\{\because\ y = x$ is tangent to $y = ke^x$ at one point$\}$

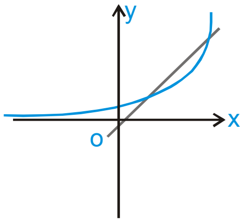
$\therefore \quad 1 = ke \qquad$ i.e. $k = 1/e$

3. (A)

Consider $y = ke^x$ and $y = x$

from above question $e^x = \frac{x}{k}$ if we decrease the value of k from $\frac{1}{e}$, then

slope of $y = \frac{x}{k}$ increases



$\therefore \quad y = e^x$ and $y = \frac{x}{k}$ intersect at two distinct points

$\therefore \quad k \in \left(0, \frac{1}{e}\right)$

24.

1. (D)

Let $0 < \alpha < \beta < 1$, and α, β are the roots of

$f(x) = x^3 - 3x + k = 0 \quad \Rightarrow \quad f(\alpha) = f(\beta) = 0$

$\Rightarrow$ f(x) satisfies RMVT

$\Rightarrow \quad f'(c) = 0 \qquad \Rightarrow \quad 3c^2 = 3$

$\Rightarrow \quad c = \pm 1$

but c must be lies between α & β.

Hence $k \in \phi$

2. (A)

Let $f(x) = \tan^{-1}x$

then for some $\alpha \in (x, y)$, $f'(\alpha) = \frac{\tan^{-1}y - \tan^{-1}x}{y - x}$ (LMVT)

$$\Rightarrow \left|\frac{1}{1+\alpha^2}\right| = \left|\frac{\tan^{-1}x - \tan^{-1}y}{x - y}\right| \left(\left|\frac{1}{1+\alpha^2}\right| \le 1\right)$$

$\Rightarrow \quad |\tan^{-1}x - \tan^{-1}y| \le |x - y|$

3. (D)

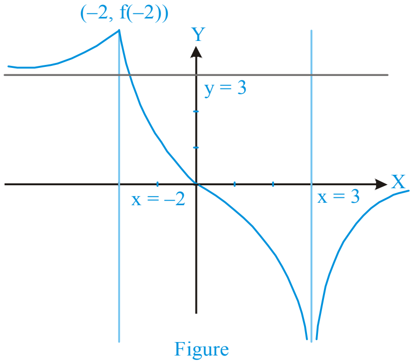
Let $f(x) = \sin x$ & $g(x) = \cos x$, Also $\sin x \ne 0$

for $x \in \left(0, \frac{\pi}{2}\right)$, then by cauchy's theorem

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(\theta)}{g'(\theta)}$$

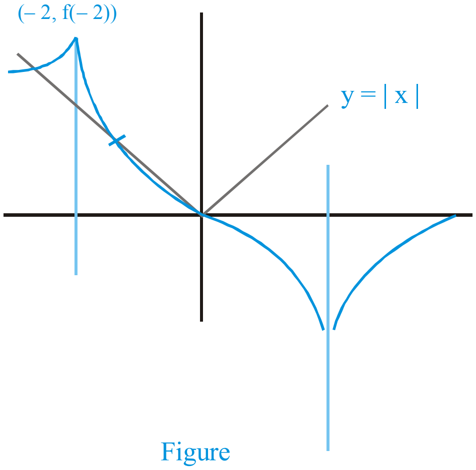
$$\Rightarrow \quad \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta$$

25. Graph of $y = f(x)$



Figure

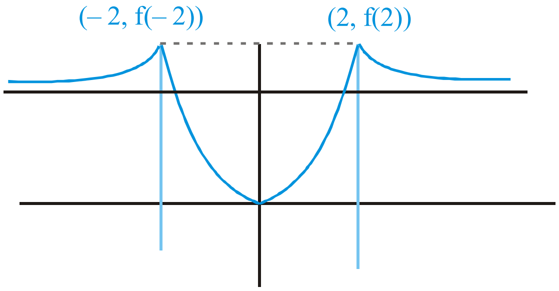
1.



Figure

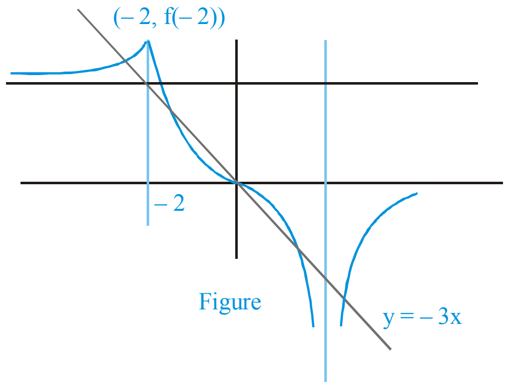
Three points of intersection. Three solutions

2.



Figure

3.



26. $y = \dfrac{1}{1-x}, x = 2 \Rightarrow y = -1$.

Let $P(2, -1)$. Tangent at P is $x - y = 3$. .....**(i)**

Chord of parabola with P as mid-point is

$(4a^2 - 5a)x + y = 8a^2 - 10a - 1$ .....**(ii)**

Comparing **(i)** and **(ii)** $\dfrac{4a^2 - 5a}{1} = \dfrac{1}{-1} = \dfrac{8a^2 - 10a - 1}{3}$

$4a^2 - 5a + 1 = 0 \Rightarrow a = 1, \dfrac{1}{4}$

If $a = 1$ then parabola is $y = -x^2 + 5x - 4$ and $P(2, -1)$ lies inside.

If $a = \dfrac{1}{4}$ then parabola is.

$y = -\dfrac{-x^2}{16} + \dfrac{5}{4}x - 4$ and P lies outside

27. (16)

For the points of intersection, we have $\dfrac{12 - y^2}{36} + \dfrac{y^2}{4} = 1$

$\Rightarrow y = \pm\sqrt{3}$ and $x = \pm 3$

Consider the point P $(3, \sqrt{3})$

Equation of the tangent at P to the circle is $3x + \sqrt{3}y = 12$

$\therefore$ slope of this tangent is $-\sqrt{3}$

Equation of the tangent at P to the ellipse is $\dfrac{x}{12} + \dfrac{\sqrt{3}}{4}y = 1$

$\therefore$ slope of this tangent is $-\dfrac{1}{3\sqrt{3}}$

if α is angle between these tangents, then

$\tan\alpha = \dfrac{2}{\sqrt{3}}$

$\therefore \alpha = \tan^{-1}\dfrac{2}{\sqrt{3}}$ $\therefore k = 4$ and hence $k^2 = 16$

29. (2)

$x = -1$ and $x = \dfrac{1}{3}$ are roots of $f'(x) = 0$

$\Rightarrow f'(x) = a(3x - 1)(x + 1) = a(3x^2 + 2x - 1)$

$\Rightarrow f(x) = a(x^3 + x^2 - x + b)$

$f(-2) = 0 \Rightarrow b = 2$

$\Rightarrow f(x) = a(x^3 + x^2 - x + 2)$

$$\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$$

$$\Rightarrow \int_{-1}^{1} a(x^3 + x^2 - x + 2) = \frac{14}{3}$$

$$\Rightarrow a\int_{-1}^{1} x^2 + 2 = \frac{14}{3}$$

$$\Rightarrow 2a\left(\frac{1}{3} + 2\right) = \frac{14}{3} \Rightarrow a = 1$$

$\therefore f(x) = x^3 + x^2 - x + 2$

30. Let $f(x) = \dfrac{\sin x}{x}$

$$f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x(x - \tan x)}{x^2} < 0 \ \forall$$

$x \in \left(0, \dfrac{\pi}{2}\right)$; $(\therefore \tan > x)$

$$f''(x) = \frac{-x^2\sin x - 2x\cos x + 2\sin x}{x^3}$$

Let $g(x) = -x^2\sin x - 2x\cos x + 2\sin x$

$\Rightarrow g'(x) = -x^2\cos x < 0 \ \forall \ x \in (0, \pi/2)$

for $x > 0$, we have $g(x) < g(0)$ i.e. $g(x) < 0$

$\therefore f'(x) < 0$ and $f''(x) < 0 \ \forall \ x \in \left(0, \dfrac{\pi}{2}\right)$

$$\Rightarrow f\left(\frac{A + B + C}{3}\right) > \left(\frac{f(A) + f(B) + f(C)}{3}\right)$$

$$\Rightarrow \frac{\sin\left(\frac{A + B + C}{3}\right)}{\frac{A + B + C}{3}} \geq \left(\frac{\frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C}}{3}\right)$$

$$\Rightarrow \frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C} \leq \frac{9\sqrt{3}}{2\pi}$$