HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

2. $\lim\limits_{x\to 0} \dfrac{x - e^x + 1 - (1-\cos 2x)}{x^2} = -\dfrac{1}{2} - 2 = -\dfrac{5}{2}$;

Hence for continuity $f(0) = -\dfrac{5}{2}$

$\therefore\ [f(0)] = -3$; $\{f(0)\} = \left\{-\dfrac{5}{2}\right\} = \dfrac{1}{2}$;

Hence $[f(0)]\ \{f(0)\} = -\dfrac{3}{2} = -1.5$]

3. By theorem, if g and h are continuous functions on the open interval (a, b), then g/h is also continuous at all x in the open interval (a, b) where h (x) is not equal to zero.

6. $\lim\limits_{x\to 0^+} f(x) = 0$ & $\lim\limits_{x\to 0^-} f(x) = 1$

7. $f(1^+) = f(1^-) = f(1) = 2$ $\quad f(0) = 1,\ f(2) = 2$

$f(2^-) = 1;\quad f(2) = 2$

$\Rightarrow$ f is not continuous at x = 2

9. $\text{Limit}_{h\to 0}\ g(n+h) = \text{Limit}_{h\to 0} \dfrac{e^h - \cos 2h - h}{h^2}$

$= \text{Limit}_{h\to 0} \dfrac{e^h - h - 1}{h^2} + \text{Limit}_{h\to 0} \dfrac{(1-\cos 2h)}{4h^2}.4$

$= \dfrac{1}{2} + 2 = \dfrac{5}{2}$

$\text{Limit}_{h\to 0}\ g(n-h)$

$= \dfrac{e^{1-\{n-h\}} - \cos 2(1-\{n-h\}) - (1-\{n-h\})}{(1-\{n-h\})^2}$

$= \lim\limits_{h\to 0} \dfrac{e^h - \cos 2h - h}{h^2}\ \left(\{n-h\} = \{-h\} = 1-h\right) = \dfrac{5}{2}$

$g(n) = \dfrac{5}{2}$. Hence g(x) is continuous at $\forall x \in I$.

Hence g (x) is continuous $\forall x \in R$]

12. $h(x) = \begin{cases} \dfrac{2\cos x - \sin 2x}{(\pi - 2x)^2} & x < \dfrac{\pi}{2} \\ \dfrac{e^{-\cos x} - 1}{8x - 4\pi} & x > \dfrac{\pi}{2} \end{cases}$

LHL at $x = \pi/2$

$\lim\limits_{h\to 0} \dfrac{2\sin h - \sin 2h}{4h^2} = \lim\limits_{h\to 0} \dfrac{2\sin h(1-\cos h)}{4h^2} = 0$

RHL: $\lim\limits_{h\to 0} \dfrac{e^{\sin h} - 1}{\left((\pi/2)+h\right) - 4\pi} = \lim\limits_{h\to 0} \dfrac{e^{\sin h} - 1}{8h} \cdot \dfrac{\sin h}{\sin h} = \dfrac{1}{8}$

$\Rightarrow$ h (x) is discontinuous at $x = \pi/2$.

Irremovable discontinuity at $x = \pi/2$.

$f\left(\dfrac{\pi^+}{2}\right) = 0$ and $g\left(\dfrac{\pi^-}{2}\right) = \dfrac{1}{8}$

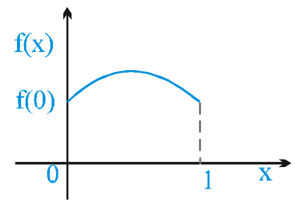
$\Rightarrow f\left(\dfrac{\pi^+}{2}\right) \neq g\left(\dfrac{\pi^-}{2}\right)$

14. $g(x) = x - [x] = \{x\}$

f is continuous with $f(0) = f(1)$

$h(x) = f(g(x)) = f(\{x\})$

Let the graph of f is as shown in the figure



satisfying

$f(0) = f(1)$

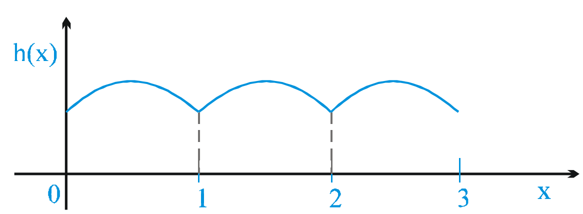
now $h(0) = f(\{0\}) = f(0) = f(1)$

$h(0.2) = f(\{0.2\}) = f(0.2)$

$h(1.5) = f(\{1.5\}) = f(0.5)$ etc.

Hence the graph of h(x) will be periodic graph as shown

$\Rightarrow$ h is continuous in R $\Rightarrow$ C



17. $\lim\limits_{x\to 0^-} \dfrac{1-\cos 4x}{x^2} = 8$

$\lim\limits_{x\to 0^+} \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} = 8 \quad \because \quad f(0)=8$

So f(x) is continuous at x = 0 when a = 8

18. $f(2^+) = 8\ ;\ f(2^-) = 16$

21. $f(x) = \lim\limits_{x\to 0} \dfrac{x\left(1+a\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}\ldots\right)\right)-b\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\ldots\right)}{x^3}$

$= \lim\limits_{x\to 0} \dfrac{(1+a-b)+x^2\left(\frac{-a}{2!}+\frac{b}{3!}\right)+\ldots}{x^2}$

$\Rightarrow \quad 1+a-b=0 \qquad$ .....(i)

and $\dfrac{-a}{2}+\dfrac{b}{6}=1 \qquad$ .....(ii)

Solving (i) and (ii) we get

$a=\dfrac{-5}{2},\ b=\dfrac{-3}{2}$

22. $f(0^+) = 0\ ;\ f(0) = 0\ ;\ f(0^-) = -1$

23. $f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = 1 \quad$ also $\quad f(0) = -c$

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(x)+f(h)+c-f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} = f'(0) = 1$

$\therefore \quad f'(x) = 1$

25. $y=\dfrac{1}{t^2+t-2}$, where $t=\dfrac{1}{x-1}$, y = f(x) is discontinuous at x = 1, where t is discontinuous and

$y=\dfrac{1}{(t+2)(t-1)}$ at t = – 2 and t = 1

$\Rightarrow \dfrac{1}{x-1} \quad \Rightarrow \quad -2x+2=1,$

$x=\dfrac{1}{2}$

$1-\dfrac{1}{x-1} \quad \Rightarrow \quad x=2$

f(g(x)) is discontinuous at $x=\dfrac{1}{2}$, 2, 1

26. $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\sqrt{a^2-ax+x^2}-\sqrt{a^2+ax+x^2}}{\sqrt{a+x}-\sqrt{a-x}}$

on rationalizing both Nr. & Dr. we get

$\lim\limits_{x\to 0} f(x) = -\sqrt{a}$

So $\quad f(0) = -\sqrt{a}$

27. for continuity $\lim\limits_{x\to 0} \dfrac{1-e^x}{x} = f(0)\ ;$

Hence $\quad f(0) = -\lim\limits_{h\to 0} \dfrac{e^x-1}{-x} = -1$

$f'(0^+) = \lim\limits_{h\to 0} \dfrac{\frac{1-e^h}{h}+1}{h} = \lim\limits_{h\to 0} \dfrac{1-e^h+h}{h^2} = \dfrac{1-h-\left[1+\frac{h}{1!}+\frac{h^2}{2!}+\ldots\ldots\right]}{h^2}$

$= -\dfrac{1}{2}$

$f'(0^-) = \lim\limits_{h\to 0} \dfrac{\frac{1-e^{-h}}{-h}+1}{-h} = \lim\limits_{h\to 0} \dfrac{1-e^{-h}-h}{h^2}$

$= \dfrac{1-h-\left[1-\frac{h}{1!}+\frac{h^2}{2!}-\ldots\ldots\right]}{h^2}$

$= -\dfrac{1}{2}$

Hence $\quad f(x) = \begin{cases} \dfrac{1-e^{-x}}{x} & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$

28. $(x-\sqrt{3})\, f(x) = -x^2+2x-2\sqrt{3}+3$

$f(x) = \dfrac{-x^2+2x-2\sqrt{3}+3}{x-\sqrt{3}}$

$= \dfrac{(x-\sqrt{3})(2-\sqrt{3}-x)}{x-\sqrt{3}} = 2-\sqrt{3}-x$

$f(\sqrt{3}) = 2-2\sqrt{3}$

EXERCISE - 2

Part # I : Multiple Choice

6. $f(x)=\dfrac{|x+\pi|}{\sin x}$

(A) $f(-\pi^{+}) = \lim\limits_{h\to 0}\dfrac{|-\pi+h+\pi|}{\sin(-\pi+h)} = \lim\limits_{h\to 0}\dfrac{|h|}{-\sin h} = -1$

(B) $f(-\pi^{-}) = \lim\limits_{h\to 0}\dfrac{|-\pi-h+\pi|}{\sin(-\pi-h)} = \lim\limits_{h\to 0}\dfrac{|h|}{\sin h} = 1$

(C) $f(-\pi^{+}) \neq f(-\pi^{-})$ So $\lim\limits_{x\to -\pi} f(x)$ does not exist

(D) for $\lim\limits_{x\to \pi} f(x)$

LHL $= \lim\limits_{x\to \pi^{-}}\dfrac{|x+\pi|}{\sin x} = \lim\limits_{h\to 0}\dfrac{2\pi-h}{\sinh} = \dfrac{2\pi}{0} = \infty$

RHL $= \lim\limits_{x\to \pi^{+}}\dfrac{|x+\pi|}{\sin x} = \lim\limits_{h\to 0}\dfrac{2\pi+h}{-\sinh} = -\dfrac{2\pi}{0} = -\infty$

LHL $\neq$ RHL

So $\lim\limits_{x\to \pi} f(x)$ does not exist.

7. $\lim\limits_{x\to 0^{+}}(x+1)e^{-[2/x]} = \lim\limits_{x\to 0^{+}}\dfrac{x+1}{e^{2/x}} = \dfrac{1}{e^{\infty}} = 0$

$\lim\limits_{x\to 0^{-}}(x+1)e^{-\left(-\frac{1}{x}+\frac{1}{x}\right)} = 1$

Hence continuous for $x \in I-\{0\}$

10. (i) $\tan f(x) = \tan\left(\dfrac{x}{2}-1\right) \qquad x \in [0,\pi]$

$0 \le x \le \pi \Rightarrow -1 \le \dfrac{x}{2}-1 \le \dfrac{\pi}{2}-1$

By graph we say tan(f(x)) is continuous in $[0, \pi]$

(ii) $\dfrac{1}{f(x)} = \dfrac{2}{x-2}$ is not defined at $x = 2 \in [0, \pi]$

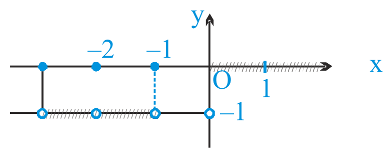
(iii) $y = \dfrac{x-2}{2}$

$\Rightarrow f^{-1}(x) = 2x+2$ is continuous in R.

11. (A) f(x) is continuous no where
(B) g(x) is continuous at x = 1/2
(C) h(x) is continuous at x = 0
(D) k(x) is continuous at x = 0

13. $[\,|x|\,]-|[x]| = \begin{cases} 0 & x=-1 \\ -1 & -1<x<0 \\ 0 & 0\le x\le 1 \\ 0 & 1<x\le 2 \end{cases}$

$\Rightarrow$ range is $\{0,-1\}$
The graph is



15. RHL $= \lim\limits_{x\to 0^{+}}\left(3-\left[\cot^{-1}\left(\dfrac{2x^3-3}{x^2}\right)\right]\right)$

$= 3-[\cot^{-1}(-\infty)] = 3-3 = 0$

LHL $= \lim\limits_{h\to 0}\{(0-h)^2\}\cos\left(e^{\left(\frac{1}{0-h}\right)}\right)$

$= \lim\limits_{h\to 0}(0-h)^2\cos(e^{-\infty}) = 0$

17. Given f is continuous in [a, b] ....**(i)**
g is continuous in [b, c] ....**(ii)**
$f(b) = g(b)$ ....**(iii)**

$h(x) = \begin{cases} f(x) & \text{for } x\in[a,b) \\ = f(b) = g(b) & \text{for } x = b \\ = g(x) & \text{for } x\in(b,c] \end{cases}$ ....**(iv)**

h (x) is continuous in $[a, b) \cup (b, c]$ [using **(i)**, **(ii)**]
also $f(b^-) = f(b)$; $g(b^+) = g(b)$ ....(5)
$\therefore$ $h(b^-) = f(b^-) = f(b) = g(b) = g(b^+) = h(b^+)$
[using **(iv)**, **(v)**]
now, verify each alternative. Of course! $g(b^-)$ and $f(b^+)$ are undefined.

$h(b^-) = f(b^-) = f(b) = g(b) = g(b^+)$
and $h(b^+) = g(b^+) = g(b) = f(b) = f(b^-)$
hence $h(b^-) = h(b^+) = f(b) = g(b)$
and h (b) is not defined $\Rightarrow$ **(A)**

18. (A) LHL = –1 & RHL = 0
(B) LHL = 1 & RHL = 2/3
(C) LHL = –1 & RHL = 2/3
(D) LHL = $-2\log_2 3$ & RHL = $2\log_3 2$

21. $\lim_{h\to 0} f(x+h) = \lim_{h\to 0} f(x) + f(h)$

$= f(x) + \lim_{h\to 0} f(h)$

Hence if $h \to 0$ $f(h) = 0$

$\Rightarrow$ 'f' is continuous otherwise discontinuous

22. $f(x) = [x]$ and $g(x) = \begin{cases} 0, & x \in I \\ x^2, & x \in R - I \end{cases}$

$\lim_{x\to 1} g(x) = \lim_{x\to 1} x^2 = 1$, but $g(1) = 0$

$\lim_{x\to 1} f(x) = \lim_{x\to 1} [x]$ does not exist since

LHL = 0 and RHL = 1

$gof(x) = g([x]) = 0$

$\Rightarrow$ gof(x) is continous for all values of x

$fog = \begin{cases} 0, & x \in I \\ [x^2], & x \in R - I \end{cases}$

$fog(1) = 0$, $\lim_{x\to 1^-} fog(x) = 0$, $\lim_{x\to 1^+} fog(x) = 1$

fog is not continous at x = 1

23. $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} b([x]^2 + [x]) + 1$

$= \lim_{h\to 0} b([-1+h]^2 + [-1+h]) + 1$

$= b((-1)^2 - 1) + 1 = 1$

$\Rightarrow$ $b \in R$

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \sin(\pi(x+a))$

$= \lim_{h\to 0} \sin(\pi(-1-h+a)) = -\sin \pi a$

$\sin \pi a = -1$

$\pi a = 2n\pi + \frac{3\pi}{2}$ $\Rightarrow$ $a = 2n + \frac{3}{2}$

Also option **(C)** is subset of option **(A)**

Part # II : Assertion & Reason

3. **Statement - 1**

$f(x) = \{\tan x\} - [\tan x]$

$f(x) = \tan x - 2[\tan x] = \begin{cases} \tan x, & 0 \le x < \frac{\pi}{4} \\ \tan x - 2, & \frac{\pi}{4} \le x < \tan^{-1} 2 \end{cases}$

obviously at $x = \frac{\pi}{3}$ f(x) is continuous. **(True)**

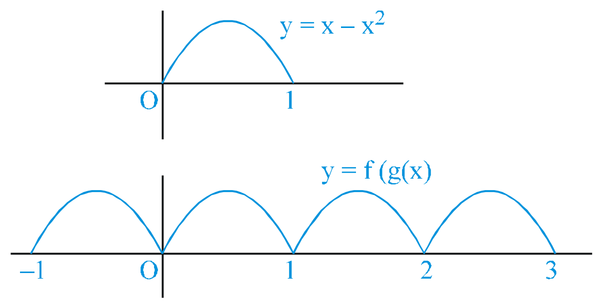
Statement - 2

y = f(x) & y = g(x) both are continuous at x = a

then y = f(x) ± (g(x) will also be continuous at x = a **(True)**

Statement-1 can be explained with the help of statement - 2.

5.



9. f(x) is discontinuous at x = 0 and $f(x) < 0 \; \forall \; x \in [-\alpha, 0)$ and $f(x) > 0 \; \forall \; x \in [0, \alpha]$

EXERCISE - 3

Part # I : Matrix Match Type

2. **(A)** $\lim_{h\to 0} \sin\{1-h\} = \cos 1 + a$

$\Rightarrow \lim_{h\to 0} \sin(1-h) - \cos 1 = a$

$\Rightarrow a = \sin 1 - \cos 1$

Now $|k| = \dfrac{\sin 1 - \cos 1}{\sqrt{2}\left(\sin 1.\dfrac{1}{\sqrt{2}} - \cos 1.\dfrac{1}{\sqrt{2}}\right)} = 1$

$k = \pm 1$

(B) $f(0) = \lim_{x\to 0} \dfrac{2\sin^2\left(\dfrac{\sin x}{2}\right)}{x^2\left(\dfrac{\sin x}{2}\right)^2} \times \left(\dfrac{\sin x}{2}\right)^2$

$\Rightarrow f(0) = \dfrac{1}{2}$

(C) function should have same rule for Q & Q'

$\Rightarrow x = 1 - x \quad \Rightarrow \quad x = \dfrac{1}{2}$

(D) $f(x) = x + \{-x\} + [x]$

x is continuous at $x \in R$

Check at $x = I$ (where I is integer),

$f(I^+) = 2I + 1$

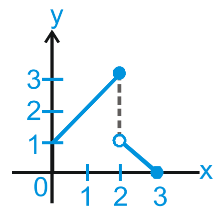
$f(I^-) = 2I - 1$

So f(x) is discontinuous at every integer i.e., 1, 0, –1

Part # II : Comprehension

Comprehension # 2

$$f(x) = \begin{cases} x+1 & 0 \le x \le 2 \\ -x+3 & 2 < x < 3 \end{cases}$$



1. f(x) is discontinuous at $x = 2$

2. $fof(x) = \begin{cases} x+2 & 0 \le x \le 1 \\ -x+2 & 1 < x \le 2 \\ -x+4 & 2 < x < 3 \end{cases}$

fof(x) is discontinuous at $x = 1, 2$

3. $f(19) = f(3 \times 6 + 1) = f(1) = 2$

EXERCISE - 4
Subjective Type

1. $a = -1\ b = 1$

2. **(i)** continuous at x = 1 **(ii)** continuous
(iii) discontinuous **(iv)** continuous at x = 1, 2

3. non-removable - finite type

$$f(0^-) = \lim_{h\to 0}\left(-\frac{2^{-1/h}-1}{2^{-1/h}+1}\right) = 1$$

$$f(0^+) = \lim_{h\to 0}\left(-\frac{2^{1/h}-1}{2^{1/h}+1}\right) = -1$$

$\Rightarrow$ LHL $\neq$ RHL $\Rightarrow$ Non removable-finite discontinuity

4. gof is discontinuous at x = 0, 1 and –1

5. $a = \frac{1}{2}$, $b = 4$

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^-} \frac{1-\sin^3 x}{3\cos^2 x} = \lim_{h\to 0}\frac{1-\cos^3 h}{3\sin^2 h}$$

$$\left[\text{put } x = \frac{\pi}{2} - h\right] = \lim_{h\to 0}\frac{1-\left(1-\frac{h^2}{2!}+\frac{h^4}{4!}-....\right)^3}{3h^2} = \frac{1}{2}$$

and $\displaystyle\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}^+}\frac{b(1-\sin x)}{(\pi-2x)^2}$

$$= \lim_{h\to 0}\frac{b(1-\cos h)}{4h^2} = \frac{b}{8}$$

So $\frac{1}{2} = \frac{b}{8} = a$

6. $a = \frac{1}{\sqrt{2}}$, $g(0) = \frac{(\ell n 2)^2}{8}$

$g(0^-)$

$$= \lim_{h\to 0}\frac{1-a^{-h}+(-h)a^{-h}\ell n(a)}{a^{-h}(-h)^2} = \lim_{h\to 0}\frac{a^h-1-h\ \ell na}{h^2}$$

$$= \lim_{h\to 0}\frac{1+h\ell na+\frac{h^2}{2!}(\ell na)^2+...-1-h\ell na}{h^2} = \frac{(\ell na)^2}{2}$$

$$g(0^+) = \lim_{h\to 0}\frac{2^h a^h - h\ell n 2 - h\ell n a - 1}{h^2}$$

$$= \lim_{h\to 0}\frac{1+h\ell n(2a)+\frac{h^2}{2!}(\ell n 2a)^2+...-h\ell na-1}{h^2} = \frac{(\ell n 2a)^2}{2!}$$

Now g(x) is continuous so

$(\ell na)^2 = (\ell n 2a)^2$

$\Rightarrow (\ell na)^2 = (\ell n\, 2)^2 + (\ell n\, a)^2 + 2\ell n\, 2\ell n\, a$

$\Rightarrow \ell n\, a = \frac{-1}{2}\ell n\, 2 \quad \Rightarrow a = \frac{1}{\sqrt{2}}$

$$g(0) = \frac{\left(\log\left(\frac{1}{\sqrt{2}}\right)\right)^2}{2} = \frac{1}{8}(\ell n\, 2)^2$$

7. $a = 0\ ;\ b = -1$

8. $a = -3/2,\ b \neq 0,\ c = 1/2$

$$\lim_{x\to 0^-}\frac{\sin(a+1)x+\sin x}{x} = a+2$$

and $\displaystyle\lim_{x\to 0^+}\frac{x+bx^2-x}{bx^{3/2}(\sqrt{x+bx^2}+\sqrt{x})} = \frac{1}{2}$ as $b \neq 0$

according to question

$c = \frac{1}{2}$ & $a + 2 = \frac{1}{2} \Rightarrow a = \frac{-3}{2}$

9. $f(0^+) = \frac{\pi}{2}$; $f(0^-) = \frac{\pi}{4\sqrt{2}} \Rightarrow$ 'f' is dicontinuous at x = 0 ;

$g(0^+) = g(0^-) = g(0) = \frac{\pi}{2} \Rightarrow$ 'g' is continuous at x = 0

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0}\frac{\left(\frac{\pi}{2}-\sin^{-1}(1-\{h\}^2)\right).\sin^{-1}(1-\{h\})}{\sqrt{2}(\{h\}-\{h\}^3)}$$

$$= \lim_{h\to 0}\frac{\left(\frac{\pi}{2}-\sin^{-1}(1-h^2)\right)\sin^{-1}(1-h)}{\sqrt{2}(h-h^3)}$$

$$= \lim_{h\to 0}\frac{\cos^{-1}(1-h^2)}{\sqrt{2}(1-h^2)}\times\frac{\sin^{-1}(1-h)}{h} = \frac{\pi}{2}$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0}\frac{\left(\frac{\pi}{2}-\sin^{-1}(1-(1-h)^2\right)\sin^{-1}(1-(1-h))}{\sqrt{2}((1-h)-(1-h)^3)}$$

$$= \lim_{h\to 0}\frac{\frac{\pi}{2}\sin^{-1}h}{\sqrt{2}(1-h)(2-h)h} = \frac{\pi}{4\sqrt{2}}$$

So f(x) is discontinuous at x = 0

Now $g(x) = \begin{cases} \dfrac{\pi}{2} & ; \ x \ge 0 \\ 2\sqrt{2}\dfrac{\pi}{4\sqrt{2}} & ; \ x < 0 \end{cases}$

$g(x) = \begin{cases} \dfrac{\pi}{2} & ; \ x \ge 0 \\ \dfrac{\pi}{2} & ; \ x < 0 \end{cases}$

So g(x) is continuous at x = 0

10. $f(1) = \lim_{n\to\infty} \dfrac{\log 3 - 1^{2n}\sin 1}{1^{2n}+1} = \dfrac{\log 3 - \sin 1}{2}$

$f(1^+) = \lim_{h\to 0}\lim_{n\to\infty} \dfrac{\log(3+h) - (1+h)^{2n} - \sin(1+h)}{(1+h)^{2n}+1} = -\sin 1$

$f(1^-) = \lim_{h\to 0}\lim_{n\to\infty} \dfrac{\log(3+h) - (1-h)^{2n} - \sin(1+h)}{(1-h)^{2n}+1} = \log 3$

discontinous at x = 1

11. $f(0) = 0$

$f(0^+) = \lim_{h\to 0} \dfrac{h}{h+1} + \dfrac{h}{(h+1)(2h+1)} + \dfrac{h}{(2h+1)(3h+1)} + \ldots\ldots\ldots\ldots\infty$

$= \lim_{h\to 0}\left\{1 - \dfrac{1}{h+1} + \dfrac{1}{(h+1)} - \dfrac{1}{2h+1} + \dfrac{1}{2h+1} - \dfrac{1}{3h+1} + \ldots\ldots\ldots\ldots\infty\right\}$

f(x) is not continous at x = 0 since $f(0) \ne f(0^+)$

12. f is continuous in $-1 \le x \le 1$

13. **(A)** –2, 2, 3 **(B)** K = 5 **(C)** even

$f(x) = (x+2)(x-2)(x-3)$

$h(x) = \begin{cases} (x+2)(x-2) & , \ x \ne 3 \\ k & , \ x = 3 \end{cases}$ for continuity

14. Since g is onto continuous function so by reference of intermediate value theorem we get required result.

$k = \lim_{x\to 3} h(x) = 5$

$h(x) = (x+2)(x-2) = x^2 - 4$ which is even $\forall\ x \in R$

15. $A = -4, B = 5, f(0) = 1$

$f(x) = \lim_{x\to 0} \dfrac{\sin 3x + A\sin 2x + B\sin x}{x^5}$

$= \lim_{x\to 0} \dfrac{\sin x}{x}\left(\dfrac{3 - 4\sin^2 x + 2A\cos x + B}{x^4}\right)$

$= \lim_{x\to 0} \dfrac{1 + 2\cos 2x + 2A\cos x + B}{x^4}$

$= \lim_{x\to 0} \dfrac{1}{x^4}\left(1 + 2\left(1 - \dfrac{(2x)^2}{2!} + \dfrac{(2x)^4}{4!} + \ldots\right) + 2A\left(1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots\right) + B\right)$

$= \lim_{x\to 0} \dfrac{1}{x^4}(3 + 2A + B + x^2(-4 - A) + x^4\left(\dfrac{4}{3} + \dfrac{A}{12}\right) + \ldots)$

$\Rightarrow$ $2A + B + 3 = 0$ and $-4 - A = 0$

$\Rightarrow$ $A = -4,\ B = 5$

and $f(0) = 1$

16. $k = 0$; $g(x) = \begin{cases} \ln(\tan x) & \text{if } 0 < x < \dfrac{\pi}{4} \\ 0 & \text{if } \dfrac{\pi}{4} \le x < \dfrac{\pi}{2} \end{cases}$.

Hence g(x) is continuous everywhere.

$f(x) = \sum_{r=1}^{n} \dfrac{\sin\left(\frac{x}{2^r}\right)}{\cos\left(\frac{x}{2^r}\right)\cos\left(\frac{x}{2^{r-1}}\right)} = \sum_{r=1}^{n} \dfrac{\sin\left(\frac{x}{2^{r-1}} - \frac{x}{2^r}\right)}{\cos\left(\frac{x}{2^r}\right)\cos\left(\frac{x}{2^{r-1}}\right)}$

$= \sum_{r=1}^{n}\left(\tan\left(\dfrac{x}{2^{r-1}}\right) - \tan\left(\dfrac{x}{2^r}\right)\right)$

$= \tan x - \tan\dfrac{x}{2} + \tan\dfrac{x}{2} - \tan\dfrac{x}{4} + \ldots - \tan\left(\dfrac{x}{2^n}\right)$

$f(x) = \tan x - \tan\left(\dfrac{x}{2^n}\right)$

Now $g(x) = \lim_{n\to\infty} \dfrac{\ell n\tan x - (\tan x)^n[\sin(\tan\frac{x}{2})]}{1 + (\tan x)^n}$

$g(x) = \begin{cases} \ell n(\tan x) & \text{when } x < \frac{\pi}{4} \\ -[\sin(\tan\frac{x}{2})] & \text{when } x > \frac{\pi}{4} \end{cases}$

$g\left(\dfrac{\pi}{4} - h\right) = \lim_{h\to 0} \ell n(\tan(\dfrac{\pi}{4} - h)) = \ell n\, 1 = 0$

$\Rightarrow$ K = 0 and g(x) is continuous in $(0, \dfrac{\pi}{2})$

17. **(i)** $x \in R - \{2, 3\}$
(ii) $x \in R - \{-1, 1\}$
(iii) $x \in R$
(iv) $x \in R - \{(2n+1), n \in I\}$

18. **(i)** continuous every where in its domain
(ii) continuous every where in its domain

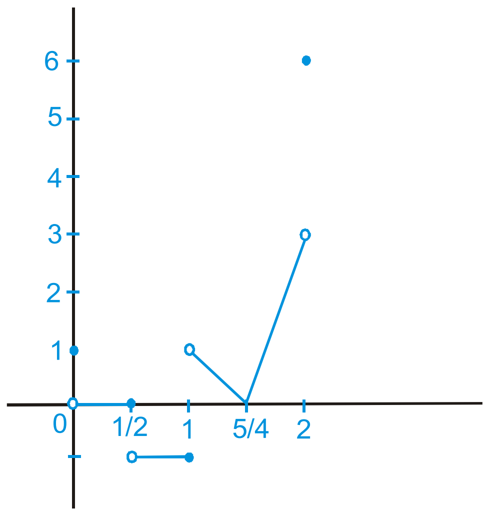
19. $a = e^{-1}$

20. discontinuous at all integral values in $[-2, 2]$

21. continuous every where except at $x = 0$

22. The function f is continuous everywhere in [0, 2] except for $x = 0, 1/2, 1$ & 2

$$f(x) = \begin{cases} 1 & , \quad x = 0 \\ 0 & , \quad 0 < x \le 1/2 \\ -1 & , \quad 1/2 < x \le 1 \\ 5-4x & , \quad 1 < x < 5/4 \\ 4x-5 & , \quad 5/4 \le x < 2 \\ 6 & , \quad x = 2 \end{cases}$$



f(x) is discontinous at $x = 0, 1/2, 1, 2$ in [0, 2]

23. $y_n(x)$ is continuous at $x = 0$ for all n and y (x) is discontinuous at $x = 0$

$$y_n(x) = x^2 \frac{\left(\frac{1}{(1+x^2)^n} - 1\right)}{\frac{1}{1+x^2} - 1}$$

$$= (1+x^2)\left(1 - \frac{1}{(1+x^2)^n}\right) \quad \text{when } x \ne 0, n \in N = 0$$

when $x = 0, n \in N$

$$y(x) = \lim_{n\to\infty} y_n(x) = \begin{cases} 1+x^2 & x \ne 0 \\ 0 & x = 0 \end{cases}$$

so y(x) is discontinuous at $x = 0$

25. discontinuous at $n\pi \pm \frac{\pi}{4}, (2n+1)\frac{\pi}{2}, n \in I$

27. $A = 1$; $f(2) = 1/2$

28. $g(x) = 2 + x$; $0 \le x \le 1$,
$= 2 - x$; $1 < x \le 2$,
$= 4 - x$; $2 < x \le 3$,
g is discontinuous at $x = 1$ & $x = 2$

29. $-\frac{7}{3}, -2, 0$

$u = \frac{1}{x+2}$ is discontinuous at $x = -2$

$$f(u) = \frac{3}{2u^2 + 5u - 3} = \frac{3}{2u^2 + 6u - u - 3} = \frac{3}{(2u-1)(u+3)} \text{ is}$$

discontinuous at $u = \frac{1}{2}$ & -3

$\therefore \frac{1}{x+2} = \frac{1}{2}$ and $\frac{1}{x+2} = -3$

$\Rightarrow x = 0$ and $x = -\frac{7}{3}$

Hence $y = f(u)$ is discontinous at $x = -\frac{7}{3}, -2, 0$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

2. $f(x)=\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x\neq 0 \\ 0, & x=0 \end{cases}$ $\therefore$ $|x|=\begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$

so $f(x)=\begin{cases} xe^{-2/x}, & x>0 \\ x, & x<0 \\ 0, & x=0 \end{cases}$

(I) continuous at x = 0

$$\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(0-h)=f(0)$$

$$\lim_{h\to 0} f(0+h)=\lim_{h\to 0} h\times e^{-2/h}=0$$

$$\lim_{h\to 0} f(0-h)=\lim_{h\to 0} =0 \qquad f(0)=0$$

f(x) is continuous at x = 0 or f(x) is continuous for all x

(II) differentiability at x = 0

L.H.D. $=Lf'(0)=\lim_{h\to 0}\dfrac{f(0-h)-f(0)}{-h}=\dfrac{-h-0}{-h}=1$

R.H.D. $Rf'(0)=\lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$

$\Rightarrow \dfrac{h\times e^{-2/h}-0}{h}=e^{-2/h}=0$

$Lf'(0)\neq Rf'(0)$

f(x) is not differentiable at x = 0

So that f(x) is cont at x = 0 but not differentiable at x = 0

3. $f(x)=\dfrac{1-\tan x}{4x-\pi}$ $x\neq\pi/4$ $x\in[0,\pi/2]$

f(x) is continuous at $x\in[0,\pi/2]$

So at $x=\pi/4$

$$\lim_{h\to 0} f\left(\frac{\pi}{4}+h\right)=\lim_{h\to 0} f\left(\frac{\pi}{4}-h\right)=f\left(\frac{\pi}{4}\right)$$

So $\lim_{h\to 0} f\left(\frac{\pi}{4}+h\right)=f\left(\frac{\pi}{4}\right)$

$$\lim_{h\to 0} f\left(\frac{\pi}{4}+h\right)=\frac{1-\tan x}{4x-\pi}$$

$$=\lim_{h\to 0}\frac{1-\tan\left(\frac{\pi}{4}+h\right)}{4\left(\frac{\pi}{4}+h\right)-\pi}$$

$$=\lim_{h\to 0}\frac{1-\left(\dfrac{\tan\frac{\pi}{4}+\tan h}{1-\tan\frac{\pi}{4}\tan h}\right)}{\pi+4h-\pi}$$

$$=\lim_{h\to 0}\frac{1-\tan h-1-\tan h}{(1-\tan h)\times 4h}$$

$$=\lim_{h\to 0}\frac{-2}{4}\left(\frac{\tan h}{h}\right)=-\frac{1}{2}$$

$$\lim_{h\to 0} f\left(\frac{\pi}{4}+h\right)=-\frac{1}{2}=f\left(\frac{\pi}{4}\right)$$

$$f\left(\frac{\pi}{4}\right)=-\frac{1}{2}$$

4. $f(x)=\dfrac{1}{x}-\dfrac{2}{e^{2x}-1}$ can be continuous at x = 0

So $\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(0-h)=f(0)$

$$\lim_{h\to 0} f(0+h)=\frac{1}{x}-\frac{2}{e^{2x}-1}$$

$$\lim_{h\to 0}\frac{1}{h}-\frac{2}{e^{2h}-1}$$

$$\lim_{h\to 0}\frac{(e^{2h}-1)-2h}{h\times(e^{2h}-1)}=\frac{0}{0}\text{ form}$$

$$\lim_{h\to 0}\frac{e^{2h}\times 2-0-2}{h\times e^{2h}\times 2+e^{2h}-1}=\frac{0}{0}\text{ form}$$

$$\lim_{h\to 0}\frac{2\times 2e^{2h}}{2e^{2h}+h\times e^{2h}\times 2\times 2+e^{2h}\times 2}=\frac{4}{4}=1$$

$f(0)=1$

5. LHL $=\lim_{x\to 0}\dfrac{\sin(p+1)x}{x}+\dfrac{\sin x}{x}$

$=(p+1)+1=p+2$

LHL = f(0) $\Rightarrow$ $\boxed{p+2=q}$ .....(i)

RHL $=\lim_{x\to 0}\dfrac{x^2}{x^{3/2}(\sqrt{x+x^2}+\sqrt{x})}=\dfrac{1}{2}$

$p+2=q=\dfrac{1}{2}$ $\Rightarrow$ $q=\dfrac{1}{2}$, $p=\dfrac{-3}{2}$

6. $f_1(x) = x\ ;\ x \in R$ is continuous.

$$f_2(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & ;\ x \neq 0 \\ 0 & ;\ x = 0 \end{cases}$$

$\lim\limits_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist

$\therefore$ $f_2(x)$ is discontinuous on R.

Now, $f(x) = \begin{cases} f_1(x).f_2(x) & ;\ x \neq 0 \\ 0 & ;\ x = 0 \end{cases}$

$$\Rightarrow \lim_{x\to 0} f_1(x).f_2(x) = \lim_{x\to 0} x.\sin\left(\frac{1}{x}\right)$$

$$= \lim_{x\to 0} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = 0 = f(0)$$

$\therefore$ $f(x)$ is continuous on R

$\therefore$ Statement–1 is true, statement–2 is false.

7. $f(x) = |x - 2| + |x - 5|\ ;\ x \in R$

f(x) is continuous in [2, 5] and differentiable is (2, 5) and $f(2) = f(5) = 3$.

$\therefore$ By Rolle's theorem $f'(x) = 0$ for at least one $x \in (2, 5)$.

$$f'(x) = \frac{|x-2|}{x-2} + \frac{|x-5|}{x-5}$$

$f'(4) = 0$ but $f'(x) = 0\ \forall\ x \in (2, 5)$

Part # II : IIT-JEE ADVANCED

2. For f to be continuous :

$f(2n^-) = f(2n^+)$.

$\Rightarrow$ $b_n + \cos 2n\pi = a_n + \sin 2n\pi$

$\Rightarrow$ $b_n + 1 = a_n$ $\Rightarrow$ $a_n - b_n = 1$

($\therefore$ B is correct)

$$\text{Also}\quad f(x) = \begin{cases} b_n + \cos\pi x & (2n-1, 2n) \\ a_n + \sin\pi x & [2n, 2n+1] \\ b_{n+1} + \cos\pi x & (2n+1, 2n+2) \\ a_n + \sin\pi x & [2n+2, 2n+3] \end{cases}$$

Again $f((2n+1)^-) = f((2n+1)^+)$

$\Rightarrow$ $a_n = b_{n+1} - 1$ $\Rightarrow$ $a_n - b_{n+1} = -1$

$\Rightarrow$ $a_{n-1} - b_n = -1$ ($\therefore$ D is correct)

MOCK TEST

1. $$\lim_{x\to 0} \frac{(a^2 - ax + x^2 - a^2 - ax - x^2)}{(a + x - a + x)} \times \frac{(\sqrt{a+x} + \sqrt{a-x})}{(\sqrt{a^2 - ax + x^2} + \sqrt{a^2 + ax + x^2})}$$

$$= \lim_{x\to 0} -\frac{2ax}{2x}\left(\frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a^2 - ax + x^2} + \sqrt{a^2 + ax + x^2}}\right)$$

$$= -\frac{\sqrt{a}}{a} = -\sqrt{a}$$

2. **(A)**

$f(x) = [x]\,(\sin kx)^p$

$(\sin kx)^p$ is continuous and differentiable function $\forall\ x \in R,\ k \in R$ and $p > 0$.

$[x]$ is discoutinuous at $x \in I$

For $k = n\pi,\ n \in I$

$f(x) = [x]\,(\sin(n\pi x))^p$

$\lim\limits_{x\to a} f(x) = 0,\ a \in I$

and $f(a) = 0$

So $f(x)$ becomes coutinuous for all $x \in R$

3. $$\text{R.H.L} = \lim_{h\to 0^+} f\left(\frac{\pi}{2} + h\right) = \lim_{h\to 0^+} \frac{\sin(1 - \sinh)}{h} \to \infty$$

$$\because\ \text{L.H.L} = \lim_{h\to 0^+} f\left(\frac{\pi}{2} - h\right) = \lim_{h\to 0^+} \frac{\sin(\sinh)}{-h}$$

$$= \lim_{h\to 0^+}\left(\frac{\sin(\sinh)}{\sinh} \times \frac{\sinh}{-h}\right) = 1 \times -1 = -1$$

$\therefore$ L.H.L $\neq$ R.H.L

4. **(B)**

$$\lim_{x\to 0^+} \frac{e^{e/x} - e^{-e/x}}{e^{1/x} + e^{-1/x}} = \lim_{x\to 0^+} \frac{e^{\frac{e-1}{ex}}(1 - e^{-2e/x})}{(1 + e^{-2/x})} = +\infty$$

$$\lim_{x\to 0^-} \frac{e^{e/x} - e^{-e/x}}{e^{1/x} + e^{-1/x}} = \lim_{x\to 0^-} \frac{e^{-e/x}(e^{2e/x} - 1)}{e^{-e/x}(e^{+2/x} + 1)}$$

$$= \lim_{x\to 0^-} e^{-\left(\frac{e-1}{x}\right)}\left(\frac{e^{2e/x} - 1}{e^{2/x} + 1}\right) = -\infty$$

limit doesn't exist So $f(x)$ is discoutinous

5. $\because$ L.H.L. $= \lim_{x\to0^-} f(x) = \lim_{x\to0^-} \frac{e^{[x]+|x|}-2}{[x]+|x|} = \frac{e^{-1}-2}{-1}$

and R.H.L $= \lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{e^{[x]+|x|}-2}{[x]+|x|}$

$= \lim_{x\to0^+} \frac{e^x-2}{x} \to -\infty$ $\quad\therefore$ L.H.L $\neq$ R.H.L

$\therefore$ **(D)**

6. **(C)**

$\lim_{x\to a} f(x) = \lim_{x\to a} x = a, x \in Q$

$\lim_{x\to a} f(x) = \lim_{x\to a} (-x) = -a, x \in R \sim Q$

Limit exists $\Leftrightarrow$ a = 0

7. $f(x) = \left[\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right]$

discontinuity may arise at the points where

$\sin\left(x+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $\sin\left(x+\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

and $\sin\left(x+\frac{\pi}{4}\right) = 0$

$x = \frac{\pi}{2}$; $x = \frac{\pi}{2}, \frac{3\pi}{2}$; $x = \frac{3\pi}{4}, \frac{7\pi}{4}$ five points

$\therefore$ **(B)**

8. **(B)**

$$f(g(x)) = \begin{cases} 1 & , \quad x < -1 \\ 0 & , \quad x = -1 \\ -1 & , \quad -1 < x < 0 \\ 0 & , \quad x = 0 \\ 1 & , \quad 0 < x < 1 \\ 0 & , \quad x = 1 \\ -1 & , \quad x > 1 \end{cases}$$

$\therefore$ points of discontinuity are x = – 1, 0, 1

9. $\lim_{x\to0^+} x^2\left[\frac{1}{x^2}\right] = \lim_{x\to0^+} x^2\left(\frac{1}{x^2} - \left\{\frac{1}{x^2}\right\}\right)$

$\Rightarrow \lim_{x\to0^+}\left(1 \to x^2\left\{\frac{1}{x^2}\right\}\right) = 0$

similarly $\lim_{x\to0^-} f(x) = 1$

$\therefore$ **(C)**

10. **(C)**

S_1 **:** False (take f(x) = 0, x $\in$ R

S_2 **:** Domain of f(x) is {2}

$\therefore$ f(x) is not continuous at x = 2

S_3 **:** $e^{-|x|}$ not differentiable at x = 0

S_4 **:** Derivative of $|f|^2$ is 0 where ever f(x) is 0 and the derivative of $|f|^2$ is $2|f(x)|.\frac{f(x)}{|f(x)|} = 2f(x)$ where ever $f(x) \neq 0$

12. **(B, D)**

(A) $\lim_{x\to1} f(x)$ does not exist

(B) $\lim_{x\to1} f(x) = \frac{2}{3}$

$\therefore$ f(x) has removable discontinuity at x = 1

(C) $\lim_{x\to1} f(x)$ does not exist

(D) $\lim_{x\to1} f(x) = \frac{-1}{2\sqrt{2}}$

$\therefore$ f(x) has removable discontinuity at x = 1

13. $\lim_{x\to0^+} (x+1)\, e^{-[2/x]} = \lim_{x\to0^+} \frac{x+1}{e^{2/x}} = \frac{1}{e^{\infty}} = 0$

$\lim_{x\to0^-} (x+1)\, e^{-\left(-\frac{1}{x}+\frac{1}{x}\right)} = 1$

Hence continuous for x $\in$ I – {0}

14. **(A)** f(x) is continuous no where

(B) g(x) is continuous at x = 1/2

(C) h(x) is continuous at x = 0

(D) k(x) is continuous at x = 0

15. $\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+} b([x]^2+[x])+1$

$= \lim_{h\to0} b([-1+h]^2+[-1+h])+1$

$= b((-1)^2-1)+1 = 1 \Rightarrow b \in R$

$\lim_{x\to-1^-} f(x) = \lim_{x\to-1^-} \sin(\pi(x+a))$

$= \lim_{h\to0} \sin(\pi(-1-h+a)) = -\sin\pi a$

$\sin\pi a = -1$

$\pi a = 2n\pi + \frac{3\pi}{2} \quad \Rightarrow \quad a = 2n + \frac{3}{2}$

Also option **(C)** is subset of option **(A)**

17. (A)

$$\lim_{x\to 0^+}(\sin x+[x])=0$$

$$\lim_{x\to 0^-}(\sin x+[x])=-1$$

Limit doesn't exist

$$\lim_{x\to a}(f(x)+h(x))=\lim_{x\to a}f(x)+\lim_{x\to a}h(x)$$

$$\neq f(a)+h(a)$$

$\therefore$ f(x) + h (x) is discontinuous function

19. (A)

Statement-II : [C & D]

$$f\left(\lim_{x\to a}g(x)\right)=f(b)=\lim_{x\to b}f(x)=\lim_{g(x)\to b}f(g(x))$$

$$=\lim_{x\to a}f(g(x))$$

$$\therefore \quad \lim_{x\to a}f(g(x))=f\left(\lim_{x\to a}g(x)\right)$$

$\therefore$ Statement is true

Statement-I :

Since f is continuous on R

and $f(x)=f\left(\dfrac{x}{3}\right)=f\left(\dfrac{x}{3^2}\right)\ldots\ldots\ldots=f\left(\dfrac{x}{3^n}\right)$

and $\lim\limits_{n\to\infty}\dfrac{x}{3^n}=0$

$$\therefore \quad \lim_{n\to\infty}f(x)=\lim_{n\to\infty}f\left(\frac{x}{3^n}\right)=f\left(\lim_{x\to 0}\frac{x}{3^n}\right)=f(0)$$

$\therefore$ f is a constant function

$\therefore$ Statement is true

22. (A) $f(x)=\dfrac{\tan\left(\frac{\pi}{4}-x\right)}{\cos 2x}$, $(x\neq\pi/4)$, is continuous at $x=\pi/4$.

Therefore, $f\left(\dfrac{\pi}{4}\right)=\lim\limits_{x\to\frac{\pi}{4}}f(x)$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\tan\left(\frac{\pi}{4}-x\right)}{\cot 2x}$$

Now, by appliying L'Hospital rule,

$$\lim_{x\to\frac{\pi}{4}}\frac{-\sec^2\left(\frac{\pi}{4}-x\right)}{-2\,\mathrm{cosec}^2(2x)}=\frac{1}{2}$$

(B) We have,

$$\text{LHL}=\lim_{h\to 4^-}f(x)$$

$$=\lim_{h\to 0}f(4-h)$$

$$=\lim_{h\to 0}\frac{4-h-4}{|4-h-4|}+a$$

$$=\lim_{h\to 0}\left(-\frac{h}{h}+a\right)=a-1$$

$$\text{RHL}=\lim_{x\to 4^+}f(x)$$

$$=\lim_{h\to 0}f(4+h)$$

$$=\lim_{h\to 0}\frac{4+h-4}{|4+h-4|}+b=b+1$$

$\therefore \quad f(4)=a+b$

Since f(x) is continuous at $x=4$.

$$\lim_{x\to 4^-}f(x)=f(4)=\lim_{x\to 4^+}f(x)$$

or $a-1=a+b=b+1$ or $b=-1$ and $a=1$

(C) $\lim\limits_{x\to 0}\dfrac{x-e^x+1-(1-\cos 2x)}{x^2}$

$$=\lim_{x\to 0}\left[\frac{x-e^x+1}{x^2}-\frac{(1-\cos 2x)}{x^2}\right]$$

$$=\lim_{x\to 0}\left[\frac{x+1-\left(1+x+\frac{x^2}{2}\right)}{x^2}-\frac{2\sin^2 x}{x^2}\right]$$

(using expansion of e^x)

$$=-\frac{1}{2}-2=-\frac{5}{2}$$

Hence, for continuity, $f(0)=-\dfrac{5}{2}$

Now, $[f(0)]=-3$; $\{f(0)\}=\left\{-\dfrac{5}{2}\right\}=\dfrac{1}{2}$

Hence, $[f(0)]\,\{f(0)\}=-\dfrac{3}{2}=-1.5.$

(D) f(x) is discontinuous at x = 1 and x = 2.

Therefore, f(f(x)) may be discontinuous when f(x) = 1 or 2. Now,

$1 - x = 1 \quad \Rightarrow \quad x = 0$, where f(x) is continuous

$x + 2 = 1 \quad \Rightarrow \quad x = -1 \notin (1, 2)$

$4 - x = 1 \quad \Rightarrow \quad x = 3 \in [2, 4]$

Now,

$1 - x = 2 \quad \Rightarrow \quad x = -1 \notin [0, 1]$

$x + 2 = 2 \quad \Rightarrow \quad x = 0 \notin (1, 2]$

$4 - x = 2 \quad \Rightarrow \quad x = 2 \in [2, 4]$

Hence, f(f(x)) is discontinuous at x = 2, 3.

23. $F(x) = \lim_{n\to\infty} \dfrac{f(x) + x^{2n} g(x)}{1 + x^{2n}}$

$$= \begin{cases} f(x), & 0 \le x^2 < 1 \\ \dfrac{f(x)+g(x)}{2}, & x^2 = 1 \\ g(x), & x^2 > 1 \end{cases}$$

$$= \begin{cases} g(x), & x < -1 \\ \dfrac{f(-1)+g(-1)}{2}, & x = -1 \\ f(x), & -1 < x < 1 \\ \dfrac{f(1)+g(1)}{2}, & x = 1 \\ g(x), & x > 1 \end{cases}$$

If F(x) is continuous $\forall$ x $\in$ R, F(x) must be made continuous at x = ± 1.

For continuity at x = –1

f(–1) = g(–1) or 1 – a + 3 = b – 1 or a + b = 5 ....**(i)**

For continuity at x = 1,

f(1) = g(1) or 1 + a + 3 = 1 + b or a – b = –3 ....**(ii)**

Solving equation **(i)** and **(ii)**, we get a = 1 and b = 4.

$f(x) = g(x) \Rightarrow x^2 + x + 3 = x + 4$ or $x = \pm 1$.

24.

1. (D)

If $f(x) = \begin{cases} x + 2.7, & x < 0 \\ 2.9, & x = 0 \\ 2x + 3, & x > 0 \end{cases}$

and $g(x) = \begin{cases} 3x + 3, & x < 0 \\ 2.8, & x = 0 \\ -x^2 + 2.7, & x > 0 \end{cases}$

then $\lim_{x\to 0^-} f(x) = 2.7$, $\lim_{x\to 0^+} f(x) = 3$

$\therefore$ $|3 - 2.7| = 0.3 < 1$ and $f(0) = 2.9$ lies in (2.7, 3)

$\therefore$ f(x) is continuous under the system S_2

g(x) is also continuous under the system S_2

under system S_1, since $\lim_{x\to 0} f(x)$ does not exist

$\therefore$ f(x) is not continuous

$\therefore$ **(i)**, **(ii)** and **(iii)** all are true

2. (D)

Let $f(x) = \begin{cases} x + 2.7, & x < 0 \\ 2.9, & x = 0 \\ 2x + 3, & x > 0 \end{cases}$

and $g(x) = \begin{cases} 3x + 3, & x < 0 \\ 2.9, & x = 0 \\ -x^2 + 2.75, & x > 0 \end{cases}$

$\therefore$ $(f+g)(x) = \begin{cases} 4x + 5.7, & x < 0 \\ 5.8, & x = 0 \\ 2x - x^2 + 5.75, & x > 0 \end{cases}$

$\therefore$ $\lim_{x\to 0^-} (f+g)(x) = 5.7$ and $\lim_{x\to 0^+} (f+g)(x) = 5.75$

$\therefore$ $\left| \lim_{x\to 0^-} (f+g) - \lim_{x\to 0^+} (f+g) \right| = .05 < 1$ is satisfied

$\therefore$ $(f+g)(0) = 5.8$ which do not lie in (5.7, 5.75)

$\therefore$ f + g is not continuous

similarly we can show that f – g and f.g are not continuous under S_2.

3. (B)

A function continuous under system S_2 may not be continuous under system S_1.

25. f(x)=

$$\begin{cases} [x], & -2 \le x \le -\frac{1}{2} \\ 2x^2-1, & -\frac{1}{2} < x \le 2 \end{cases} = \begin{cases} -2, & -2 \le x < -1 \\ -1, & -1 \le x \le -\frac{1}{2} \\ 2x^2-1, & -\frac{1}{2} < x \le 2 \end{cases}$$

$$|f(x)| = \begin{cases} 2, & -2 \le x < -1 \\ 1, & -1 \le x \le -\frac{1}{2} \\ |2x^2-1|, & -\frac{1}{2} < x \le 2 \end{cases}$$

$$= \begin{cases} 2, & -2 \le x < -1 \\ 1, & -1 \le x \le -\frac{1}{2} \\ 1-2x^2, & -\frac{1}{2} < x \le \frac{1}{\sqrt{2}} \\ 2x^2-1, & \frac{1}{\sqrt{2}} < x \le 2 \end{cases}$$

$$f(|x|) = \begin{cases} -2, & -2 \le |x| < -1 \\ -1, & -1 \le |x| \le -\frac{1}{2} \\ 2|x|^2-1, & -\frac{1}{2} < |x| \le 2 \end{cases} = 2x^2-1, -2 \le x \le 2$$

$$\therefore \quad g(x) = f(|x|) + |f(x)| = \begin{cases} 2x^2+1, & -2 \le x < -1 \\ 2x^2, & -1 \le x \le -\frac{1}{2} \\ 0, & -\frac{1}{2} < x < \frac{1}{\sqrt{2}} \\ 4x^2-2, & \frac{1}{\sqrt{2}} \le x \le 2 \end{cases}$$

$$g(-1^-) = \lim_{x\to -1} (2x^2+1) = 3, \; g(-1^+) = \lim_{x\to -1} 2x^2 = 2$$

$$g\left(-\frac{1}{2}^-\right) = \lim_{x\to \frac{1}{2}} 2x^2 = \frac{1}{2}, \; g\left(-\frac{1}{2}^+\right) = \lim_{x\to -\frac{1}{2}} 0 = 0$$

$$g\left(\frac{1}{\sqrt{2}}^-\right) = \lim_{x\to \frac{1}{\sqrt{2}}} 0 = 0, \; g\left(\frac{1}{\sqrt{2}}^+\right) = \lim_{x\to \frac{1}{\sqrt{2}}} (4x^2-2) = 0$$

Hence, g(x) is discontinuous at $x = -1, -\frac{1}{2}$.

g(x) is continuous at $x = \frac{1}{\sqrt{2}}$

Now, $g'\left(\frac{1}{\sqrt{2}}^-\right) = 0, \; g'\left(\frac{1}{\sqrt{2}}^+\right) = 8\left(\frac{1}{\sqrt{2}}\right) = \frac{8}{\sqrt{2}}$

Hence, g(x) is non-differentiable at $x = \frac{1}{\sqrt{2}}$.

26. $$\lim_{x\to 0^-} \frac{1-a^x + x.a^x \ell na}{x^2 a^x}$$

$$= \lim_{x\to 0^-} \frac{-a^x \ell na + \ell na(a^x + xa^x \ell na)}{x^2 a^x \ell na + 2x.a^x}$$

$$= \lim_{x\to 0^-} \frac{a^x (\ell na)^2}{(xa^x \ell na + 2a^x)} = \frac{(\ell na)^2}{2}$$

$$\lim_{x\to 0^+} \frac{(2a)^x - x\ell n2a - 1}{x^2}$$

$$= \lim_{x\to 0^+} \frac{(2a)^x \ell n2a - \ell n2a}{2x}$$

$$= \lim_{x\to 0^+} \frac{(2a)^x (\ell n2a)^2}{2} = \frac{(\ell n2a)^2}{2}$$

for g(x) to be continuous $(\ell na)^2 = (\ell n2a)^2$

$\Rightarrow \quad (\ell na + \ell n2a) = 0$

$\Rightarrow \quad a = \frac{1}{\sqrt{2}}$

$\therefore \quad g(0) = \frac{1}{8}(\ell n\, 2)^2$

27. $\because$ R.H.L. $= \lim_{h\to 0^+} f(0+h)$

$$= \frac{\cos^{-1}(1-\{h\}^2)\sin^{-1}(1-\{h\})}{\{h\}-\{h\}^3}$$

$$= \lim_{h\to 0^+}\frac{\cos^{-1}(1-h^2)}{h}\cdot\lim_{h\to 0^+}\frac{\sin^{-1}(1-h)}{1-h^2}$$

(putting $1-h^2 = \cos 2\theta$) $= (\sin^{-1}1)$

$$\lim_{\theta\to 0^+}\frac{\cos^{-1}(1-2\sin^2\theta)}{\sqrt{2}\sin\theta} = \frac{\pi}{2\sqrt{2}}\lim_{\theta\to 0^+}\frac{2\theta}{\sin\theta} = \frac{\pi}{\sqrt{2}}$$

$\because$ L.H.L $= \lim_{h\to 0^+} f(0-h)$

$$= \lim_{h\to 0^+}\frac{\cos^{-1}(1-\{-h\}^2)\sin^{-1}(1-\{-h\})}{\{-h\}-\{-h\}^3}$$

$$= \lim_{h\to 0^+}\frac{\cos^{-1}(h(2-h))\sin^{-1}h}{(1-h)(2-h)h}$$

$$= \lim_{h\to 0^+}\frac{\cos^{-1}(h(2-h))}{(1-h)(2-h)}\lim_{h\to 0^+}\frac{\sin^{-1}h}{h}$$

$$= \frac{\cos^{-1}0}{2} = \frac{\pi}{4}$$

since R.H.L. $\neq$ L.H.L

Therefore no value of f(0) can make f continuous at $x = 0$

29. As f is continuous on R, so $f(0) = \underset{x\to 0}{\text{limit}}\ f(x)$

Thus $f(0) = \underset{n\to\infty}{\text{limit}}\ f\left(\frac{1}{4n}\right)$

$$= \underset{n\to\infty}{\text{limit}}\left((\sin e^n)\, e^{-n^2} + \frac{1}{1+\frac{1}{n^2}}\right) = 0+1 = 1$$

30. we have

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0^+}(\sin(-h)+\cos(-h))^{\operatorname{cosec}(-h)}$$

$$= \lim_{h\to 0^+}(\cos h - \sin h)^{-\operatorname{cosec} h}$$

$$= \lim_{h\to 0^+}(1+(\cos h-\sin h-1))^{\frac{1}{(\cos h-\sin h-1)}\cdot\frac{(\cos h-\sin h-1)}{(-\sin h)}}$$

$$= \lim_{h\to 0^+} e^{\frac{\cos h-\sin h-1}{-\sin h}} = e$$

Now we have

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0^+}\frac{e^{\frac{1}{h}}+e^{2/h}+e^{3/h}}{ae^{-2+1/h}+be^{-1+3/h}}$$

$$= \lim_{h\to 0^+}\frac{e^{-\frac{2}{h}}+e^{\frac{-1}{h}}+1}{(ae^{-2})e^{-2/h}+(be^{-1})} = \frac{e}{b}$$

If ‘f’ is continuous at $x = 0$, then

$e = a = \frac{e}{b}$ gives $a = e$ and $b = 1$