# CONTINUITY

## INTRODUCTION

After conceiving the notion of limits the next element which is taken into consideration is the **continuity** of function. Qualitatively the graph of a function is said to be continuous at x = a if while travelling along the graph of the function and in crossing over the point at x = a either from L to R or from R to L one does not have to lift his pen. In case one has to lift his pen the graph of the function is said to have a break or discontinuous at x = a. Different type of situations which may come up at x = a along the graph can be :

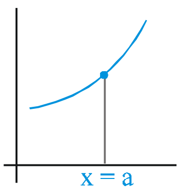
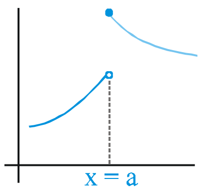


figure (1)



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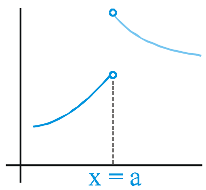

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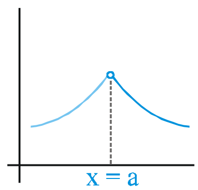

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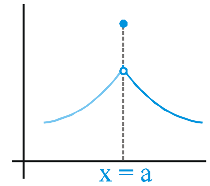

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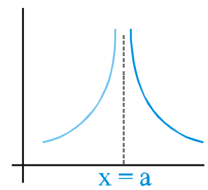


figure (6)

Figure (2) – (6) is discontinuous at x = a and in figure (1) f is continuous at x = a

## MATHEMATICAL DEFINITION OF CONTINUITY

A function f(x) is said to be continuous at x = a, if $\lim_{x \to a} f(x)$ exists and = f (a). Symbolically f is continuous at x = a if

$$\lim_{h\to 0} f(a-h) = \lim_{h\to 0} f(a+h) = f(a) = \text{a finite quantity.}$$

i.e. LHL at x = a = RHL at x = a = value of f (x) at x = a = finite quantity.

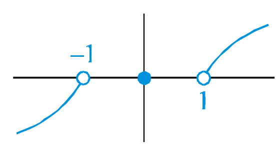
### KEY POINTS

**(i)** continuity at x = a $\Rightarrow$ existence of limit at x = a, but not the converse

**(ii)** continuity at x = a $\Rightarrow$ f is well defined at x = a, but not the converse

**(iii)** continuity at x = a is meaningful to talk if in the immediate neighbourhood of x = a, the function has a graph, but not necessarily at x = a i.e. continuity is a property of interval.

For example continuity of $f(x) = \dfrac{1}{x-1}$ at x = 1




A function f can be discontinuous due to any of the following three reasons:

**(i)** $\lim_{x\to c} f(x)$ does not exist i.e. $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$

**(ii)** f (x) is not defined at x = c

**(iii)** $\lim_{x\to c} f(x) \neq f(c)$

Geometrically, the graph of the function will exhibit a break at x= c.

**Ex.** Consider $f(x) = \begin{cases} \dfrac{8^x - 4^x - 2^x + 1}{x^2}, & x > 0 \\ e^x \sin x + \pi x + k\ell n4, & x < 0 \end{cases}$ Define the function at x = 0 if possible, so that f(x) becomes continuous at x = 0.

**Sol.** $f(0^+) = \lim_{h \to 0} \dfrac{8^h - 4^h - 2^h + 1}{h^2} = \lim_{h \to 0} \dfrac{4^h(2^h - 1) - (2^h - 1)}{h^2}$

$= \lim_{h \to 0} \dfrac{(4^h - 1)}{h} \dfrac{(2^h - 1)}{h} = \ell n4 \,.\, \ell n2$

$f(0^-) = \lim_{x \to 0^-} \left(e^x \sin x + \pi x + k\ell n4\right) = k\ell n4$

f(x) is continuous at x = 0,

$\Rightarrow$ $f(0^+) = f(0^-) = f(0)$ $\Rightarrow$ $\ell n4 . \ell n2 = k\ell n4$

$\Rightarrow$ $k = \ell n2$ $\Rightarrow$ $f(0) = (\ell n4)(\ell n2)$

## SINGLE POINT CONTINUITY

Functions which are continuous only at one point are said to exhibit single point continuity

**Example's**

**(i)** $f(x) = \begin{cases} x \text{ if } x \in Q \\ 0 \text{ if } x \notin Q \end{cases}$

continuous only at x = 0

**(ii)** $f(x) = \begin{cases} x & \text{if } x \in Q \\ 1 - x & \text{if } x \notin Q \end{cases}$

continuous only at x = 1/2

**(iii)** $f(x) = \begin{cases} x^2 & \text{if } x \in Q \\ 1 & \text{if } x \notin Q \end{cases}$

continuous only at x = 1 or – 1

**Ex.** If $f(x) = \begin{cases} x & \text{if } x \in Q \\ -x & \text{if } x \notin Q \end{cases}$, find the points where $f(x)$ is continuous

**Sol.** Let x = a be the point at which $f(x)$ is continuous.

$\Rightarrow$ $\lim_{\substack{x \to a \\ \text{through rational}}} f(x) = \lim_{\substack{x \to a \\ \text{through irrational}}} f(x)$

$\Rightarrow$ a = –a

$\Rightarrow$ a = 0 $\Rightarrow$ function is continuous at x = 0.

## CONTINUITY OF SPECIAL TYPE OF FUNCTIONS

### Continuity of Functions in which Greatest Integer Function is Involved

f(x) = [x] is discontinuous when x is an integer.

Similarly, f(x) = [g(x)] is discontinuous at all integers when g(x) is an integer, but this is true only when g(x) is monotonic [g(x) is strictly increasing or strictly decreasing].

For example, $f(x) = [\sqrt{x}]$ is discontinuous at all integers when $\sqrt{x}$ is integer, as $\sqrt{x}$ is strictly increasing (monotonic functions.)

$f(x) = [x^2]$, $x \geq 0$, is discontinuous at all integers when $x^2$ is an integer, as $x^2$ is strictly increasing $x \geq 0$.

Now, consider $f(x) = [\sin x]$, $x \in [0, 2\pi]$. $g(x) = \sin x$ is not monotonic in $[0, 2\pi]$. For this type of function, points of discontinuity can be determined easily by graphical methods. We can note that $x = 3\pi/2$, $\sin x$ takes integral value $-1$, but at $x = 3\pi/2$, $f(x) = [\sin x]$ is continuous.

**Ex.** Discuss the continuity of the following functions ([·] represent the GIF)

**(A)** $f(x) = [\log_e x]$

**(B)** $f(x) = [\sin^{-1} x]$

**(C)** $f(x) = \left[\dfrac{2}{1+x^2}\right], x \geq 0$

**Sol.** **(A)** $\log_e x$ is a monotonically increasing function.

Hence, $f(x) = [\log_e x]$ is discontinuous, where $\log_e x = k$ or $x = e^k$, $k \in Z$

Thus, $f(x)$ is discontinuous at $x = \ldots.\ e^{-2}, e^{-1}, e^0, e^1, e^2, \ldots.$

**(B)** $\sin^{-1} x$ is a monotonically increasing function.

Hence, $f(x) = [\sin^{-1} x]$ is discontinuous where $\sin^{-1} x$ is an integer.

Therefore, $\sin^{-1} x = -1, 0, 1$ or $x = -\sin 1, 0, \sin 1$.

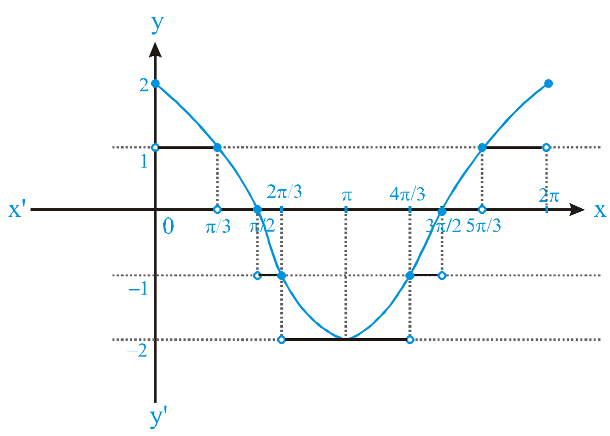
**(C)** $\dfrac{2}{1+x^2}$, $x \geq 0$, is a monotonically decreasing function.

Hence, $f(x) = \left[\dfrac{2}{1+x^2}\right]$, $x \geq 0$, is discontinuous, when $\dfrac{2}{1+x^2}$ is an integer.

Therefore, $\dfrac{2}{1+x^2} = 1, 2$ or $x = 1, 0$

**Ex.** Draw the graph and find the points of discontinuity for $f(x) = [2 \cos x]$, $x \in [0, 2\pi]$. ([·] represent the greatest integer function.)

**Sol.** $f(x) = [2 \cos x]$




Clearly, from the graph given in fig. f(x) is discontinuity at $x = 0, \dfrac{\pi}{3}, \dfrac{\pi}{2}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{3\pi}{2}, \dfrac{5\pi}{3}, 2\pi$

**Ex.** Discuss the continuity of $f(x) = \begin{cases} x\{x\}+1, & 0 \le x < 1 \\ 2-\{x\}, & 1 \le x \le 2 \end{cases}$ where (x) denotes the fractional part function.

**Sol.** $f(0) = f(0^+) = 1$

$f(2) = 2$ and $f(2^-) = 1$

Hence, f(x) is discontinuous at x = 2.

Also, $f(1^+) = 2$, $f(1^-) = 1 + 1 = 2$, and $f(1) = 2$

Hence, f(x) is continuous at x = 1

## Continuity of functions in which Signum function is involved

We know that f(x) = sgn(x) is discontinuous at x = 0.

In general, f(x) = sgn(g(x)) is discontinuous x = a if g(a) = 0.

**Ex.** Discuss the continuity of

**(A)** $f(x) = \text{sgn}(x^3 - x)$ **(B)** $f(x) = \text{sgn}(2 \cos x - 1)$

**(C)** $f(x) = \text{sgn}(x^2 - 2x + 3)$

**Sol.** **(A)** $f(x) = \text{sgn}(x^3 - x)$

Here, $x^3 - x = 0 \Rightarrow x = 0, -1, 1.$

Hence, f(x) is discontinuous at x = 0, –1, 1

**(B)** $f(x) = \text{sgn}(2 \cos x - 1)$

Here, $2 \cos x - 1 = 0 \Rightarrow \cos x = 1/2 \Rightarrow x = 2n\pi + (\pi/3), n \in Z$, where f(x) is discontinuous.

**(C)** $f(x) = \text{sgn}(x^2 - 2x + 3)$

Here, $x^2 - 2x + 3 > 0$ for all x.

Thus, f(x) = 1 for all x.

Hence, it is continuous for all x.

**Ex.** Discuss the continuity of $f(x) = |x| \text{ sgn } (x^3 - x)$

**Sol.** $\text{Sgn}(x^3 - x)$ is discontinuous when $x^3 - x = 0$ or $x = 0, \pm 1$.

But $f(0) = f(0^+) = f(0^-) = 0$.

Hence, f(x) is continuous at x = 0.

Hence, $f(x) = |x| \text{ sgn}(x^3 - x)$ is discontinuous at $x = \pm 1$ only.

## Continuity of Functions Involving Limit $\lim\limits_{n \to \infty} a^n$

We know that $\lim\limits_{n \to \infty} a^n = \begin{cases} 0, & 0 \le a < 1 \\ 1, & a = 1 \\ \infty, & a > 1 \end{cases}$

**Ex.** Discuss the continuity of $f(x) = \lim_{n\to\infty} \dfrac{x^{2n}-1}{x^{2n}+1}$

**Sol.** $f(x) = \lim_{n\to\infty} \dfrac{(x^2)^n - 1}{(x^2)^n + 1}$

$$= \lim_{n\to\infty} \frac{1 - \dfrac{1}{(x^2)^n}}{1 + \dfrac{1}{(x^2)^n}} = \begin{cases} -1, & 0 \le x^2 < 1 \\ 0, & x^2 = 1 \\ 1, & x^2 > 1 \end{cases} = \begin{cases} 1, & x < -1 \\ 0, & x = -1 \\ -1, & -1 < x < 1 \\ 0, & x = 1 \\ 1, & x > 1 \end{cases}$$

Thus, f(x) is discontinuous at $x = \pm 1$.

## THEOREMS ON CONTINUITY

**(i)** If f & g are two functions which are continuous at x = c, then the functions defined by: $F_1(x) = f(x) \pm g(x)$ ; $F_2(x) = K\, f(x)$, K is any real number ; $F_3(x) = f(x).g(x)$ are also continuous at x = c.

Further, if g (c) is not zero, then $F_4(x) = \dfrac{f(x)}{g(x)}$ is also continuous at x = c.

**(ii)** If f(x) is continuous & g(x) is discontinuous at x = a, then the product function $\phi(x) = f(x).\, g(x)$ may or may not be continuous but sum or difference function $\phi(x) = f(x) \pm g(x)$ will necessarily be discontinuous at x = a.

**e.g.** $f(x) = x$ & $g(x) = \begin{cases} \sin\frac{\pi}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$

**(iii)** If f (x) and g(x) both are discontinuous at x = a, then the product function $\phi(x) = f(x).\, g(x)$ is not necessarily be discontinuous at x = a.

**e.g.** $f(x) = g(x) = \begin{cases} 1 & , x \ge 0 \\ -1 & , x < 0 \end{cases}$

and atmost one out of f(x) + g(x) and f(x) – g(x) is continuous at x = a.

**Ex.** If $f(x) = \begin{cases} \sin\dfrac{\pi x}{2} & , x < 1 \\ [x] & , x \ge 1 \end{cases}$, then find whether f(x) is continuous or not at x = 1,

where [ . ] is greatest integer function.

**Sol.** $f(x) = \begin{cases} \sin\dfrac{\pi x}{2} & , x < 1 \\ [x] & , x \ge 1 \end{cases}$

For continuity at x = 1, we determine f(1), $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

Now, $f(1) = [1] = 1$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sin\dfrac{\pi x}{2} = \sin\dfrac{\pi}{2} = 1$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} [x] = 1$

so $f(1) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 1$

$\therefore$ f(x) is continuous at x = 1

**Ex.** Let $f(x) = \begin{cases} \dfrac{a(1-x\sin x)+b\cos x+5}{x^2} & x<0 \\ 3 & x=0 \\ \left(1+\left(\dfrac{cx+dx^3}{x^2}\right)\right)^{\frac{1}{x}} & x>0 \end{cases}$

If f is continuous at x = 0, then find out the values of a, b, c and d.

**Sol.** Since f(x) is continuous at x = 0, so at x = 0, both left and right limits must exist and both must be equal to 3.

Now

$$\lim_{x\to 0^-}\frac{a(1-x\sin x)+b\cos x+5}{x^2}=\lim_{x\to 0^-}\frac{(a+b+5)+\left(-a-\frac{b}{2}\right)x^2+\ldots}{x^2}=3$$ (By the expansions of sinx and cosx)

If $\lim_{x\to 0^-} f(x)$ exists then a + b + 5 = 0 and $-a-\frac{b}{2}=3$ $\Rightarrow$ a = –1 and b = – 4

since $\lim_{x\to 0^+}\left(1+\left(\frac{cx+dx^3}{x^2}\right)\right)^{\frac{1}{x}}$ exists $\Rightarrow$ $\lim_{x\to 0^+}\frac{cx+dx^3}{x^2}=0$ $\Rightarrow$ c = 0

Now $\lim_{x\to 0^+}(1+dx)^{\frac{1}{x}} = \lim_{x\to 0^+}\left[(1+dx)^{\frac{1}{dx}}\right]^d = e^d$

So $e^d = 3$ $\Rightarrow$ d = ℓn 3,

Hence a = – 1, b = – 4, c = 0 and d = ℓn 3.

**Ex.** If f(x) = [sin(x – 1)] – {sin(x – 1)}. Comment on continuity of f(x) at $x=\frac{\pi}{2}+1$

(where [ . ] denotes G.I.F. and { . } denotes fractional part function).

**Sol.** f(x) = [sin (x – 1)] – {sin (x – 1)}

Let g(x) = [ sin (x – 1)] + {sin (x – 1)} = sin (x – 1)

which is continuous at $x=\frac{\pi}{2}+1$

as [sin (x – 1)] and { sin (x – 1)} both are discontinuous at $x=\frac{\pi}{2}+1$

∴ At most one of f(x) or g(x) can be continuous at $x=\frac{\pi}{2}+1$

As g(x) is continuous at $x=\frac{\pi}{2}+1$, therefore, f(x) must be discontinuous

Alternatively, check the continuity of f(x) by evaluating $\lim_{x\to\frac{\pi}{2}+1} f(x)$ and $f\left(\frac{\pi}{2}+1\right)$.

## CONTINUITY IN AN INTERVAL

**(A)** A function f is said to be continuous in (a, b) if f is continuous at each & every point $\in$(a, b).

**(B)** A function f is said to be continuous in a closed interval [ a, b ] if:

**(i)** f is continuous in the open interval (a, b),

**(ii)** f is right continuous at 'a' i.e. $\lim\limits_{x\to a^+} f(x) = f(a)$ = a finite quantity and

**(iii)** f is left continuous at 'b' i.e. $\lim\limits_{x\to b^-} f(x) = f(b)$ = a finite quantity.

**(C)** All Polynomial functions, Trigonometrical functions, Exponential and Logarithmic functions are continuous at every point of their respective domains.

**On the basis of above facts continuity of a function should be checked at the following points**

**(i)** Continuity of a function should be checked at the points where definition of a function changes.

**(ii)** Continuity of {f(x)} and [f(x)] should be checked at all points where f(x) becomes integer.

**(iii)** Continuity of sgn (f(x)) should be checked at the points where f(x) = 0 (if f(x) = 0 in any open interval containing a, then x = a is not a point of discontinuity)

**(iv)** In case of composite function f(g(x)) continuity should be checked at all possible points of discontinuity of g(x) and at the points where g(x) = c, where x = c is a possible point of discontinuity of f(x).

**Ex.** Discuss the continuity of $f(x) = \begin{cases} |x+1| & , \quad x < -2 \\ 2x+3 & , \quad -2 \le x < 0 \\ x^2+3 & , \quad 0 \le x < 3 \\ x^3-15 & , \quad x \ge 3 \end{cases}$

**Sol.** We write f(x) as $f(x) = \begin{cases} -x-1 & , \quad x < -2 \\ 2x+3 & , \quad -2 \le x < 0 \\ x^2+3 & , \quad 0 \le x < 3 \\ x^3-15 & , \quad x \ge 3 \end{cases}$

As we can see, f(x) is defined as a polynomial function in each of intervals (– ∞, –2), (–2, 0), (0, 3) and (3, ∞). Therefore, it is continuous in each of these four open intervals. Thus we check the continuity at x = –2, 0, 3.

At the point x = –2

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} (-x-1) = +2-1 = 1$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} (2x+3) = 2.(-2)+3 = -1$$

Therefore, $\lim\limits_{x\to -2} f(x)$ does not exist and hence f(x) is discontinuous at x = –2.

At the point x = 0

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2x+3) = 3$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^2+3) = 3$$

$$f(0) = 0^2 + 3 = 3$$

Therefore f(x) is continuous at x = 0.

At the point x = 3

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2+3) = 3^2 + 3 = 12$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x^3-15) = 3^3 - 15 = 12$$

$$f(3) = 3^3 - 15 = 12$$

Therefore, f(x) is continuous at x = 3.

We find that f(x) is continuous at all points in R except at x = – 2

**Ex.** If $f(x) = \begin{cases} [\sin \pi x] & ; \quad 0 \le x < 1 \\ \left\{x - \frac{2}{3}\right\} \cdot \operatorname{sgn}\left(x - \frac{5}{4}\right) & ; \quad 1 \le x \le 2 \end{cases}$, where { . } represents fractional part function and [ . ] is greatest integer function, then comment on the continuity of function in the interval [0, 2].

**Sol.** **(i)** Continuity should be checked at the end-points of intervals of each definition i.e. x = 0, 1, 2

**(ii)** For [sin πx], continuity should be checked at all values of x at which sin πx ∈ I

i.e. $x = 0, \frac{1}{2}$

**(iii)** For $\left\{x - \frac{2}{3}\right\} \cdot \operatorname{sgn}\left(x - \frac{5}{4}\right)$, continuity should be checked when $x - \frac{5}{4} = 0$

(as sgn (x) is discontinuous at x = 0) i.e. $x = \frac{5}{4}$ and when $x - \frac{2}{3} \in I$

i.e. $x = \frac{5}{3}$ (as {x} is discontinuous when x ∈ I)

∴ overall discontinuity should be checked at $x = 0, \frac{1}{2}, 1, \frac{5}{4}, \frac{5}{3}$ and 2 check the discontinuity your self.

discontinuous at $x = \frac{1}{2}, 1\ \frac{5}{4}, \frac{5}{3}$

## Properties of Continuous Function in [a, b]

**(i)** If a function f is continuous on a closed interval [a, b] then it is bounded.

**(ii)** A continuous function whose domain is some closed interval must have its range also in closed interval.

**(iii)** If f is continuous and onto on [a, b] and is onto then $f^{-1}$(from the range of f ) is also continuous.

**(iv)** **Some Discontinuous Functions**

| Functions | Points of discontinuity |
|---|---|
| [x], {x} | Every Integer |
| | [illegible] |
| | [illegible] |
| [illegible] | |

**(v) Some continuous functions**

| Function f(x) | Interval in which f(x) is continuous |
|---|---|
| Constant function | $(-\infty, \infty)$ |
| $x^n$, n is an integer $\geq 0$ | $(-\infty, \infty)$ |
| $x^{-n}$, n is a positive integer | $(-\infty, \infty) - \{0\}$ |
| $\|x - a\|$ | $(-\infty, \infty)$ |
| $p(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n$ | $(-\infty, \infty)$ |
| $\frac{p(x)}{q(x)}$, where p(x) and q(x) are polynomial in x | $(-\infty, \infty) - \{x : q(x) = 0\}$ |
| sinx, cosx, $e^x$ | $(-\infty, \infty)$ |
| tanx, secx | $(-\infty, \infty) - \{(2n + 1)\pi/2 : n \in I\}$ |
| cotx, cosecx | $(-\infty, \infty) - \{n\pi : n \in I\}$ |
| $\ell nx$ | $(0, \infty)$ |

## CONTINUITY OF COMPOSITE FUNCTIONS

If f is continuous at x = c and g is continuous at x = f(c),
Then the composite g[f(x)] is continuous at x = c.
f(x) = f(g(x)) is discontinuous also at those values of x where g(x) is discontinuous.

eg. $f(x) = \frac{x \sin x}{x^2 + 2}$ & $g(x) = |x|$ are continuous at x = 0, hence the composite function $(gof)(x) = \left|\frac{x \sin x}{x^2 + 2}\right|$ will also be continuous at x = 0.

For example, $f(x) = \frac{1}{1 - x}$ is discontinuous at x = 1.

Now, $f(f(x)) = \frac{1}{1 - \frac{1}{1 - x}} = \frac{x - 1}{x}$ is not only discontinuous at x = 0 but also at x = 1.

Now, $f(f(f(x))) = \frac{\frac{x-1}{x} - 1}{\frac{x-1}{x}} = x$ seems to be continuous, but it is discontinuous at x = 1 and x = 0, where f(x) and f(f(x)) are discontinuous, respectively.

**Ex.** If $f(x) = \begin{cases} x - 2, & x \leq 0 \\ 4 - x^2, & x > 0 \end{cases}$, then discuss the continuity of y = f(f(x)).

**Sol.** f(x) is discontinuous at x = 0.

Hence, f(f(x)) may be discontinuous at x = 0.

$f(f(0+)) = f(4) = 4 - 16 = 12$

and $f(f(0-)) = f(-2) = -4$

Hence, f(x) is discontinuous at x = 0.

f(f(x)) is also discontinuous when f(x) = 0. Therefore,

$x - 2 = 0$ when $x \le 0$ or $x^2 - 4 = 0$ when $x > 0$

So, it is discontinuous at x = 2.

Also, we can see that $f(f(2)) = 0$, $f(f(2+)) = f(0^-) = -2$, and $f(f(2-)) = f(0+) = 4$.

Hence, f(f(x)) is discontinuous at x = 0 and x = 2.

## REASONS OF DISCONTINUITY

A function can be discontinuous due to the following reasons.

**(i)** $\lim_{x \to a} f(x)$ does not exist ( f(a) may or may not be defined)

i.e. $\lim_{h \to 0} f(a+h) \neq \lim_{h \to 0} f(a-h)$

e.g. $f(x) = [x]$ ; $f(x) = \text{sgn}\, x$ , $f(x) = \dfrac{x}{x-1}$

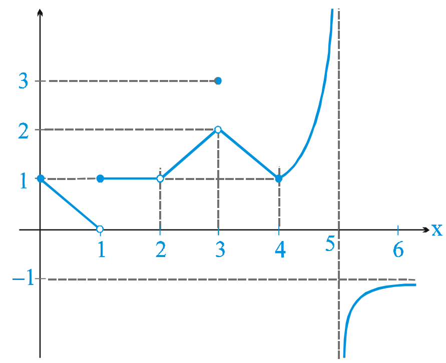
**(ii)** $\lim_{x \to a} f(x)$ exist but is not equal to f(a)

i.e. $\lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(a-h) \neq f(a)$

$$f(x) = \begin{cases} (1-x)\tan\dfrac{\pi x}{2} & \text{if } x \neq 1 \\ \dfrac{\pi}{2} & \text{if } x = 1 \end{cases}$$

**(iii)** f(a) is not defined $\qquad f(x) = \dfrac{1}{x-1}$

To understand explicitly the reasons of discontinuity. Consider the following graph of a function. Note that




- f is continuous at x = 0 and x = 4
- f is discontinuous at x = 1 as limit does not exist
- f is discontinuous at x = 2 as f (2) is not defined although limit exist.
- f is discontinuous at x = 3 as $\lim_{x \to 3} f(x) \neq f(3)$
- f is discontinuous at x = 5 as neither the limit exist nor f is defined at x = 5

It should be remembered that all polynomial functions, trigonometric function, exponential and logarithmic functions are continuous in their domain.

## TYPES OF DISCONTINUITIES

**Type - 1 (Removable type of discontinuities) : -** In case $\lim_{x\to a} f(x)$ exists but is not equal to f(a) then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that $\lim_{x\to a} f(x) = f(a)$ & make it continuous at x = a. Removable type of discontinuity can be further classified as:

**(A) Missing point discontinuity :**

Where $\lim_{x\to a} f(x)$ exists but f(a) is not defined.

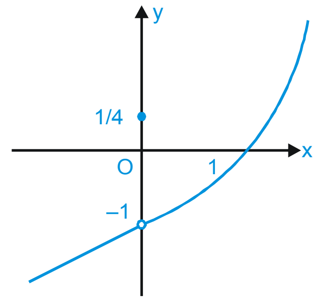
**(B) Isolated point discontinuity :**

Where $\lim_{x\to a} f(x)$ exists & f(a) also exists but ; $\lim_{x\to a} f(x) \neq f(a)$.

**Ex.** Examine the function , $f(x) = \begin{cases} x-1 & , \quad x<0 \\ 1/4 & , \quad x=0 \\ x^2-1 & , \quad x>0 \end{cases}$. Discuss the continuity, and if discontinuous remove the discontinuity by redefining the function (if possible).

**Sol.** Graph of f(x) is shown, from graph it is seen that

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = -1$ , but f(0) = 1/4



y = f(x) before redefining

Thus, f(x) has removable discontinuity and f(x) could be made continuous by taking f(0) = – 1

$$\Rightarrow \qquad f(x) = \begin{cases} x-1 & , \quad x<0 \\ -1 & , \quad x=0 \\ x^2-1 & , \quad x>0 \end{cases}$$

**Type - 2 (Non-Removable type of discontinuities)**

In case $\lim_{x\to a} f(x)$ does not exist then it is not possible to make the function continuous by redefining it. Such a discontinuity is known as non-removable discontinuity or discontinuity of the 2nd kind. Non-removable type of discontinuity can be further classified as :

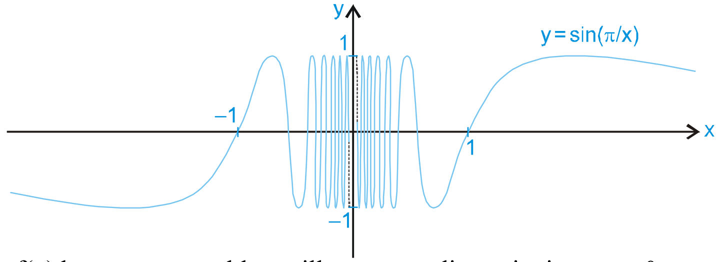
**(i) Finite type discontinuity :** In such type of discontinuity left hand limit and right hand limit at a point exists but are not equal.

**(ii) Infinite type discontinuity :** In such type of discontinuity atleast one of the limit viz. LHL and RHL is tending to infinity.

**(iii) Oscillatory type discontinuity :**

e.g. $f(x) = \sin\frac{\pi}{x}$ at x = 0

$f(x) = \sin\frac{\pi}{x}$



f(x) has non removable oscillatory type discontinuity at x = 0

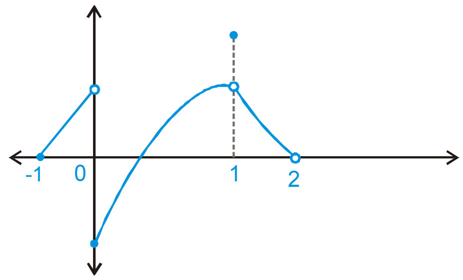
**Example** From the adjacent graph note that

**(i)** f is continuous at $x = -1$

**(ii)** f has isolated discontinuity at $x = 1$

**(iii)** f has missing point discontinuity at $x = 2$

**(iv)** f has non removable (finite type) discontinuity at the origin.




In case of non-removable (finite type) discontinuity the non-negative difference between the value of the RHL at $x = a$ & LHL at $x = a$ is called the jump of discontinuity. A function having a finite number of jumps in a given interval I is called a piece wise continuous or sectionally continuous function in this interval.

**Ex.** Show that the function, $f(x) = \begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1} & ; \text{ when } x \neq 0 \\ 0, & ; \text{ when } x = 0 \end{cases}$ has non-removable discontinuity at $x = 0$.

**Sol.** We have, $f(x) = \begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1} & ; \text{ when } x \neq 0 \\ 0, & ; \text{ when } x = 0 \end{cases}$

$\Rightarrow$ $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{h \to 0} f(0 + h) = \lim\limits_{h \to 0} \dfrac{e^{\frac{1}{h}} - 1}{e^{\frac{1}{h}} + 1} = \lim\limits_{h \to 0} \dfrac{1 - \dfrac{1}{e^{1/h}}}{1 + \dfrac{1}{e^{1/h}}} = 1$ $\quad [\because \; e^{1/h} \to \infty]$

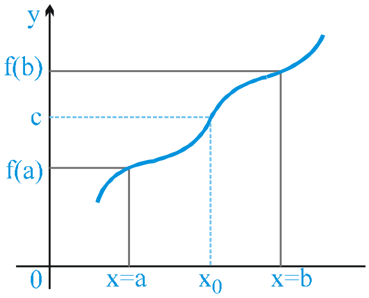
$\Rightarrow$ $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{h \to 0} \dfrac{e^{-1/h} - 1}{e^{-1/h} + 1} = \dfrac{0 - 1}{0 + 1} = -1$ $\quad [\because \; h \to 0 \; ; \; e^{-1/h} \to 0]$

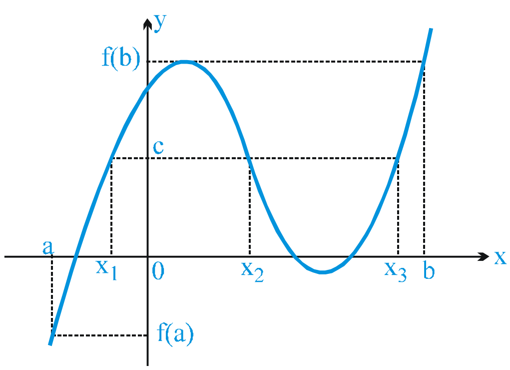
$\lim\limits_{x \to 0^-} f(x) = -1$

$\Rightarrow$ $\lim\limits_{x \to 0^+} f(x) \neq \lim\limits_{x \to 0^-} f(x)$. Thus f(x) has non-removable discontinuity.

## INTERMEDIATE VALUE THEOREM

If f is continuous on [a, b] and $f(a) \neq f(b)$ then for any value $c \in (f(a), f(b))$, there is at least one number $x_0$ in (a, b) for which $f(x_0) = c$



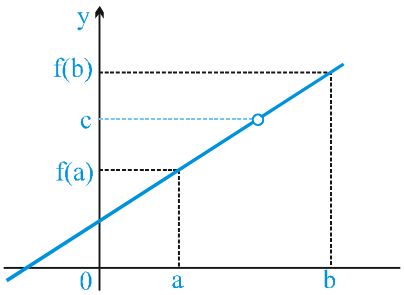



Figure-2

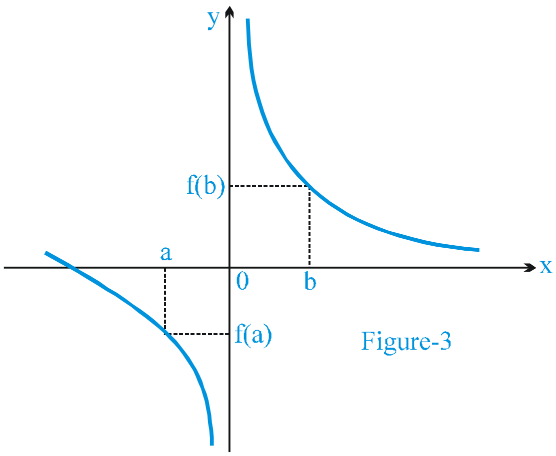
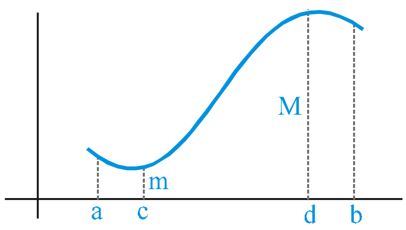



Figure-3

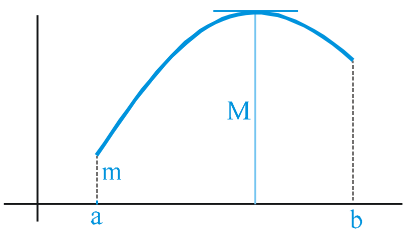
**(i)** Continuity through the interval [a, b] is essential for the validity of this theorem.

**(ii)** In figure-3, f (a) and f (b) are of opposite sign but f (x) has no root in (a, b) as f is continuous.

**EXTREME VALUE THEOREM :** If f is continuous on [a, b] then f takes on, a least value of m and a greatest value M on this interval.




Minimum value 'm' occurs at x = c and maximum value M occurs at x = d. c, d ∈ (a, b)




Minimum value 'm' occurs at the end point x = a and the maximum value M occurs inside the interval

To see that continuity is necessary for the extreme value theorem to be true refer the graph shown.

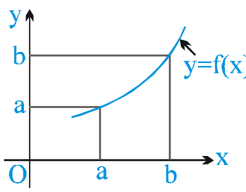
There is a discontinuity at x = c interval. The function has a minimum value at the left end point x = a and f has no maximum value.

**Ex.** Show that the function $f(x) = (x - a)^2 (x - b)^2 + x$ takes the value $\frac{a+b}{2}$ for some value of $x \in [a, b]$

**Sol.** f (a) = a ; f (b) = b ; Also find f is continuous in [a, b] and $\frac{a+b}{2} \in [a, b]$

Hence using intermediate value theroem

$\exists$ some $c \in [a, b]$ such that $f(c) = \frac{a+b}{2}$



**Ex.** Given that a > b > c > d, then prove that the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ will have real and distinct roots.

**Sol.** $(x - a)(x - c) + 2(x - b)(x - d) = 0$

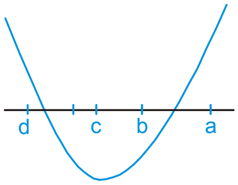
$f(x) = (x - a)(x - c) + 2(x - b)(x - d)$

$f(a) = (a - a)(a - c) + 2(a - b)(a - d) = +\text{ ve}$

$f(b) = (b - a)(b - c) + 0 = -\text{ve}$

$f(c) = 0 + 2(c - b)(c - d) = -\text{ve}$

$f(d) = (d - a)(d - c) + 0 = +\text{ve}$




Hence $(x - a)(x - c) + 2(x - b)(x - d) = 0$

Have real and distinct roots

# TIPS & FORMULAS

## 1. Continuous Functions

A function f(x) is said to be continuous at x = a, if $\lim_{x \to a} f(x) = f(a)$. Symbolically f is continuous at x = a

if $\lim_{h \to 0} f(a - h) = \lim_{h \to 0} f(a + h) = f(a)$

## 2. Continuity of the Function in an Interval

**(A)** A function is said to be continuous in (a, b) if f is continuous at each & every point belonging to (a, b).

**(B)** A function is said to be continuous in a closed interval [a, b] if :

- **(i)** f is continuous in the open interval (a,b)
- **(ii)** f is right continuous at 'a' i.e. $\lim_{x \to a^+} f(x) = f(a)$ = a finite quantity
- **(iii)** f is left continuous at 'b' i.e. $\lim_{x \to b^-} f(x) = f(b)$ = a finite quantity

**Note**

**(i)** All polynomials, trigonometrical functions, exponential & logarithmic functions are continuous in their domains.

**(ii)** If $f$ (x) & g(x) are two functions that are continuous at x = c then the function defined by :
$F_1(x) = f(x) \pm g(x)$ ; $F_2(x) = K\,f(x)$, where K is any real number ; $F_3(x) = f(x)\,.\,g(x)$ are also continuous at x = c.

Further, if g(c) is not zero, then $F_4(x) = \dfrac{f(x)}{g(x)}$ is also continuous at x = c.

**(iii)** If f and g are continuous then fog and gof are also continuous.

**(iv)** If f and g are discontinuous at x = c, then f + g, f - g, f.g may still be continuous.

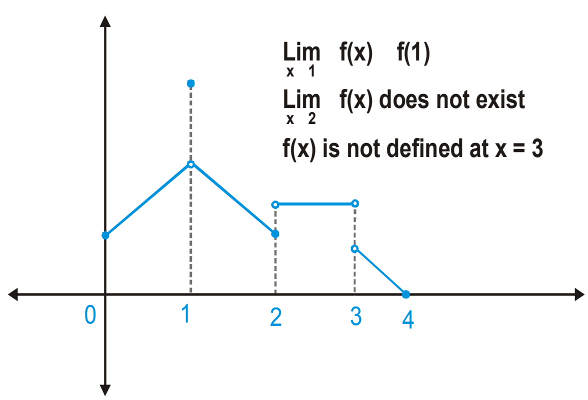
## 3. Reasons of Discontinuity :

**(A)** Limit does not exist

i.e. $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

**(B)** f(x) is not defined at x = a

**(C)** $\lim_{x \to a} f(x) \neq f(a)$




Geometrically, the graph of the function will exhibit a break at x = a, if the function is discontinuous at x = a. The graph as shown is discontinuous at x = 1 , 2 and 3.

## 4. Types of Discontinuities

**Type-1 : (Removable type of discontinuities) : -** In case $\lim_{x \to a} f(x)$ exists but is not equal to f(a) then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that $\lim_{x \to a} f(x) = f(a)$ & make it continuous at x = a. Removable type of discontinuity can be further classified as:

**(A) Missing Point Discontinuity :**

Where $\lim_{x \to a} f(x)$ exists but f(a) is not defined.

**(B) Isolated Point Discontinuity :**

Where $\lim_{x \to a} f(x)$ exists & f(a) also exists but ; $\lim_{x \to a} f(x) \neq f(a)$.

**Type-2 : (Non-Removable Type of Discontinuities) :-**

In case $\lim_{x \to a} f(x)$ does not exist then it is not possible to make the function continuous by redefining it. Such a discontinuity is known as non-removable discontinuity or discontinuity of the 2nd kind. Non-removable type of discontinuity can be further classified as :

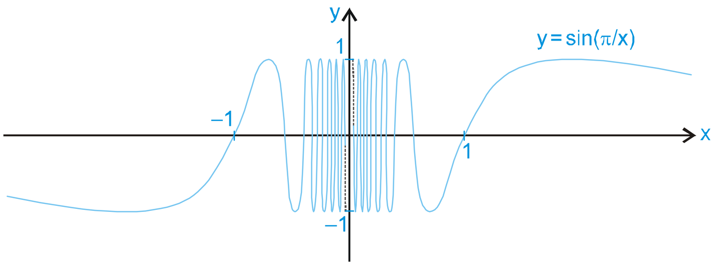
**(i) Finite Type Discontinuity :** In such type of discontinuity left hand limit and right hand limit at a point exists but are not equal.

**(ii) Infinite Type Discontinuity :** In such type of discontinuity atleast one of the limit viz. LHL and RHL is tending to infinity.

**(iii) Oscillatory Type Discontinuity :**

e.g. $f(x) = \sin\frac{\pi}{x}$ at $x = 0$

$$f(x) = \sin\frac{\pi}{x}$$

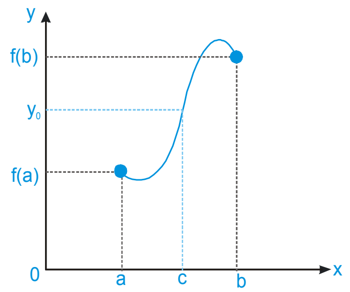



f(x) has non removable oscillatory type discontinuity at x = 0

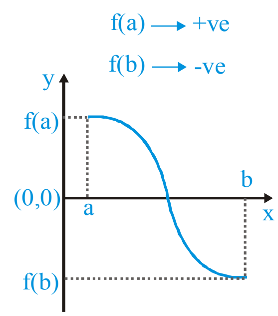
**Note :** In case of non-removable (finite type) discontinuity the non-negative difference between the value of the RHL at $x = a$ & LHL at $x = a$ is called the jump of discontinuity. A function having a finite number of jumps in a given interval I is called a piece wise continuous or sectionally continuous function in this interval.

## 5. The Intermediate Value Theorem

Suppose f(x) is continuous on an interval I, and a and b are any two points of I. Then if $y_0$ is a number between f(a) and f(b), there exists a number c between a and b such that $f(c) = y_0$



The function f, being continuous on [a,b] takes on every value between f(a) and f(b)



f(x) = 0 in (a, b)

Note that a function f which is continuous in [a,b] possesses the following properties :

**(i)** If f(a) & f(b) posses opposite signs, then there exists atleast one root of the equation f(x) = 0 in the open interval (a,b).

**(ii)** If K is any real number between f(a) & f(b), then there exists atleast one root of the equation f(x) = K in the open interval (a,b).