# HINTS & SOLUTIONS

## EXERCISE - 1

### Single Choice

**1.** Consider the graph of $h(x) = \max(x, x^2)$ at $x = 0$ and $x = 1$ for D: $h(x) = \max.(x^2, -x^2)]$

**3.** Let $x = \frac{2}{3}$ which is rational

$\Rightarrow \quad h\left(\frac{2}{3}\right) = \frac{1}{3}$

$\lim\limits_{t\to 0} h\left(\frac{2}{3}+t\right) = 0 \quad \Rightarrow$ discontinuous at $x \in Q$

Let $x = \sqrt{2} \notin Q$

$h\left(\sqrt{2}\right) = 0 \qquad$ consider $\sqrt{2} = 1.41401235839$

$h\left(\sqrt{2}\right) = h\left(\frac{1.414023583}{10^{10}}\right) = \frac{1}{10^{10}} \to 0$

Hence h is continuous for all irrational $\Rightarrow$ A

**5.** $f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}$ .......(i)

$f(0) = 0, \; f'(0) = 3$

Put $x = 3x$ and $y = 0$

$f(x) = \frac{f(3x)}{3}$ ........(ii)

$\lim\limits_{h\to 0} f(x+h) = \lim\limits_{h\to 0} f\left(\frac{3x+3h}{3}\right) = \lim\limits_{h\to 0} \frac{f(3x)+f(3h)}{3}$

$= \frac{f(3x)}{3} = f(x)$

Similarly we can prove $\lim\limits_{h\to 0} f(x-h) = f(x)$

$\Rightarrow$ f(x) is continuous for all x in R

Given that $f'(0) = 3$

$\Rightarrow \quad \lim\limits_{h\to 0} \frac{f(h)}{h} = \lim\limits_{h\to 0} \frac{f(-h)}{-h} = 3$

**7.** $$\begin{cases} x - 2k\pi & \text{for } 2k\pi - \frac{\pi}{2} \le x \le 2k\pi + \frac{\pi}{2} \\ (2k+1)\pi - x & \text{for } 2k\pi + \frac{\pi}{2} \le x < 2k\pi + \frac{3\pi}{2} \end{cases}$$

**8.** $f'(0^-) = \lim\limits_{h\to 0} \dfrac{-h\left(\frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}}\right) - 0}{-h} = -1$

$f'(0^+) = \lim\limits_{h\to 0} \dfrac{h\left(\frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}\right)}{h} = 1$

Since $f'(0^-) \ne f'(0^+)$

So f(x) is not differentiable.

**12.** In the immediate neighborhood of $x = \pi/2$, $\sin x > \sin^3 x$

$\Rightarrow \quad |\sin x - \sin^3 x| = \sin x - \sin^3 x$

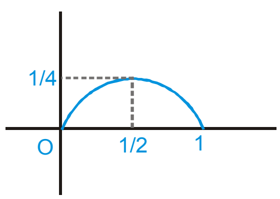
Hence for $x \ne \pi/2$,

$f(x) = \left[\frac{2(\sin x - \sin^3 x) + \sin x - \sin^3 x}{2(\sin x - \sin^3 x) - \sin x + \sin^3 x}\right]$

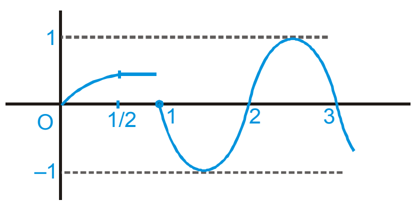
$= \frac{3\sin x - 3\sin^3 x}{\sin x - \sin^3 x} = 3$

Hence f is continuous and diff. at $x = \pi/2$ ]

**13.**




$$y = g(x) = \begin{cases} x - x^2 & 0 \le x \le 1/2 \\ 1/4 & 1/2 < x \le 1 \\ \sin \pi x & x > 1 \end{cases}$$

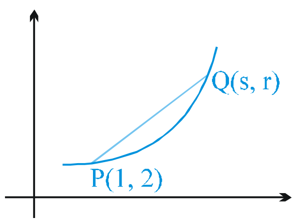



**14.** $x^{1/3}$ is not differentiable at $x = 0$

**16.** I By definition $f'(1)$ is the limit of the slope of the secant line when $s \to 1$.

Thus $f'(1) = \lim\limits_{s\to 1} \dfrac{s^2+2s-3}{s-1}$

$= \lim\limits_{s\to 1} \dfrac{(s-1)(s+3)}{s-1}$




$= \lim\limits_{s\to 1}(s+3) = 4 \Rightarrow$ (D)

II By substituting x = s into the equation of the secant line, and cancelling by s – 1 again, we get

$y = s^2 + 2s - 1$. This is f (s), and its derivative is $f'(s) = 2s+2$, so $f'(1) = 4$.]

**18.** f'(x)

$= \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(h)+|x|h+xh^2}{h}$

where $x = h$ and $y = x$

$\because f(0) = 0;$ hence $f'(x)$

$= \lim\limits_{h\to 0}\left(\dfrac{f(h)-f(0)}{h} + |x| + xh\right)$

$f'(x) = f'(0) + |x| = |x|$

**19.** $\lim\limits_{x\to 2^-} f(x) = \dfrac{3}{5} = f(2) \neq \lim\limits_{x\to 2^+} f(x) = 1$

f(x) is not continous at x = 2

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) = f(3) = \dfrac{9}{2}$

Now LHD (x =3) is

$\lim\limits_{h\to 0} \dfrac{\frac{1}{4}((3-h)^3-(3-h)^2)-\frac{9}{2}}{-h}$

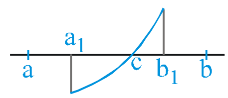
$= \lim\limits_{h\to 0} \dfrac{h^2-8h+21}{4} = \dfrac{21}{4}$

and RHD $(x=3)$ is $\lim\limits_{h\to 0} \dfrac{\frac{9}{4}(|h-1|+|-1-h|)-\frac{9}{2}}{h} = 0$

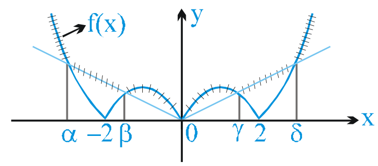
f(x) is not differentiable at x =2 and x = 3

**20.** I and II are false. The function f (x) = 1/x, 0 < x <1, is a counter example.

Statement III is true. Apply the intermediate value theorem to f on the closed interval $[a_1, b_1]$




**22.** $f(x)$ is non differentiable at $x = \alpha, \beta, 0, \gamma, \delta$




and $g(x)$ is non differentiable at $x = \alpha, \beta, 0, -2, 2 \Rightarrow$ **(B)**

**23.** $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1} x^2 e^{2(x-1)} = 1$

$f(1) = 1$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1} a\,\text{sgn}(x+1)\cos 2(x-1) + bx^2 = a.1.1 + b$

for continuity a + b = 1

LHD $(x=1)$ is $\lim\limits_{h\to 0} \dfrac{(1-h)^2 e^{-2h} - 1}{h}$

$= \lim\limits_{h\to 0} 2e^{-2h} + he^{-2h} + \left(\dfrac{e^{-2h}-1}{h}\right)$

$= 2 + 0 + 2 = 4$

RHD $(x=1)$ is $\lim\limits_{h\to 0} \dfrac{a\,\text{sgn}(2+h)\cos 2h + b(1+h)^2 - 1}{h}$

$= \lim\limits_{h\to 0} \dfrac{a\cos 2h + b + bh^2 + 2bh - (a+b)}{h}$

$= \lim\limits_{h\to 0} a\left(\dfrac{\cos 2h - 1}{h}\right) + bh + 2b = 2b$

f(x) is differentiable at x = 1 if 2b = 4

$\Rightarrow b = 2 \quad a = -1$

**24.** $g(x) = \begin{cases} 3x^2 - 4\sqrt{x} + 1 & \text{for } x < 1 \\ ax + b & \text{for } x \geq 1 \end{cases}$

for differentiability at $x = 1, g'(1^+) = g'(1^{-1})$

$a = 6x - \dfrac{4}{2\sqrt{x}} \Rightarrow a = 6 - 2 = 4$

for continuity at $x = 1$, $g(1^+) = g(1^-)$

$a + b = 3 - 4 + 1 \Rightarrow a + b = 0 \Rightarrow b = -4$

$a = 4$, and $b = -4$

**26.** $f\left(\frac{x+y}{3}\right)=\frac{4-2f(x)-2f(y)}{3}\quad \forall, x, y\in R$ ....**(i)**

differentiate w.r.t. y

$\frac{1}{3}f'\left(\frac{x+y}{3}\right)=-\frac{2}{3}f'(y)$

replace x with 3x and y with 0

$f'(x)=-2f'(0)$

put $x=0$ we get $f'(0)=0$

$\Rightarrow f'(x)=0$

$\Rightarrow f(x)=\text{constant} \quad \Rightarrow f(x)=f(0)$

in equation **(i)** put x = 0 = y it gives $f(0)=\frac{4}{7}$

$\Rightarrow f(x)=\frac{4}{7}$

**27.** $g(x)=\frac{\sin\frac{\pi[x]}{4}}{[x]}$

obv. cont. at x = 3/2

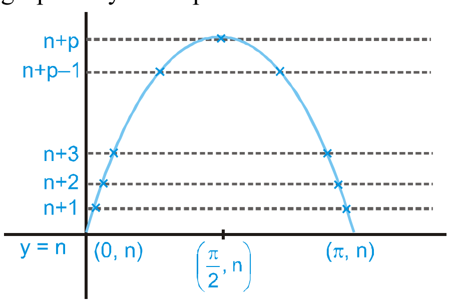
at $x=2$ $\quad f(2^-)=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

$f(2)=\frac{\sin\frac{\pi}{2}}{2}=\frac{1}{2}$

Hence discontinuous at x = 2

**28.** $f(x)=[n+p\sin x], \quad x\in(0,\pi)$

graph of $y=n+p\sin x$




obviously

f(x) = [n + psinx] is discontinous at points mark in above curve

$\Rightarrow$ number of such points $\Rightarrow (p-1)+1+p-1=2p-1$

**29.** $\lim_{h\to 0}|f(x+h)-f(x)|\le(x+h-x)^2$

$\Rightarrow \lim_{h\to 0}|f(x+h)-f(x)|\le|h|^2$

$\Rightarrow \lim_{h\to 0}\left|\frac{f(x+h)-f(x)}{h}\right|\le 0 \quad \Rightarrow f'(x)=0$

$\Rightarrow$ f(x) is constant function $\quad \Rightarrow f(1)=0$

**30.** not derivable at x = 0 and 2

## EXERCISE - 2

### Part # I : Multiple Choice

**4.** **(C)** is false and is True only if $f'(a)=0$ limit is $2f'(a)$. In

**(D)** same logic limit is $\frac{1}{2}f'(a)$

**5.** $f(x)=\begin{cases}|x-3| & x\ge 1\\ \frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4} & x<1\end{cases}$

Clearly it is continuous at x = 1

$f(1^+)=f(1^+)=f(1)$

at $\quad x=3 \quad f(3^+)=f(3^+)=f(3)=0$

It is continuous at x = 3

$f(1^+)=\lim_{n\to 0}\frac{f(1+n)-f(1)}{-n}=-1$

$f(1^-)=\lim_{n\to 0}\frac{f(1-n)-f(1)}{-h}$

$=\frac{n^2-2n+1-6+6n+13-8}{-4h}$

$=\lim_{n\to 0}\frac{n^2 4h}{-4h}=\frac{n+4}{-4}$

$\Rightarrow -1$

**6.** $f(x)=\frac{\sin\frac{\pi}{4}}{1}=\frac{1}{\sqrt{2}}\quad ; \quad 1\le x<2$

$=\frac{\sin\frac{\pi}{2}}{2}=\frac{1}{2}\quad ; \quad 2\le x<3$

Hence f(x) is continuous at $\frac{3}{2}$, differentiable at $\frac{4}{3}$ & discontinuous at 2.

**7.** $f'(0^+)=\frac{1}{\sqrt{2}}$ ; $f'(0^-)=-\frac{1}{\sqrt{2}}$ ;

$f(x)=\frac{\sqrt{x^2}}{\sqrt{1+\sqrt{1-x^2}}}=\frac{|x|}{\sqrt{1+\sqrt{1-x^2}}}$

**9.** Range is $R^+\cup\{0\}$ $\Rightarrow$ B is not correct

f is not differentiable at x = – 1

as $f(x)=\begin{cases}x^3+1 & \text{if } x\ge -1\\ -(x^3+1) & \text{if } x<-1\end{cases}$

$$\Rightarrow \quad f'(x) = \begin{cases} 3x^2 & \text{if } x > -1 \\ -3x^2 & \text{if } x < -1 \end{cases}$$

$f'(-1^+) = 3; \quad f'(-1^-) = -3$

$\Rightarrow$ $f$ is not differentiable at $x = -1$

also since f is not bijective hence it has no inverse

$\Rightarrow$ **(C)**

**10.** $$f(x) = \begin{cases} (x+1)(2x-1), & x < -1 \\ (x+1)(1-2x), & -1 \le x \le 0 \\ x+1, & 0 \le x < 1 \\ (x+1)(2x-1), & x \ge 1 \end{cases}$$

$$f'(x) = \begin{cases} 4x+1, & x < -1 \\ -4x-1, & -1 \le x \le 0 \\ 1, & 0 \le x < 1 \\ 4x+1, & x \ge 1 \end{cases}$$

Function is not differentiable at $x = -1$, 0 and 1.

**11.** $f(x) = \left|x - \frac{1}{2}\right| + |x-1| + \tan x$

$\left|x - \frac{1}{2}\right|$ is non-differentiable at $x = \frac{1}{2}$

$\Rightarrow$ $|x-1|$ is non-differentiable at $x = 1$

$\tan x$ is non-differentiable at $x = \frac{\pi}{2}$

**12.** $$f'(0^+) = \lim_{h\to 0} \frac{h \ln(\cos h)}{h \ln(1+h^2)} = \lim_{h\to 0} \frac{\ln(\cos h)^{1/h^2}}{\frac{\ln(1+h^2)}{h^2}}$$

$$= \lim_{h\to 0} \frac{1}{h^2}(\cos h - 1) = -\frac{1}{2}; \quad f'(0^-) = -\frac{1}{2}$$

hence f is continuous and derivable at $x = 0$

**15.** $$f(x) = \begin{cases} 0 & 0 < x < 1 \\ 0 & x = 0 \text{ or } 1 \text{ or } -1 \\ 0 & -1 < x < 0 \end{cases}$$

$\Rightarrow$ $f(x) = 0$ for all in $[-1, 1]$

**18.** $f(x) = \sum_{k=0}^{n} a_k |x|^k = a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3 + \ldots\ldots + a_n|x|^n$

$f(0) = a_0$ we know that $\lim_{x\to 0} |x| = 0$

$\lim_{x\to 0} f(x) = a_0$

f(x) is continous for $x = 0$

$|x|^n$ is differentiable if $n \ne 1, \quad n \in N$

f(x) is not differentiable at $x = 0$, due to presence of $|x|$

If all $a_{2k+1} = 0$, f(x) does not contains $|x|$

$\Rightarrow$ f(x) is differentiable at $x = 0$

**20.** $$H(x) = \begin{cases} \cos x \; ; & 0 \le x \le \frac{\pi}{2} \\ \frac{\pi}{2} - x \; ; & \frac{\pi}{2} < x \le 3 \end{cases}$$

$$H'\left(\frac{\pi^-}{2}\right) = -\sin x = -1 \quad \Rightarrow \quad H'\left(\frac{\pi^+}{2}\right) = -1$$

Hence H(x) is continuous and derivable in [0, 3] & has maximum value 1 in [0, 3]

## Part # II : Assertion & Reason

**2.** $y = |\ln x|$ not differentiable at $x = 1$

$y = \big|\cos|x|\big|$ is not differentiable at $x = \frac{\pi}{2}, \frac{3\pi}{2}$

$y = \cos^{-1}(\text{sgn}\, x) = \cos^{-1}(1) = 0$ differentiable $\forall\ x \in (0, 2\pi)$

**3.** $f'(0^+) = \frac{h \sin h - 0}{h} = 0$

$f'(0^-) = \frac{h \sin(-h) - 0}{-h} = 0$

f(x) is diff. at $x = 0$

e.g. $x|x|$ is derivable at $x = 0$

**6.** **Statement-1** $f(x) = \text{sgn}(\cos x)$

at $x = \frac{\pi}{2}$, $\cos x = 0$

$\therefore$ f(x) is discontinuous & non differentiable at $x = \frac{\pi}{2}$

**Statement - 2** $g(x) = [\cos x]$

at $x = \frac{\pi}{2}$, $\cos x = 0$.

$\therefore$ g(x) is discontinuous & hencenon differentiable

at $x = \frac{\pi}{2}$. **(True)**

**7.** Consider $g(x) = x^3$ at $x = 0$; $g(0) = 0$

$|g(x)|$ is derivable as $x = 0$

actually nothing definite can be said. Also for

$g(x) = x - 1$ with $g(1) = 0$

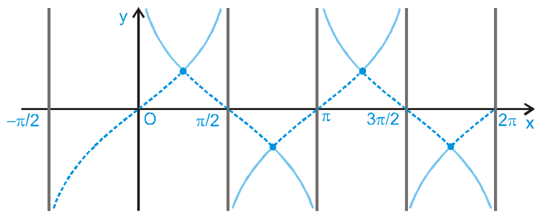
then $|g(x)|$ not derivable at $x = 1$

## EXERCISE - 3

### Part # I : Matrix Match Type

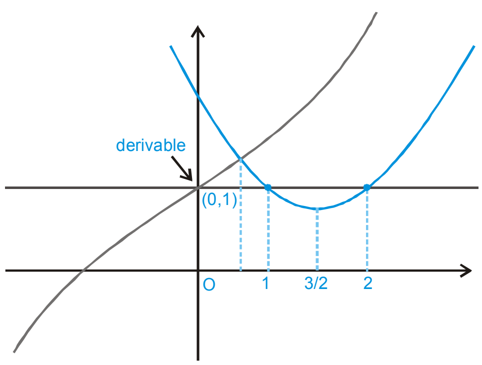
**1.(A)** $f(x) = \dfrac{\tan x + \cot x}{2} - \left|\dfrac{\tan x - \cot x}{2}\right|$

$f(x) = \begin{cases} \cot x\ , & \tan x \ge \cot x \\ \tan x\ , & \tan x < \cot x \end{cases}$




There are 4 points where the function is continuous but not differentiable in $(0, 2\pi)$

**(B)**




**(C)** $f(x) = (x+4)^{1/3}$

$f'(x) = \dfrac{1}{3}(x+4)^{-2/3}$

Not derivable at $x = -4$

**(D)** $f(x) = \begin{cases} -\dfrac{\pi}{2}\ell n\left(\dfrac{x.2}{\pi}\right) + \dfrac{\pi}{2}, & 0 < x \le \dfrac{\pi}{2} \\ \pi - x, & \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \end{cases}$

$f'(x) = \begin{cases} -\dfrac{\pi}{2x} & 0 < x < \dfrac{\pi}{2} \\ -1 & \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \end{cases}$

$f'\left(\dfrac{\pi^-}{2}\right) = f'\left(\dfrac{\pi^+}{2}\right) = -1$

function differentiable at $x = \dfrac{\pi}{2}$

**2.** Consider the graph of $2\cos x$ in $(-\pi, \pi)$. $2\cos x$ is integer at 9 points.

$[2\cos x]$ is discontinuous at 7 points in $(-\pi, \pi)$

Similarly from graph of 2sinx, we can observe that $[2\sin x]$ is discontinuous at 7 points

(continuous at $-\pi/2, \pi$)

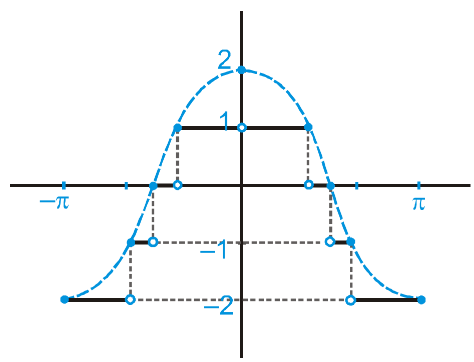
$[2\tan x/2]$ is discontinuous at 4 points

(continuous at $-\pi/2$)

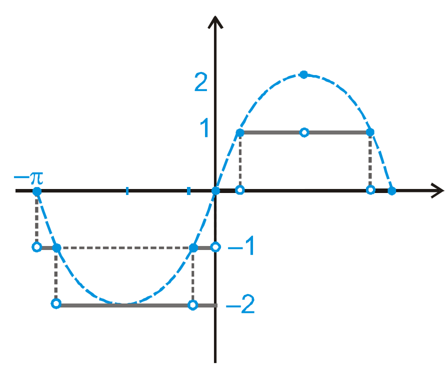
$[3\operatorname{cosec} x/3]$ is discontinuous at 4 points

(continuous at $\pi/2$)

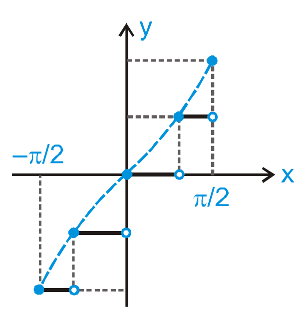
$y = [2\cos x]$



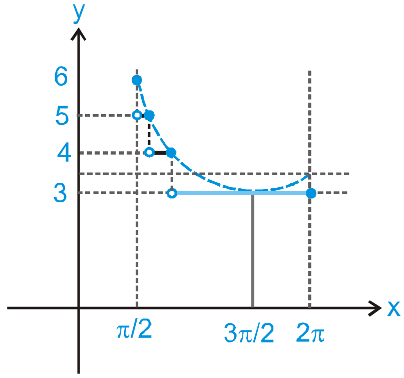

$y = [2\sin x]$




$y = [2\tan x/2]$



$y = [3 \operatorname{cosec} x/3]$




**5.** **(A)** $f'(0) = \lim_{h \to 0} \frac{\cos h - 0}{h}$ does not exist. Obviously

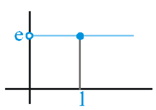
$f(0) = f(0^-) = f(0^+) = 1$

Hence continuous and not derivable

**(B)** $g(x) = 0$ for all x, hence continuous and derivable

**(C)** as $0 \le \{f(x)\} < 1$, hence $h(x) = \sqrt{\{x\}^2} = \{x\}$ which is discontinuous hence non derivable all $x \in I$

**(D)** $\lim_{x \to 1} x^{\frac{1}{\ln x}} = \lim_{x \to 1} x^{\log_x e} = e = f(1)$




Hence k (x) is constant for all x > 0 hence continuous and differentiable at x = 1.

**6.** **(A)** $f(x) = \begin{cases} 1 - 1 = 0 & ; \quad 1 < x \le 2 \\ 0 & ; \quad x = 1 \\ 1 - x & ; \quad 0 \le x < 1 \\ -\sin \pi x & ; \quad -1 \le x < 0 \end{cases}$

at x = 0, f(x) is not continuous & not differentiable

at x = 1, f(x) is continuous & not differentiable

at x = 2 and – 1, f(x) is continuous & differentiable

**(C)** $f(x) = \frac{x}{x+1}$, not defined at x = – 1

$g(x) = \frac{f(x)}{f(x) + 2}$

g(x) is not defined at f(x) = – 2

$\frac{x}{x+1} = -2 \quad \Rightarrow x = \frac{-2}{3}$

Also x = 0 is not in the domain of f(x)

So, at 3 points g(x) is not differentiable.

## Part # II : Comprehension

### Comprehension–2

$$\text{LHD} = \lim_{h \to 0} \frac{\dfrac{-\sinh + \tanh + \cosh - 1}{2h^2 + \ln(2-h) - \tanh} - 0}{-h}$$

$$= \lim_{h \to 0} \frac{\dfrac{\sinh}{h} - \dfrac{\tanh}{h} + \dfrac{1 - \cosh}{h^2} \times h}{2h^2 + \ln(2-h) - \tanh} = 0$$

$$f'(0^+) = \text{RHD} = \sqrt{2} = h \times \frac{e^{h^2} - 1}{h^2} = 0$$

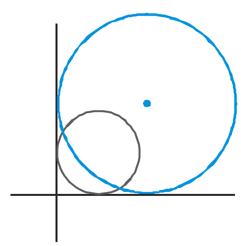
$L_1 \equiv y = 0$ and $L_2 \equiv x = 0$

**1.** $(x - r)^2 + (y - r)^2 = r^2$ (family of circle)

$x^2 + y^2 - 2rx - 2ry + r^2 = 0$

$2(r_1 r_2 + r_1 r_2) = r_1^2 + r_2^2$ or $4r_1 r_2 = r_1^2 + r_2^2$

$\left(\frac{r_2}{r_1}\right)^2 - 4\left(\frac{r_2}{r_1}\right) + 1 = 0$

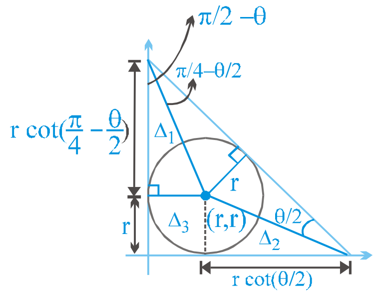


$\frac{r_2}{r_1} = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$

**2.** $2[\Delta_1 + \Delta_2 + \Delta_3]$

$\Delta = 2 \times \frac{1}{2}\left(\cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + \cot\frac{\theta}{2} + 1\right) \qquad \left[\text{using } \frac{1}{2}ab\right]$

$$\Delta = \frac{\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)} + \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} + 1$$



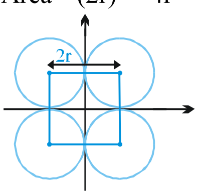

$$\Delta = 1+\frac{2\sin\frac{\pi}{4}}{2\sin\frac{\theta}{2}\cdot\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}$$

$$\Delta = 1+\frac{\sqrt{2}}{\cos\left(\theta-\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{4}\right)}$$

$\Delta$ is minimum if numerator is maximum when $\theta=\frac{\pi}{4}$

$$\Delta_{min}=1+\frac{\sqrt{2}}{1-\frac{1}{\sqrt{2}}}=1+\frac{2}{\sqrt{2}-1}=1+2(\sqrt{2}+1)=3+2\sqrt{2}$$

**3.** Area $=(2r)^2=4r^2$




## Comprehension 4

$f(-x)=\frac{1}{f(x)}$

But $x=0 \Rightarrow f^2(0)=1 \Rightarrow f(0)=1$

$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}$

$f'(x)=f(x)\lim_{h\to 0}\frac{f(h)-f(0)}{h}$

$\Rightarrow \frac{f'(x)}{f(x)}=-1 \Rightarrow \int\frac{f'(x)}{f(x)}dx=-x+c$

$\Rightarrow \ell n\, f(x)=-x+c$

$f(x)=\lambda e^{-x}$

at $x=0, \lambda=1$

$\therefore f(x)=e^{-x}$

**1.** Range of f(x) is $R^+$

**2.** Range of f(|x|) is (0, 1]

**3.** f(x) is decreasing function

**4.** $f'(x)=-e^{-x}=-f(x)$

## EXERCISE - 4

### Subjective Type

**1.** continuous but not differentiable at $x=0$; differentiable & continuous at $x=\pi/2$

**2.** continuous but not differentiable at $x=0$

**3.** f is not derivable at all integral values in $-1<x\le 3$

**4.** f(x) is continuous but not derivable at $x=0$

**5.** $f(x)=\begin{cases} ax^2-b & ; \quad -1<x<1 \\ \frac{-1}{x} & ; \quad x\ge 1 \\ \frac{1}{x} & ; \quad x\le -1\end{cases}$

Now f(x) is differentiable at $x=1$

$\Rightarrow 2ax=\frac{1}{x^2}$ at $x=1$

$\Rightarrow a=\frac{1}{2}$

Also f(x) is continuous at $x=1$

$\Rightarrow a(1)^2-b=-1 \quad \Rightarrow b=\frac{3}{2}$

**6.** $a\neq 1, b=0, p=1/3$ and $q=-1$

**7.** $f:R\to R$ and $f(x+y)=f(x)\,f(y)\ \forall\ x,y\in R$

Put $x=y=0$ $f(0)=f^2(0)$ since $f(0)\neq 0$

$\Rightarrow f(0)=1$

$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}f(x)\frac{[f(h)-1]}{h}$

$=f(x)\,f'(0)$

Let $f(x)=y \Rightarrow \frac{dy}{dx}=y.f'(0)$

On solving $\ell ny=x\,f'(0)+c$

$y=f(x)=e^c.\,e^{x\,f'(0)}$

$\because f(0)=1 \Rightarrow c=0$

Thus $f(x)=e^{x.f'(0)}\ \forall\ x\in R$

**8.** $(fog)(x) = x+1$ for $-2 \le x \le -1$, $-(x+1)$ for $-1 < x \le 0$ & $x-1$ for $0 < x \le 2$. $(fog)(x)$ is continuous at $x = -1$, $(gof)(x) = x+1$ for $-1 \le x \le 1$ & $3-x$ for $1 < x \le 3$. $(gof)(x)$ is not differentiable at $x = 1$

**9.** **(A)** $f'(0) = \dfrac{h^m \sin\left(\frac{1}{h}\right) - 0}{h} = h^{m-1}\sin\left(\frac{1}{h}\right)$

$\Rightarrow$ $m - 1 > 0$ for derivable

$f'(x) = mx^{m-1}\sin\left(\frac{1}{x}\right) - x^{m-2}\cos\left(\frac{1}{x}\right)$

$f'(x)$ to be discontinuous at $x = 0$, $m \in (1, 2]$

**(B)** Clearly for $f(x)$ to be derivable, & its derivative continuous at $x = 0$, $m \in (2, \infty)$

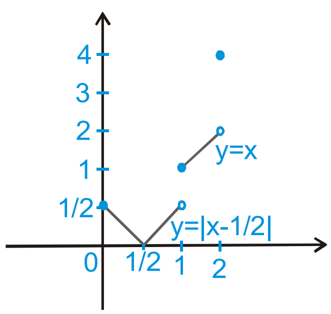
**10.** $f(x)$ is continuous but not differentiable at $x = n\pi$, $n \in I$, $f(x)$ is not periodic.

**11.** f is discontinuous at $x = 2$, f is not differentiable at $x = 1, 3/2, 2$

**12.** $2x\,e^x$

**13.** f is continuous but not derivable at $x = 1/2$, f is neither differentiable nor continuous at $x = 1$ & $x = 2$

y = f(x)




**14.** $f'(0) = \lim_{h\to 0}\dfrac{f(0+h) - f(0)}{h}$ $\Rightarrow$ $1 = \lim_{h\to 0}\dfrac{f(h)}{h}$

$\therefore$ $\lim_{x\to 0}\dfrac{f(x)}{x} = 1$ ; $\lim_{x\to 0}\dfrac{f\left(\frac{x}{2}\right)}{\frac{x}{2}\times 2} = \dfrac{1}{2}$

and similarly so on.

On substituting value we get required result.

**15.** $f(x + y^n) = f(x) + (f(y))^n$

$f(0+0) = f(0) + (f(0))^n$ $\Rightarrow$ $f(0) = 0$

also $f'(0) = \lim_{h\to 0}\dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\dfrac{f(h)}{h}$

Let $I = f'(0) = \lim_{h\to 0}\dfrac{f(0 + (h^{1/n})^n) - f(0)}{(h^{1/n})^n}$

$= \lim_{h\to 0}\dfrac{f((h^{1/n}))^n}{(h^{1/n})^n} = \lim_{h\to 0}\left(\dfrac{f(h^{1/n})}{h^{1/n}}\right)^n = I^n$

$\Rightarrow$ $I = I^n$ or $I = 0, 1, -1$

Since $f'(0) \ge 0$ & $f(x)$ is not identically zero

So $I = 1$ $\therefore$ $f'(0) = 1$ .....**(i)**

Thus $f'(x) = \lim_{h\to 0}\dfrac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0}\dfrac{f(x + (h^{1/n})^n) - f(x)}{(h^{1/n})^n}$

$= \lim_{h\to 0}\dfrac{f(x) + (f(h^{1/n}))^n - f(x)}{(h^{1/n})^n}$

$= \lim_{h\to 0}\left(\dfrac{f(h^{1/n})}{h^{1/n}}\right)^n = (f'(0))^n$

$\Rightarrow$ $f'(x) = 1$ (using **(i)**)

Integrating both side

$f(x) = x + c$

$f(x) = x$ $[f(0) = 0]$

$f(10) = 10$

**16.** $y = f(x) = x\sin 1/x.\ \sin\dfrac{1}{x\sin 1/x}$

when $x \ne 0, \dfrac{1}{r\pi}$, $r = 1,2,3$

$y = 0, x = 0, \dfrac{1}{r\pi}$ where $r = 1,2,3,$...............

Let $t = x\sin 1/x$ as $x \to 0^+$, $t \to 0$

and as $x \to \dfrac{1}{r\pi}$, $t \to 0$

$y = t\sin 1/t$

$\lim_{x\to 0} y = \lim_{t\to 0} t\sin t = 0 = f(0)$

also $\lim_{x\to\frac{1}{r\pi}} y = \lim_{t\to 0} t\sin t = 0 \quad = f\left(\frac{1}{r\pi}\right)$

$\Rightarrow$ f(x) is continous at x = 0 and $\frac{1}{r\pi}$

$\Rightarrow$ f(x) is continous $\forall\ x \in [0, 1]$

We know that $t = x\sin 1/x$ is not differentiable at x = 0

therefore $y = t\sin 1/t = x\sin 1/x.\ \sin\dfrac{1}{x\sin\frac{1}{x}}$ is not differentiable at x = 0

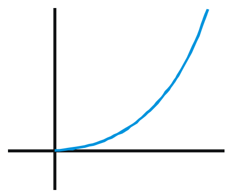
**17.** For $t \ge 0$

$x = 2t - t = t$

$y = t^2 + t^2 = 2t^2$

For $t < 0$

$x = 3t, y = 0$



Also when

$0 \le x < 1 \Rightarrow 0 \le t < 1$ $\quad [\therefore\ x = t]$

$-1 \le x < 0 \Rightarrow \dfrac{-1}{3} \le t < 0$ $\quad [\therefore\ x = 3t]$

f is continuous and differentiable at x = 0

**18.** $f(1^-) = \lim_{h\to 0} \cos^{-1}\left(\text{sgn}\left(\dfrac{2[1-h]}{3(1-h)-[1-h]}\right)\right) = \dfrac{\pi}{2}$

$f(1^+) = \lim_{h\to 0} \cos^{-1}\left(\text{sgn}\left(\dfrac{2[1+h]}{3(1+h)-[1+h]}\right)\right)$

$= \lim_{h\to 0} \cos^{-1}\left(\text{sgn}\left(\dfrac{2}{2}\right)\right) = 0$

Hence f(x) is not continuous & not derivable at x = 1

Now at x = – 1

$f(-1^-) = \lim_{h\to 0} \cos^{-1}\left(\text{sgn}\left(\dfrac{2[-1-h]}{3(-1-h)-[-1-h]}\right)\right)$

$= \lim_{h\to 0} \cos^{-1}\left(\text{sgn}\left(\dfrac{-4}{-3+2}\right)\right) = \cos^{-1}1 = 0$

Also $f(-1^+) = \lim_{h\to 0} \cos^{-1}\left(\text{sgn}\left(\dfrac{2[-1+h]}{3(-1+h)-[-1+h]}\right)\right)$

$= \lim_{h\to 0} \cos^{-1}\left(\text{sgn}\left(\dfrac{-2}{-3+1}\right)\right) = \cos^{-1}1 = 0$

Hence f(x) is continuous & differentiable at x = –1

**19.** Differentiability at x = 1

$$f'(1^-) = \lim_{h\to 0} \frac{\dfrac{\sin[(1-h)^2]\,\pi}{(1-h)^2 - 3(1-h) + 8} + a\,(1-h)^3 + b - (a+b)}{-h}$$

$$= \lim_{h\to 0} \frac{a(1-h)^3 - a}{-h} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{h\to 0} \frac{3a\,(1-h)^2}{1}$$

$f'(1^-) = 3a$

$$f'(1^+) = \lim_{h\to 0} \frac{2\cos(1+h)\,\pi + \tan^{-1}(1+h) - a - b}{h}$$

$$= \lim_{h\to 0} \frac{(-2\cos\pi h + \tan^{-1}(1+h) - a - b)}{h}$$

Function is differentiable

$\therefore\ -2 + \dfrac{\pi}{4} = a + b$ .....**(i)**

$$= \lim_{h\to 0} \frac{-2\cos\pi h + \tan^{-1}(1+h) - 2 + \pi/2}{h}$$

$$= \lim_{h\to 0} 2\pi\sin\pi h + \frac{1}{1+(1+h)^2} = \frac{1}{2}$$

Now $f'(1^-) = f'(1^+)$

$3a = \dfrac{1}{2}$

$a = \dfrac{1}{6}$ .....**(ii)**

by **(i)** and **(ii)** $b = \dfrac{\pi}{4} - \dfrac{13}{6}$

**20.** $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} \ge \lim_{h\to 0} \dfrac{\ell n\left(\frac{x+h}{x}\right) + x + h - x}{h}$

$\ge \lim_{h\to 0} \dfrac{\ell n\left(1 + \frac{h}{x}\right)}{h} + 1 \ge \dfrac{1}{x} + 1$ .....**(i)**

$$f'(x)=\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}\le \lim_{h\to 0}\frac{\ell n\left(\frac{x-h}{x}\right)+x-h-x}{-h}$$

$$\le \lim_{h\to 0}\frac{\ell n\left(1-\frac{h}{x}\right)}{-h}+1\le\frac{1}{x}+1 \qquad .....\textbf{(ii)}$$

from **(i)** and **(ii)**

$$\Rightarrow \quad f'(x)=\frac{1}{x}+1$$

$$\therefore \sum_{n=1}^{100} g\left(\frac{1}{n}\right)=g\left(\frac{1}{1}\right)+g\left(\frac{1}{2}\right)+....+g\left(\frac{1}{100}\right)$$

$$=(1+2+3+.....100)+100=5150$$

## EXERCISE - 5

### Part # I : AIEEE/JEE-MAIN

**1.** $f(x+y)=f(x).f(y) \quad \forall\, x, y$

$\therefore \quad f(5+0)=f(5).f(0) \quad \{\because f(5)=2\}$

$\therefore \quad f(0)=1$

Now $f'(5)=\lim_{h\to 0}\frac{f(5+h)-f(5)}{h}$

$$=\lim_{h\to 0}\frac{f(5)f(h)-f(5)}{h}$$

$$=f(5)\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$=f(5)\,f'(0)=2\times 3\Rightarrow 6$$

**4.** Apply L Hospital rule

$$\lim_{h\to 0}\frac{f'(1+h)}{1}=5 \quad \Rightarrow \quad f'(1)=5$$

**5.** $|f(x)-f(y)|\le|x-y|^2$

$$\Rightarrow \frac{|f(x)-f(y)|}{|x-y|}\le |x-y|$$

$$\Rightarrow \lim_{x\to y}\left|\frac{f(x)-f(y)}{x-y}\right|\le \lim_{x\to y}|x-y|$$

$\Rightarrow \quad f'(x)\le 0 \quad \Rightarrow \quad f'(x)=0$

$\Rightarrow \quad f(x)$ is continuous function

$\therefore \quad f(1)=0=f(0)$

**6.** $f(x)=\frac{x}{1+|x|}$ is differentiable

$$f(x)=\begin{cases}\frac{x}{1-x}, & x<0\\ 0, & x=0\\ \frac{x}{1+x}, & x>0\end{cases}$$

L.H.D. $=\lim_{h\to 0}\frac{f(0-h)-f(0)}{h}$

$$\text{L.H.D.}=\frac{\frac{-h}{1+h}-0}{-h}=1$$

R.H.D $= \dfrac{f(0+h)-f(0)}{h} = \dfrac{\frac{h}{1+h}-0}{h} = 1$

so differentiable at $(-\infty, \infty)$

**7.** $gof(x) = \begin{cases} \sin x^2 & ; \quad x \geq 0 \\ -\sin x^2 & ; \quad x < 0 \end{cases}$

gof(x) is continuous (LHL = RHL = 0) = f(0)

$gof'(x) = \begin{cases} 2x\cos x^2 & . \quad x > 0 \\ -2x\cos x^2 & ; \quad x < 0 \end{cases}$

LHD = 0 RHD = 0

gof(x) is differentiable

Now $gof''(x) = \begin{cases} 2[\cos x^2 - x\sin x^2 . 2x] & ; \quad x > 0 \\ -2[\cos x^2 - x\sin x^2 . 2x] & ; \quad x < 0 \end{cases}$

LHD = – 2, RHD = 2

Not differentiable.

**8.** $\lim_{x\to a} \dfrac{x^2 f(a) - a^2 f(x)}{x-a} \quad \left[\dfrac{0}{0} \text{ form}\right]$

Use L'Hospital rule

$= \lim_{x\to a} \dfrac{2x f(a) - a^2 f'(x)}{1}$

$= 2af(a) - a^2 f'(a)$

**9.** $f(x) = |x-2| + |x-5|$ ; $x \in R$

f(x) is continuous in [2, 5] and differentiable is (2, 5) and f(2) = f(5) = 3.

$\therefore$ By Rolle's theorem f'(x) = 0 for at least one $x \in (2, 5)$.

$f'(x) = \dfrac{|x-2|}{x-2} + \dfrac{|x-5|}{x-5}$

$f'(4) = 0$ but $f'(x) = 0 \; \forall \; x \in (2, 5)$

## Part # II : IIT-JEE ADVANCED

**2.** Let us first prove that

**(I)** g is continuous at $\alpha$ and

$f(x) - f(\alpha) = g(x)(x-\alpha), \forall x \in R$

$\Rightarrow$ f(x) is differentiable at $\alpha$.

Since g is continuous at $x = \alpha$

and $g(x) = \dfrac{f(x)-f(\alpha)}{x-\alpha}$

We should have, $\lim_{x\to\alpha} g(x) = g(\alpha)$

$\Rightarrow \lim_{x\to\alpha} \dfrac{f(x)-f(\alpha)}{x-\alpha} = g(\alpha) \Rightarrow f'(x) = g(\alpha)$

$\Rightarrow$ $f'(\alpha)$ exists and is equal to $g(\alpha)$.

Conversely now we prove.

**(II)** f(x) is differentiable at $x = \alpha$

$\Rightarrow$ g is continuous at

$x = \alpha$ and $f(x) - f(\alpha) = g(x)(x-\alpha) \; \forall \; x \in R$.

$\because$ f(x) is differentiable at $x = \alpha$

$\therefore \lim_{x\to\alpha} \dfrac{f(x)-f(\alpha)}{x-\alpha} = f'(\alpha)$

exists and is finite.

Let us define, $g(x) = \begin{cases} \dfrac{f(x)-f(\alpha)}{x-\alpha}, & x \neq \alpha \\ f'(\alpha), & x = \alpha \end{cases}$

Then, $f(x) - f(\alpha) = (x-\alpha)\, g(x), \forall x \neq \alpha$

Now for continuity of g(x) at $x = \alpha$

$\lim_{x\to\alpha} g(x) = \lim_{x\to\alpha} \dfrac{f(x)-f(\alpha)}{x-\alpha} = f'(\alpha) = g(\alpha)$

$\therefore$ g is continuous at $x = \alpha$.

**4.** Given that f ; R $\to$ R such that

$f(1) = 3$ and $f'(1) = 6$

Then $\lim_{x\to 0}\left[\dfrac{f(1+x)}{f(1)}\right]^{1/x} = e^{\lim_{x\to 0}\frac{1}{x}[\log f(1+x) - \log f(1)]}$

$= e^{\lim_{x\to 0}\frac{\frac{1}{f(1+x)}f'(1+x)}{1}}$ [Using L' Hospital rule]

$= e^{\frac{f'(1)}{f(1)}} = e^{6/3} = e^2$

**5.** Given that

$$f(x)=\begin{cases} x+a & \text{if } x<0 \\ |x-1| & \text{if } x\ge 0 \end{cases} = \begin{cases} x+a & \text{if } x<0 \\ 1-x & \text{if } 0\le x<1 \\ x-1 & \text{if } x\ge 1 \end{cases}$$

and $g(x)=\begin{cases} (x+1) & \text{if } x<0 \\ (x-1)^2+b & \text{if } x\ge 0 \end{cases}$

where $a, b \ge 0$

Then $(gof)(x)=g[f(x)]$

$$=\begin{cases} f(x)+1 & \text{if } f(x)<0 \\ [f(x)-1]^2+b & \text{if} (f)(x)\ge 0 \end{cases}$$

(Using definition of g(x))

Now, $f(x)<0$ when $x+a<0$ i.e. $x<-a$

$f(x)=0$ when $x=-a$ or $x=1$

$f(x)>0$ when $-a<x<1$ or $x>1$

$$g(f(x))=\begin{cases} f(x)+1 & \text{if } x<-a \\ [f(x)-1]^2+b & \text{if } x=-a \text{ or } x=1 \\ [f(x)-1]^2+b & \text{if } -a<x<0 \\ [f(x)-1]^2+b & \text{if } 0\le x<1 \\ [f(x)-1]^2+b & \text{if } x>1 \end{cases}$$

[Keeping in mind that x = 0 and 1 are also the breaking pt's because of definition of f(x)]

$$\therefore g[f(x)]=\begin{cases} x+a+1 & \text{if } x<-a \\ (x+a-1)^2+b & \text{if } -a\le x<0 \\ ((1-x)-1)^2+b & \text{if } 0\le x\le 1 \\ (x-1-1)^2+b & x>1 \end{cases}$$

(Substituting the value of f(x) under different conditions)

$$\therefore g[f(x)]=\begin{cases} x+a+1 & \text{if } x<-a \\ (x+a-1)^2+b & \text{if } -a\le x<0=F(x)\text{(say)} \\ x^2+b & \text{if } 0\le x\le 1 \\ (x-2)^2+b & \text{if } x>1 \end{cases}$$

Now given that $gof(x)\equiv F(x)$ is continuous for all real numbers, therefore it will be continuous at – a.

$\Rightarrow$ L.H.L. = R.H.L. = f(–a)

$\Rightarrow \lim_{h\to 0} F(-a-h)=\lim_{h\to 0} F(-a+h)=F(-a)$

Now, $\lim_{h\to 0} F(-a-h)=\lim_{h\to 0} a-h+a+1=1$

$\lim_{h\to 0} F(-a+h)=\lim_{h\to 0} (-a+h+a-1)^2+b=1+b$

$F(-a)=1+b$

Thus we should have $1=1+b \Rightarrow b=0$

Again for continuity at $x=0$

L.H.L. = f(0)

$\Rightarrow \lim_{h\to 0} f(0-h)=f(0)$

$\Rightarrow \lim_{h\to 0} (-h+a-1)^2+b=b$

$\Rightarrow (a-1)^2=0 \Rightarrow a=1$

For a = 1 and b = 0, gof becomes

$$gof(x)=\begin{cases} x+2, & x<-1 \\ x^2, & -1\le x\le 1 \\ (x-2)^2 & x>1 \end{cases}$$

Now to check differentiability of gof(x) at x = 0

We see $gof(x)=x^2=F(x)$

$\Rightarrow F'(x)=2x$ which exists clearly at $x=0$

Hence gof is differentiable at x = 0.

**6.** Given that $f:[-2a, 2a]\to R$

f is an odd function.

Lf' at x = a is 0.

$$\Rightarrow \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}=0$$

$$\Rightarrow \lim_{h\to 0} \frac{f(a-h)-f(a)}{h}=0 \quad \text{....(1)}$$

To find Lf' at x = – a which is given by

$$\lim_{h\to 0} \frac{f(-a-h)-f(-a)}{-h}=\lim_{h\to 0} \frac{-f(a+h)+f(a)}{-h}$$

$[\because f(-x)=-f(x)]$

$$= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Again for $x \in [a, 2a]$

$$f(x) = f(2a-x)$$

$\therefore$ $f(a+h) = f(2a-a-h) = f(a-h)$

substituting this values in last expression we get

$$Lf'(-a) = \lim_{h\to 0}\frac{f(a-h)-f(a)}{h} = 0$$

[Using eq$^n$ (1)]

Hence $Lf'(-a) = 0$

**9.** $p = -1$ Now

$$\lim_{x\to 1^+}\frac{(x-1)^n}{\log\cos^m(x-1)} = \lim_{x\to 1^+}\frac{(x-1)^n}{\log[\cos^m(x-1)-1+1]}$$

$$= \lim_{x\to 1^+}\frac{(x-1)^n}{\cos^m(x-1)-1}$$

$$= \lim_{x\to 1^+}\frac{n(x-1)^{n-1}}{m\cos^{m-1}(x-1)\sin(x-1)}$$

$$= \frac{-n}{m}\lim_{x\to 1^+}\frac{(x-1)}{\sin(x-1)}\cdot\frac{1}{\cos^{m-1}(x-1)}\times(x-1)^{n-2}$$

$$= \frac{-n}{m}\lim_{x\to 1^+}(x-1)^{n-2} = -1 \text{ (Given)}$$

$\Rightarrow$ $n = 2$ and $m = 2$

**10.** $f(x+y) = f(x) + f(y)$

$f(0) = 0$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)+f(h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(h)}{h} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$f'(x) = f'(0) = k$ (k is constant)

$\Rightarrow$ $f(x) = kx$, hence $f(x)$ is continuous and $f'(x)$ is constant $\forall\ x \in R$

**11.** $f\left(-\frac{\pi}{2}^-\right) = 0,\ f\left(-\frac{\pi}{2}^+\right) = 0$

$$f'(x) = \begin{cases} -1 & x \le \frac{\pi}{2} \\ \sin x & -\frac{\pi}{2} < x \le 0 \\ 1 & 0 < x \le 1 \\ \frac{1}{x} & x > 1 \end{cases}$$

$f'(0^-) = 0$, $f'(0^+) = 1$ $\therefore$ not differentiable at $x = 0$

$f'(1^-) = 1$, $f'(1^+) = 1$ $\therefore$ differentiable at $x = 1$

as $-\frac{3}{2} \in \left(-\frac{\pi}{2}, 0\right)$

$f'(x) = \sin x$ which is differentiable at $x = -\frac{3}{2}$

**12.** At $x = 0$

$$R.H.D = \lim_{h\to 0}\frac{(0+h)-(0)}{h} = \lim_{h\to 0}\frac{h^2\left|\cos\frac{\pi}{h}\right|-0}{h}$$

$$= \lim_{h\to 0} h\left|\cos\frac{\pi}{h}\right| = 0\times\cos(\infty) = 0\times \text{finite} = 0$$

$$LHD := \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{h^2\cos\left(\frac{\pi}{-h}\right)-0}{-h} = \lim_{h\to 0} -h\cos\left(\frac{\pi}{h}\right)$$

$= 0$

$\because$ LHD = RHD at $x = 0$

$\Rightarrow$ $f(x)$ is differentiable at $x = 0$

At $x = 2$

$$RHD = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0}\frac{(2+h)^2.\cos\left(\frac{\pi}{2+h}\right)-0}{h}$$

$$= 4\lim_{h\to 0}\frac{\cos\left(\frac{\pi}{2+h}\right)}{h}$$

$$= -4\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{2+h}\right).\left(-\frac{\pi}{(2+h)^2}\right)}{1} = \pi$$

LHD : $\lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$

$$= \lim_{h\to 0}\frac{(2-h)^2\left(-\cos\left(\frac{\pi}{2-h}\right)\right)}{-h}$$

$$= \lim_{h\to 0}\frac{4\left(\sin\frac{\pi}{(2-h)}\right)\left(\frac{\pi}{(2-h)^2}\right)}{-1} = -\pi.$$

LHD $\neq$ RHD at $x = 2$

$\therefore$ Not differentiable at $x = 2$.

**19.** $f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$

**(A)** If $a = 0, b = 1$, $f(x) = |x|\sin(|x^3 + x|)$

$\Rightarrow f(x) = x\sin(x^3 + x) \quad \forall x \in R$

Hence f(x) is differentiable.

**(B, C)** If $a = 1, b = 0$, $f(x) = \cos(|x^3 - x|)$

$f(x) = \cos(x^3 - x)$

Which is differentiable at $x = 1$ and $x = 0$.

**(D)** If $a = 1, b = 1$ $f(x) = \cos(x^3 - x) + |x|\sin(|x^3 + x|)$

$= \cos(x^3 - x) + x\sin(x^3 + x)$

Which is differentiable at $x = 1$

**20.** if $\lim_{x\to 2}\frac{f(x)g(x)}{f'(x)g'(x)} = 1$

if $\lim_{x\to 2}\frac{f'(x)g(x) + g'(x)f(x)}{f''(x)g'(x) + f'(x)g''(x)} = 1$

As Limit 1 $\Rightarrow \frac{f'(2)g(2) + g'(2)f(2)}{f''(2)g'(2) + f'(x)g''(2)} = 1$

$\Rightarrow \frac{g'(2).f(2)}{f''(2)g'(2)} = 1 \quad \Rightarrow \quad f''(2) = f(2)$

Hence option (D)

As $f''(2) = f(2)$ and range of $f(x) \in (0, \infty)$

$\Rightarrow f''(2) > 0$

$\Rightarrow$ f has local min. at $x = 2$

Hence (A)

**21.** $f(x) = [x^2 - 3] = [x^2] - 3$

$f(x)$ is discontinuous at $x = 1, \sqrt{2}, \sqrt{3}, 2$

$g(x) = (|x| + |4x - 7|)([x^2] - 3)$

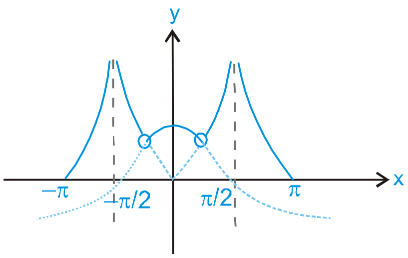
$$g(x) = \begin{cases} 15x - 21 & x < 0 \\ 9x - 21 & 0 \le x \le 1 \\ 6x - 14 & 1 \le x < \sqrt{2} \\ 3x - 7 & \sqrt{2} \le x < \sqrt{3} \\ 0 & \sqrt{3} \le x < 2 \\ 3 & x = 2 \end{cases}$$

$\therefore$ g(x) is not differentiable,

at $x = 0, 1, \sqrt{2}, \sqrt{3}$

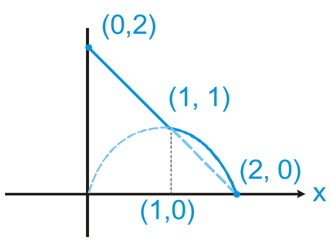
## MOCK TEST

**1. (A)**




The functions is not differentiable at two points between $x = -\pi/2$ & $x = \pi/2$ also function is not continuous at $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ hence at four points function is not differentiable.

**2. (D)**




**3. (B)**

$$\frac{f(2x+2y)}{f(2x-2y)} = \frac{\sin(x+y)}{\sin(x-y)}$$

$$\Rightarrow \frac{f(\alpha)}{\sin\frac{\alpha}{2}} = \frac{f(\beta)}{\sin\frac{\beta}{2}} = k$$

$$f(x) = k\sin\frac{x}{2}$$

$$f'(x) = \frac{k}{2}\cos\frac{x}{2},\ f''(x) = \frac{-k}{4}\sin\frac{x}{2}$$

$$4f''(x) + f(x) = 0$$

**4.** $\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+}\left([x^2] + \sqrt{\{x\}^2}\right) \Rightarrow \lim\limits_{x\to1^+}(1+0) = 1$

$\because \lim\limits_{x\to1^-} f(x) \Rightarrow \lim\limits_{x\to1^-}(0+1) = 1$ and $f(1) = 1$

$\Rightarrow \lim\limits_{x\to1} f(x) = f(1)$

$\therefore$ continuous at $x = 1$

similarly we check for another integers

**5. (B)**

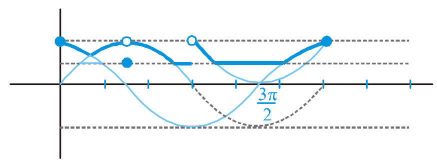
$f: R \to R, \quad g(x) = \frac{1}{f(x)}$

$g'(x) = -\frac{1}{f(x)^2}\cdot f'(x)$

$\Rightarrow$ g is one - one if f is one - one.

**6. (D)**

From figure




There are 4 points

**7. (A)**

$\lim\limits_{x\to0} f(x) = 0$

$\{\because \lim\limits_{x\to0} x^2 = 0$ and $\{e^{1/x}\}$ is a bounded function$\}$

$\therefore \quad k = 0$

$$\lim_{x\to0}\frac{f(0+x)-f(0)}{x} = \lim_{x\to0} x\{e^{1/x}\} = 0$$

**[C & D]**

$\therefore \quad f'(0) = 0$

not continuous at $x = \log_2 e, \log_3 e, \ldots$ etc.

**8.** Function

$f(x) = (x^2-1)\,|x^2-3x+2| + \cos(|x|)$ ....... **(i)**

**Imp Note :** In differentiable of $|f(x)|$ we have to consider critical points for which $f(x) = 0$

$|x|$ is not differentiablity at $x = 0$

but $\cos|x| = \begin{cases} \cos(-x), & \text{if } x < 0 \\ \cos x, & \text{if } x \ge 0 \end{cases}$

$\Rightarrow \cos|x| = \begin{cases} \cos x, & \text{if } x < 0 \\ \cos x, & \text{if } x \ge 0 \end{cases}$

Therefore it is differentiable at $x = 0$

Next, $|x^2-3x+2| = |(x-1)(x-2)|$

$$= \begin{cases} (x-1)(x-2), & \text{if } x < 1 \\ -(x-1)(x-2), & \text{if } 1 \le x < 2 \\ (x-1)(x-2), & \text{if } 2 \le x \end{cases}$$

Therefore,

$$f(x)=\begin{cases}(x^2-1)(x-1)(x-2)+\cos x, & \text{if } -\infty<x<1\\ -(x^2-1)(x-1)(x-2)+\cos x, & \text{if } 1\le x<2\\ (x^2-1)\,(x-1)(x-2)+\cos x, & \text{if } 2\le x<\infty\end{cases}$$

Now, x = 1, 2 are critical point for differentiability Because f(x) is differentiable on other points in its domain

**Differentiability at x = 1**

$$L\,f'(1)=\lim_{x\to1-0}\frac{f(x)-f(1)}{x-1}$$

$$=\lim_{x\to1-0}\left[(x^2-1)(x-2)+\frac{\cos x-\cos 1}{x-1}\right]$$

= 0 – sin 1 = – sin 1

$$\left(\because\ \lim_{x\to1-0}\frac{\cos x-\cos 1}{x-1}=\frac{d}{dx}(\cos x)\right.$$

at x = 1 – 0

= – sin x at x = 1 – 0 = – sin x at x = 1 = – sin1

$$\text{and } R\,f'(1)=\lim_{x\to1+0}\frac{f(x)-f(1)}{x-1}$$

$$=\lim_{x\to1+0}\left[-(x^2-1)(x-2)+\frac{\cos x-\cos 1}{x-1}\right]$$

= 0 – sin 1 = – sin 1 (same approach)

$\because$ Lf′ (1) = Rf′ (1).

Therefore, function is differentiable at x = 1.

$$\text{Again } Lf'(2)=\lim_{x\to2-0}\frac{f(x)-f(2)}{x-2}$$

$$=\lim_{x\to2-0}\left[-(x^2-1)(x-1)+\frac{\cos x-\cos 2}{x-2}\right]$$

= – (4 – 1) (2 – 1) – sin 2 = – 3 – sin 2

$$\text{and } R\,f'(2)=\lim_{x\to2+0}\frac{f(x)-f(2)}{x-2}$$

$$=\lim_{x\to2+0}\left[(x^2-1)(x-1)+\frac{\cos x-\cos 2}{x-2}\right]$$

= $(2^2 – 1)$ – sin 2 = 3 – sin 2

So L f′ (2) ≠ R f′ (2), f is not differentiable at x = 2

Therefore, (d) is the answer.

**9. (B)**

$S_1$ : $\lim_{x\to a} f(x)=f(a)$

$\therefore$ $\lim_{x\to a}[f(x)]$ is an integer and

$\lim_{x\to a}[f(x)]=\lim_{x\to a}f(x)=f(a)$

$\therefore$ $S_1$ is true

$S_2$ : Derivative of cos |x| at x = 0 is 0 but derivative of |x| does not exist at x = 0

$\therefore$ $S_2$ is false

$S_3$ : False : Consider the function $x^{1/3}$

$S_4$ : Let $f(x)=\begin{cases}1, & x\in Q\\ -1, & x\in R\sim Q\end{cases}$ and

$$g(x)=\begin{cases}0, & x\in Q\\ 1, & x\in R\sim Q\end{cases}$$

then $gof(x)=0,\ \forall x\in R$

$\therefore$ $S_4$ is true

**10. (A)**

$S_1$ : $\lim_{h\to0^+}\frac{f(a+h)-f(a)}{h}$ exists finitely

$$\therefore\ \lim_{h\to0^+}f(a+h)-f(a)=\lim_{h\to0^+}\left(\frac{f(a+h)-f(a)}{h}\right)h=0$$

$\Rightarrow$ $\lim_{h\to0^+}f(a+h)=f(a)$ Similarly $\lim_{h\to0^-}f(a+h)=f(a)$

$\therefore$ f is continuous at x = a

$S_2$ : Function is not differentiable at $5x=(2n+1)\frac{\pi}{2}$ only, which are not in its domain.

$S_3$ : Let $f(x)=\frac{1}{x^2}$ & $g(x)=-\frac{1}{x^2}$, $\lim_{x\to0}(f(x)+g(x))$ exists whenever $\lim_{x\to0}f(x)$ and $\lim_{x\to0}g(x)$ does not exist.

$S_4$ : Not necessary.

**11. (B,D)**

**(A)** $\lim_{x\to1} f(x)$ does not exist

**(B)** $\lim_{x\to1} f(x) = \frac{2}{3}$

$\therefore$ f(x) has removable discontinuity at x = 1

**(C)** $\lim_{x\to1} f(x)$ does not exist

**(D)** $\lim_{x\to1} f(x) = \frac{-1}{2\sqrt{2}}$

$\therefore$ f(x) has removable discontinuity at x = 1

**12. (A,C,D)**

$f(x+y) = f(x) + f(y) + xy(x+y)$

$f(0) = 0$

$\therefore \lim_{h\to0} \frac{f(h)}{h} = -1$

$\therefore \lim_{h\to0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to0} \frac{f(x)+f(h)+xh(x+h)-f(x)}{h}$

$= \lim_{h\to0} \frac{f(h)}{h} + \lim_{h\to0} x(x+h) = -1 + x^2$

$\therefore f'(x) = -1 + x^2$

$\therefore f(x) = \frac{x^3}{3} - x + c$

$\therefore$ f(x) is a polynomial function, f(x) is twice differentiable for all $x \in R$ and $f'(3) = 3^2 - 1 = 8$

**13.** $\lim_{x\to0} \frac{f(x)+f\left(\frac{x}{2}\right)+f\left(\frac{x}{3}\right)+...+f\left(\frac{x}{k}\right)}{x} \quad \left(\frac{0}{0}\text{ form}\right)$

$= \lim_{x\to0} \left[f'(x)+\frac{1}{2}f'\left(\frac{x}{2}\right)+\frac{1}{3}f'\left(\frac{x}{3}\right)+...+\frac{1}{k}f'\left(\frac{x}{k}\right)\right]$

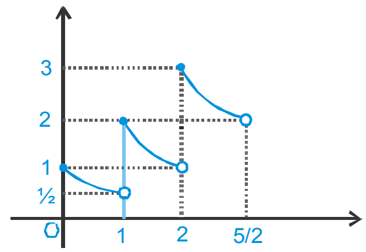
$= 1 + \frac{1}{2} + \frac{1}{3} + .... + \frac{1}{n} = \int_0^1 (1+x+x^2+...+x^{n-1})\,dx$

$= \int_0^1 \frac{1-x^n}{1-x}dx = \int_0^1 \frac{1-(1-x)^n}{x}dx$

$= {}^nC_1 - \frac{{}^nC_2}{2} + \frac{{}^nC_3}{3} - ........ + (-1)^{n-1} . \frac{{}^nC_n}{n}$

**14. (A, B, D)**

$$f(x) = \begin{cases} \frac{1}{x+1} & , \quad 0 \le x < 1 \\ \frac{2}{x} & , \quad 1 \le x < 2 \\ \frac{3}{x-1} & , \quad 2 \le x < \frac{5}{2} \end{cases}$$




clearly f(x) is discontinuous and bijective function

$\lim_{x\to1^-} f(x) = \frac{1}{2}$

$\lim_{x\to1^+} f(x) = 2$

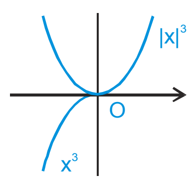
$\min\left(\lim_{x\to1^-} f(x), \lim_{x\to1^+} f(x)\right) = \frac{1}{2} \ne f(1)$

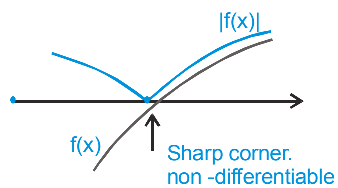
$\max(1, 2) = 2 = f(1)$

**15. (A, D)**

both $x^2, -x^{3/2}$ have their RHL = 0 and RHD = 0

**16. (C)**



**17. (A)**

$f(x) = |x| \sin x$

L.H.D $= \lim_{h\to 0} \dfrac{|0-h|\sin(0-h) - 0}{h}$

$= \lim_{h\to 0} \dfrac{-h \sin h}{h} = 0$

R.H.D $= \lim_{h\to 0} \dfrac{|0+h|\sin(0+h) - 0}{h}$

$f(x)$ is differentiable at $x = 0$

**18. (A)**

$f(2) = 4$

$f(2^-) = \lim_{x\to 2^-} |[x]\, x| = 2$

Discontinuous $\Rightarrow$ Non. differentiable

**19. (A)**

**Statement-I** $f(x) = \begin{cases} x^2 - 5x + 6 & , \quad x \le 2 \\ -x^2 + 5x - 6 & , \quad 2 \le x \le 3 \\ x^2 - 5x + 6 & , \quad x \ge 3 \end{cases}$

$f'(x) = \begin{cases} 2x - 5 & , \quad x < 2 \\ -2x + 5 & , \quad 2 < x < 3 \\ 2x - 5 & , \quad x > 3 \end{cases}$

$f'(2^-) + f'(2^+) = -1 + 1 = 0$

**Statement-II** $f(x) = \begin{cases} (x-a)(x-b) & , \quad x < a \\ -(x-a)(x-a) & , \quad a \le x \le b \\ (x-a)(x-b) & , \quad x > b \end{cases}$

$f'(x) = \begin{cases} 2x - a - b & , \quad x < a \\ -2x + a + b & , \quad a < x < b \\ 2x - a - b & , \quad x > b \end{cases}$

$\therefore \quad f'(a-) = a - b, \; f'(a+) = -a + b$

$\therefore \quad f'(a-) + f'(a+) = 0$

statement-2 explains statement–1.

**20. (D)**

**Statement-I :** $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & , \quad x \ne 0 \\ 0 & , \quad x = 0 \end{cases}$,

Since $\lim_{x\to 0} f(x) = 0$, therefore, $f(x)$ is continuous

$f'(0) = \lim_{x\to 0} \dfrac{x^2 \sin \frac{1}{x} - 0}{x} = 0$

$f'(x) = \begin{cases} 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x} & , \quad x \ne 0 \\ 0 & , \quad x = 0 \end{cases}$,

which is clearly not continuous at $x = 0$.

$\therefore$ statement is false

**Statement-II :** is true (standard result)

**21. (A) $\to$ (q), (B) $\to$ (p), (C) $\to$ (s), (D) $\to$ (p)**

**(A)** $f(x) = \begin{cases} 0 & , \quad 1 < x \le 2 \\ 1 - x & , \quad 0 \le x < 1 \\ -\sin \pi x & , \quad -1 \le x < 0 \end{cases}$

continuous at $x = 1$ but not differentiable

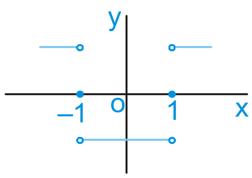
**(B)** $f'(0^-) = \lim_{h\to 0^-} \dfrac{h^2 e^{-1/h} - 0}{-h} = \lim_{h\to 0^-} (-he^{-1/h}) = 0$

**(C)** $g(x) = \dfrac{1}{1 + \frac{1}{x}(2 + 2x)} = \dfrac{x}{3x + 2}$

thus the points where $g(x)$ is not differentiable are

$x = 0, -1, -\dfrac{2}{3}$

**(D)** vertical tangents exist at $x = 1$ and $x = -1$ else where horizontal




tangents exist.

$\therefore$ number of points where tangent does not exist is 0

**22.** **(A) $\rightarrow$ (p, t, r), (B) $\rightarrow$ (p, r, s), (C) $\rightarrow$ (p, r, s), (D) $\rightarrow$ (p, r, s)**

**(A)** $f(x) = |x^3|$ is continuous and differentiable

**(B)** $f(x) = \sqrt{|x|}$ is continuous

$$f'(x) = \frac{1}{2\sqrt{|x|}} \cdot \frac{x}{|x|}$$ {does not exist at $x = 0$}

**(C)** $f(x) = |\sin^{-1} x|$ is continuous

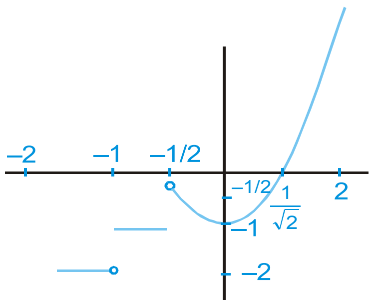
$$f'(x) = \frac{\sin^{-1} x}{|\sin^{-1} x|} \cdot \frac{1}{\sqrt{1-x^2}}$$ {does not exist at $x = 0$}

**(D)** $f(x) = \cos^{-1}|x|$ is continuous

$$f'(x) = \frac{-1}{\sqrt{1-x^2}} \cdot \frac{x}{|x|}$$ {does not exist at $x = 0$}

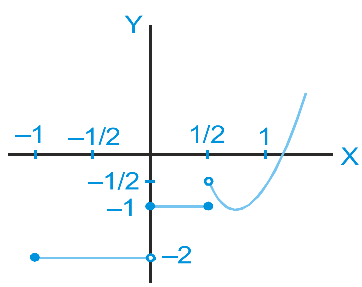
**23.**

**1. (B)**



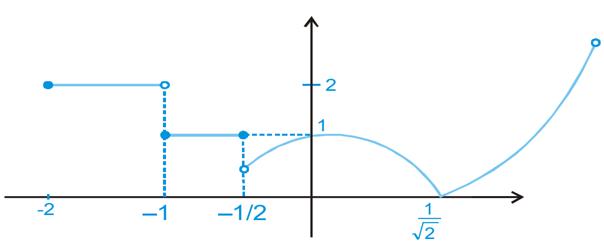

–1, –1/2 are two points of discontinuity

**2. (C)**




Discontinuous at 0, 1/2

**3. (C)**




at $-1, -1/2, 1/\sqrt{2}$ the function is not differentiable.

**24.**

**1. (C)**

$$\lim_{x \to 1^-} f(x) = a + b$$

$$\lim_{x \to 1^+} f(x) = 4b$$

for continuity $a + b = 4b$ i.e. $a = 3b$ .....**(i)**

$$f'(1^+) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0^+} \frac{2b(1+h) + 2b - (a+b)}{h}$$

$$= \lim_{h \to 0^+} \frac{3b - a + 2bh}{h} = 2b$$

$$f'(1^-) = \lim_{h \to 0^+} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0^+} \frac{a(1-h)^2 + b - a - b}{-h}$$

$$= \lim_{h \to 0^+} \frac{a(-2h + h^2)}{-h}$$

$$= 2a$$

$2a \neq 2b$, $a \neq b$

**2. (A)**

$$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} cx^2 + d = 4c + d$$

$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (dx + 3 - c) = 2d + 3 - c$$

$g(2) = 4c + d$

$\therefore$ $4c + d = 2d + 3 - c$

$\therefore$ $d = 5c - 3$

**3. (D)**

$$\lim_{x \to 3^-} f(x) = 8b, \quad \lim_{x \to 3^+} f(x) = 3(a-1) + 2a - 3 = 5a - 6$$

Since $f(x)$ is continuous at $x = 3$

$\therefore$ $8b = 5a - 6$ .........**(i)**

$$f'(3^-) = \lim_{h \to 0^+} \frac{f(3-h) - f(3)}{-h}$$

$$= \lim_{h \to 0^+} \frac{2b(3-h) + 2b - 8b}{-h} = 2b$$

$$f'(3^+) = \lim_{h \to 0^+} \frac{f(3+h) - f(3)}{h}$$

$= \lim_{h\to 0^+} \dfrac{(a-1)(3+h)+2a-3-8b}{h}$

Since f is differentiable at x = 3

$\therefore \lim_{h\to 0^+} (a-1)(3+h)+2a-3-8b = 0$

i.e. $5a - 8b - 6 = 0$

$\therefore f'(3^+) = a - 1$

thus $a - 1 = 2b$ ........**(ii)**

from (i) and (ii), we get $a = 2, b = \dfrac{1}{2}$

**25.**

**1. (A)**

$$\text{L.H.D.} = \lim_{h\to 0^-} \frac{f(-a+h)-f(-a)}{h}$$

$$= \lim_{h\to 0^-} \frac{-f(a-h)+f(a)}{h} = \lim_{h\to 0^-} \frac{f(a-h)-f(a)}{-h}$$

**2. (A)**

If f is even, then $f'(-x) = -f'(x)$

$$\therefore f''(a^+) = \lim_{h\to 0^-} \frac{f'(a-h)-f'(a)}{-h}$$

$$= \lim_{h\to 0^-} \frac{f'(a)-f'(a-h)}{h} = \lim_{h\to 0^-} \frac{f'(a)+f'(h-a)}{h}$$

**3. (B)**

$$\lim_{h\to 0} \frac{f(-x)-f(-x-h)}{h} = f'(-x) \text{ and}$$

$$\lim_{h\to 0} \frac{f(x)-f(x-h)}{-h} = -f'(x)$$

$\Rightarrow f'(-x) = -f'(x) \quad \therefore f'(x)$ is an odd function

$\therefore$ f is an even function

**26. (180)**

$$f(x) = \begin{cases} ax^3 + b, & 0 \le x \le 1 \\ 2\cos \pi x + \tan^{-1} x, & 1 < x \le 2 \end{cases}$$

as $\tan [x^2]\pi = \tan n\pi$, $n \in I$

$$f'(x) = \begin{cases} 3ax^2, & 0 < x < 1 \\ -2\pi \sin \pi x + \dfrac{1}{1+x^2}, & 1 < x < 2 \end{cases}$$

As the function is differentiable in [0, 2]

$\Rightarrow$ function is differentiable at x = 1

$\therefore f'(1^-) = f'(1^+)$

$\Rightarrow 3a = \dfrac{1}{2} \quad \Rightarrow a = \dfrac{1}{6}$

Function will also be continuous at x = 1

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$\Rightarrow a + b = -2 + \dfrac{\pi}{4}$

$\therefore b = -2 - \dfrac{1}{6} + \dfrac{\pi}{4} = \dfrac{\pi}{4} - \dfrac{13}{6}$

$\Rightarrow k_1 = 6 \quad \& \quad k_2 = 12$

$\Rightarrow k_1^2 + k_2^2 = 180$ **Ans.**

**27. (1)**

$$\lim_{x\to 0^+} \frac{|x|^p \sin\frac{1}{x} + x|\tan x|^q - 0}{x}$$

$$= \lim_{x\to 0^+} \left( x^{p-1} \sin\frac{1}{x} + |\tan x|^q \right) = 0$$

If $p - 1 > 0$ and $q > 0$ .....**(i)**

$$\lim_{x\to 0^-} \frac{|x|^p \sin\frac{1}{x} + x|\tan x|^q - 0}{x}$$

$$= \lim_{x\to 0^-} \left( (-1)^p x^{p-1} \sin\frac{1}{x} + |\tan x|^q \right) = 0$$

if $p - 1 > 0$ and $q > 0$ .....**(ii)**

$\therefore$ f(x) is differentiable if $p > 1$ and $q > 0$

i.e. $p + q > 1$

$\therefore$ least possible value of $[p + q] = 1$

**28.** Given $f(x + y^3) = f(x) + [f(y)]^3$ and $f'(0) \ge 0$

putting $x = y = 0$, we get

$f(0) = f(0) + (f(0))^3 \quad \Rightarrow f(0) = 0$

also $f'(0) = \text{limit}_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \text{limit}_{h\to 0} \dfrac{f(h)}{h}$

Let $L = f'(0) = \text{limit}_{h\to 0} \dfrac{f(0 + (h^{1/3})^3) - f(0)}{(h^{1/3})^3}$

$$= \lim_{h\to 0} \frac{\left(f(h^{1/3})\right)^3}{(h^{1/3})^3} = L^3$$

or $L = L^3$ or $L = 0, 1, -1$ as $f'(0) \geq 0$

$\Rightarrow$ $f'(0) = 0, 1$

Thus $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{f\left(x + (h^{1/3})^3\right) - f(x)}{(h^{1/3})^3}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x) + (f(h^{1/3}))^3 - f(x)}{(h^{1/3})^3}$$

$\Rightarrow$ $f'(x) = 0, 1$

Integrating both sides, we get

$f(x) = 0$ or $f(x) = x + c$

As $f(0) = 0$ , we have $f(x) = 0$ or $f(x) = x$

Now $f(x) = 0$ is imposible as $f(x)$ is not identically zero

$\therefore$ $f(x) = x$ and $f(10) = 10$

**29. (8)**

**(i)** $f(x) = \sin^{-1}(2x\sqrt{1-x^2})$

$$= \begin{cases} -\pi - 2\sin^{-1}x & , \quad -1 \leq x < -\frac{1}{\sqrt{2}} \\ 2\sin^{-1}x & , \quad -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & , \quad \frac{1}{\sqrt{2}} < x \leq 1 \end{cases}$$

$$f'(x) = \begin{cases} -\frac{2}{\sqrt{1-x^2}} & , \quad -1 < x < -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{1-x^2}} & , \quad -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \\ \frac{-2}{\sqrt{1-x^2}} & , \quad \frac{1}{\sqrt{2}} < x < 1 \end{cases}$$

$$f'(1/2) = \frac{4}{\sqrt{3}}, f'(-1/2) = \frac{4}{\sqrt{3}}$$

**30. (3)**

$f(x+y+1) = \left(\sqrt{f(x)} + \sqrt{f(y)}\right)^2$

$f(1) = 4$

$$\frac{\partial}{\partial x} f'(x+y+1) = 2\left(\sqrt{f(x)} + \sqrt{f(y)}\right) \cdot \frac{f'(x)}{2\sqrt{f(x)}}$$

$$\frac{\partial}{\partial y} f'(x+y+1) = \left(\sqrt{f(x)} + \sqrt{f(y)}\right) \cdot \frac{f'(y)}{\sqrt{f(y)}}$$

$$\frac{f'(x)}{\sqrt{f(x)}} = \frac{f'(y)}{\sqrt{f(y)}} = k$$

$$\therefore \int \frac{f'(x)}{\sqrt{f(x)}} dx = \int k\, dx$$

$\Rightarrow$ $2\sqrt{f(x)} = kx + c$

put $x = 0$

$\Rightarrow$ $4 = k + 2$ $\Rightarrow$ $k = 2$

$\therefore$ $2\sqrt{f(x)} = 2(x+1)$

$\Rightarrow$ $f(x) = (x+1)^2$