DIFFERENTIABILITY

MEANING OF DERIVATIVE

The instantaneous rate of change of a function with respect to the dependent variable is called derivative. Let 'f' be a given function of one variable and let Δx denote a number (positive or negative) to be added to the number x. Let Δf denote the corresponding change of 'f' then $\Delta f = f(x + \Delta x) - f(x)$

$$\Rightarrow \quad \frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If $\Delta f/\Delta x$ approaches a limit as Δx approaches zero, this limit is the derivative of 'f' at the point x. The derivative of a function 'f' is a function ; this function is denoted by symbols such as

$$f'(x), \frac{df}{dx}, \frac{d}{dx}f(x) \text{ or } \frac{df(x)}{dx}$$

$$\Rightarrow \quad \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The derivative evaluated at a point a, is written, $f'(a), \left.\frac{df(x)}{dx}\right|_{x=a}, f'(x)_{x=a}$, etc.

EXISTENCE OF DERIVATIVE

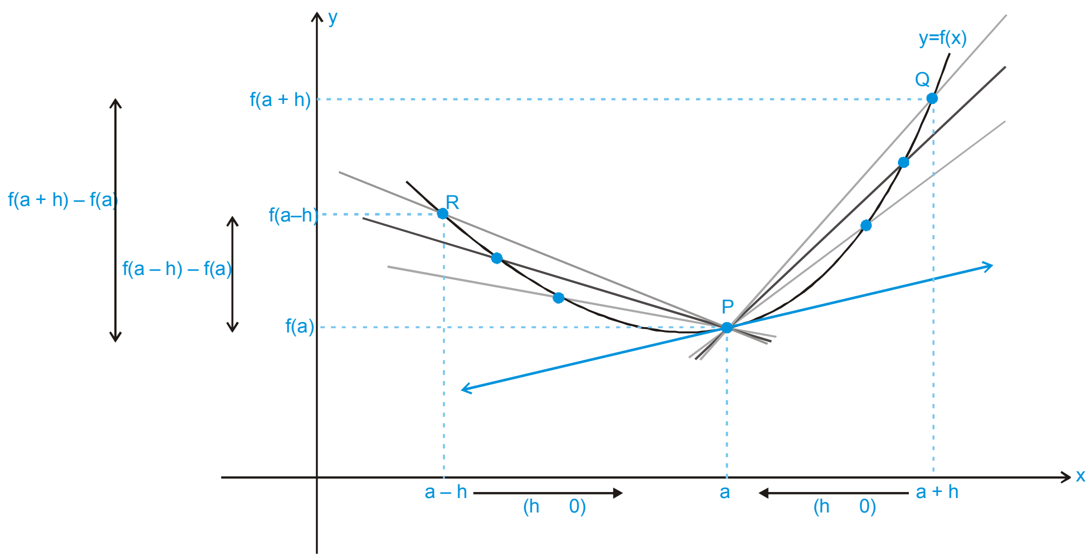
1. Right hand & Left hand Derivatives ;

By definition : $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ if it exist

(i) The right hand derivative of f at x = a

denoted by $f'(a^+)$ is defined by :

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h},$$



provided the limit exists & is finite .

(ii) The left hand derivative : of f at x = a
denoted by $f'(a^-)$ is defined by :

$$f'(a^-) = \lim_{h\to 0^+} \frac{f(a-h)-f(a)}{-h},$$

Provided the limit exists & is finite.
f is said to be derivable at x = a if $f'(a^+) = f'(a^-)$ = a finite quantity.
This geomtrically means that a unique tangent with finite slope can be drawn at x = a as shown in the figure.

Ex. Comment on the differentiability of $f(x) = \begin{cases} x & , \quad x<1 \\ x^2 & , \quad x \ge 1 \end{cases}$ at x = 1.

Sol. R.H.D. = $f'(1^+)$

$$= \text{Limit}_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$$

$$= \text{Limit}_{h\to 0^+} \frac{(1+h)^2-1}{h} = \text{Limit}_{h\to 0^+} \frac{1+h^2+2h-1}{h} = \text{Limit}_{h\to 0^+}(h+2) = 2$$

L.H.D. $= f'(1^-) = \text{Limit}_{h\to 0^+} \frac{f(1-h)-f(1)}{-h} = \text{Limit}_{h\to 0^+} \frac{1-h-1}{-h} = 1$

As L.H.D. ≠ R.H.D. Hence f(x) is not differentiable at x = 1.

Ex. If $f(x) = \begin{cases} A+Bx^2 & , \quad x<1 \\ 3Ax-b+2 & , \quad x \ge 1 \end{cases}$, then find A and B so that f(x) become differentiable at x = 1.

Sol. $f'(1^+) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h}$

$$= \lim_{h\to 0^+} \frac{3A(1+h)-B+2-3A+B-2}{h} = \lim_{h\to 0^+} \frac{3Ah}{h} = 3A$$

$$f'(1^-) = \lim_{h\to 0^+} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0^+} \frac{A+B(1-h)^2-3A+B-2}{-h}$$

$$= \lim_{h\to 0^+} \frac{(-2A+2B-2)+Bh^2-2Bh}{-h}$$

Hence for this limit to be defined

$-2A + 2B - 2 = 0$

$B = A + 1$

$\therefore$ $f'(1^-) = \lim_{h\to 0} -(Bh-2B) = 2B$

For f(x) to be differentiable at x = 1

$\therefore$ $f'(1^-) = f'(1^+)$

$\Rightarrow$ $3A = 2B = 2(A+1)$ $\qquad \because B = A + 1$

$A = 2, B = 3$

Ex. $f(x) = \begin{cases} [\cos \pi x] & x \le 1 \\ 2\{x\}-1 & x>1 \end{cases}$ comment on the derivability at x =1, where [ ] denotes greatest integer function & { } denotes fractional part function.

Sol. $f'(1^-) = \lim_{h\to 0}\dfrac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\dfrac{[\cos(\pi-\pi h)]+1}{-h} = \lim_{h\to 0}\dfrac{-1+1}{-h} = 0$

$f'(1^+) = \lim_{h\to 0}\dfrac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\dfrac{2\{1+h\}-1+1}{h} = \lim_{h\to 0}\dfrac{2h}{h} = 2$

Hence f(x) is not differentiable at x = 1.

RELATION BETWEEN DIFFERENTIABILITY & CONTINUITY

Theorem : If a function f(x) is derivable at x = a, then f(x) is continuous at x = a.

Proof: $f'(a) = \lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ exists.

Also $f(a+h)-f(a) = \dfrac{f(a+h)-f(a)}{h}.h \quad [h \neq 0]$

$\therefore \quad \lim_{h\to 0}[f(a+h)-f(a)] = \lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}.h = f'(a).0 = 0$

$\Rightarrow \quad \lim_{h\to 0}[f(a+h)-f(a)] = 0 \Rightarrow \lim_{h\to 0} f(a+h) = f(a) \Rightarrow f(x)$ is continuous at x = a.

(i) Differentiable $\Rightarrow$ Continuous ; Continuity $\not\Rightarrow$ *Differentiable* ; Not Differentiable $\not\Rightarrow$ Not Continuous But Not Continuous $\Rightarrow$ Not Differentiable

(ii) All polynomial, trigonometric, logarithmic and exponential function are continuous and differentiable in their domains.

(iii) If f(x) & g(x) are differentiable at x = a then the functions f(x) + g(x), f(x) – g(x), f(x). g(x) will also be differentiable at x =a & if g(a) ≠ 0 then the function f(x)/g(x) will also be differentiable at x = a.

If f(x) is a function such that R.H.D = $f'(a^+) = \ell$ and L.H.D. = $f'(a^-) = m$. Then

Case - I

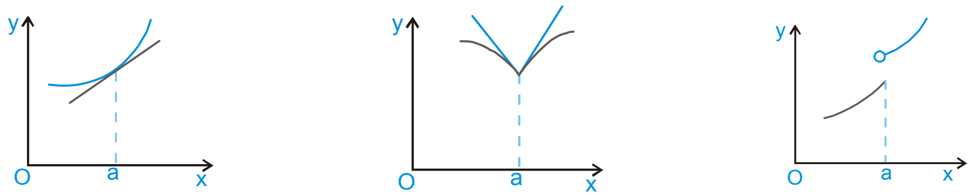
If ℓ = m = some finite value, then the function f(x) is differentiable as well as continuous.

Case - II

if $\ell \neq$ m = but both have some finite value, then the function f(x) is non differentiable but it is continuous.

Case - III

If at least one of the ℓ or m is infinite, then the function is non differentiable but we can not say about continuity of f(x).



(i) continuous and differentiable

(ii) continuous but not differentiable

(iii) neither continuous nor differentiable

Ex. If $f(x) = \begin{cases} x^2 \text{sgn}[x] + \{x\}, & 0 \le x < 2 \\ \sin x + |x-3|, & 2 \le x < 4 \end{cases}$, comment on the continuity and differentiability of f(x), where [.] is greatest integer function and {.} is fractional part function, at x = 1, 2.

Sol. Continuity at x = 1

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2 \text{sgn}[x] + \{x\}) = 1 + 0 = 1$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2 \text{sgn}[x] + \{x\})$

$= 1 \text{ sgn}(0) + 1 = 1$

$\because$ f(1) = 1

$\therefore$ L.H.L = R.H.L = f(1). Hence f(x) is continuous at x = 1.

Now for differentiability,

$$\text{R.H.D.} = f'(1^+) = \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0^+} \frac{(1+h)^2 \text{sgn}[1+h] + \{1+h\} - 1}{h}$$

$$= \lim_{h\to 0^+} \frac{(1+h)^2 + h - 1}{h} = \lim_{h\to 0^+} \frac{1 + h^2 + 2h + h - 1}{h} = \lim_{h\to 0^+} \frac{h^2 + 3h}{h} = 3$$

and $\text{L.H.D.} = f'(1^-) = \lim_{h\to 0^+} \frac{f(1-h) - f(1)}{-h}$

$$= \lim_{h\to 0^+} \frac{(1-h)^2 \text{sgn}[1-h] + 1 - h - 1}{-h} = 1$$

$\Rightarrow$ $f'(1^+) \ne f'(1^-)$.

Hence f(x) is non differentiable at x = 1.

Now at x = 2

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 \text{sgn}[x] + \{x\}) = 4 . 1 + 1 = 5$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (\sin x + |x - 3|) = 1 + \sin 2$

Hence L.H.L $\ne$ R.H.L

Hence f(x) is discontinuous at x = 2 and then f(x) also be non differentaible at x = 2.

Ex. Let $f(x) = \begin{cases} \frac{1}{|x|}, & |x| \ge 1 \\ ax^2 + b, & |x| < 1 \end{cases}$ be continuous and differentiable everywhere. Then find a and b.

Sol. $f(x) = \begin{cases} -\frac{1}{x}, & x \le -1 \\ ax^2 + b, & -1 < x < 1 \\ \frac{1}{x}, & x \ge 1 \end{cases}$. Since function is continuous everywhere

$\therefore$ LHL = RHL at x = –1

$$\text{LHL} = \lim_{h\to 0} f(-1-h) = \lim_{h\to 0} \frac{-1}{(-1-h)} = 1$$

RHL $= \lim_{h\to 0} f(-1+h) = \lim_{h\to 0} a(-1+h)^2 + b = a+b$

$\Rightarrow \quad a+b=1$ .....(i)

Again, function is differentiable at everywhere.

$\therefore \quad$ LHD = RHD at $x=-1$

LHD $= \lim_{h\to 0} \dfrac{f(-1-h)-f(-1)}{-h} = \lim_{h\to 0} \dfrac{\dfrac{-1}{-1-h} - \dfrac{1}{1}}{-h} = 1$

RHD $= \lim_{h\to 0} \dfrac{f(-1+h)-f(-1)}{+h} = \lim_{h\to 0} \dfrac{a\left(-1+h\right)^2 + b - 1}{h} = \lim_{h\to 0} \dfrac{a(1+h^2-2h)+b-1}{h}$

$= \lim_{h\to 0} \dfrac{a+b-1+ah^2-2ah}{h} = -2a$ $\quad$ [$\because$ a + b = 1 from (i)]

$\Rightarrow \quad -2a = 1$ ......(ii)

$\Rightarrow \quad a = \dfrac{-1}{2}$ & $b = \dfrac{3}{2}$ (using (i) & (ii))

How can a function fail to be differentiable ?

The function f(x) is said to be non-differentiable at x = a if

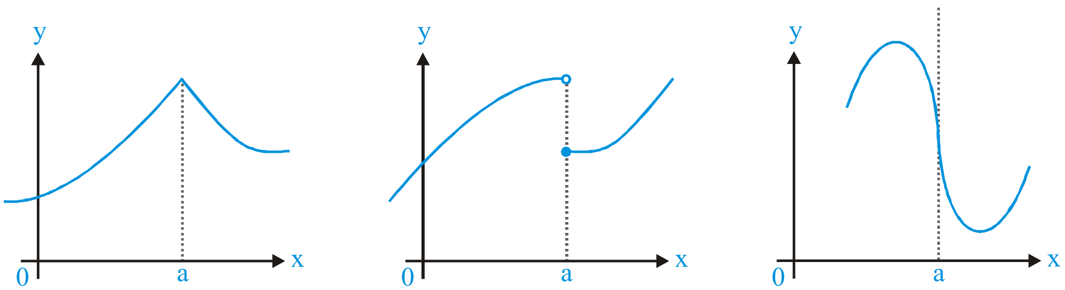
1. Both RHD and LHD exist but are not equal,
2. Either or both RHD and LHD are not finite, and
3. either or both RHD and LHD do not exist.

The function y = |x| is not differentiable at 0 as its graph changes direction abruptly when x = 0. In general, if the graph of a function has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. [To compute f '(a), we find that the left and right derivatives are different.]

If f is not continuous at a, then f is not differentiable at a. So, at any discontinuity (for instance, a jump of discontinuity), f fails to be differentiable.

A third possibility is that the curve has a vertical tangent line when x = a, that is, f is continuous at a and ∞.

This means that the tangent lines become steeper and steeper as $x \to a$. The following figures illustrate the three possiblilities that we have discussed.



A conrner (a) $\qquad$ A discontinuity (b) $\qquad$ A vertical tangent (c)

Differentiability of sum, product & composition of functions

(i) If f(x) & g(x) are differentiable at x = a, then the functions $f(x) \pm g(x)$, f(x). g(x) will also be differentiable at x = a & if $g(a) \neq 0$, then the function f(x)/g(x) will also be differentiable at x = a.

(ii) If f(x) is not differentiable at x = a & g(x) is differentiable at x = a, then the product function F(x) = f(x) . g(x) can still be differentiable at x = a

e.g. $f(x) = |x|$ and $g(x) = x^2$.

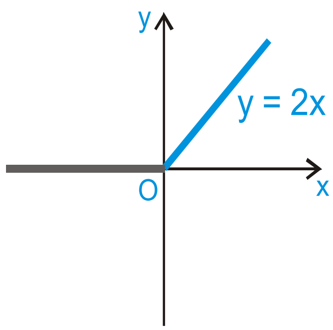
(iii) If f(x) & g(x) both are not differentiable at x = a, then the product function F(x) = f(x) . g(x) can still be differentiable at x = a e.g. $f(x) = |x|$ & $g(x) = |x|$.

(iv) If f(x) & g(x) both are non-differentiable at x = a, then the sum function F(x) = f(x) + g(x) may be a differentiable function. e.g. $f(x) = |x|$ & $g(x) = -|x|$.

(v) If f is differentiable at x = a, then $\lim_{h\to 0} \frac{f(a+g(h))-f(a+p(h))}{g(h)-p(h)} = f'(a)$, where $\lim_{h\to 0} p(h) = \lim_{h\to 0} g(h) = 0$

Ex. Discuss the differentiability of f(x) = x + |x|.

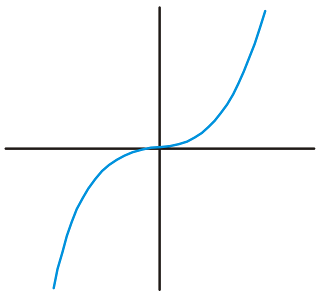
Sol.



Non-differentiable at x = 0.

Ex. Discuss the differentiability of f(x) = x|x|

Sol. $\because$ $f(x) = \begin{cases} x^2 & , \ x \geq 0 \\ -x^2 & , \ x < 0 \end{cases}$



Differentiable at x = 0

DIFFERENTIABILITY OVER AN INTERVAL

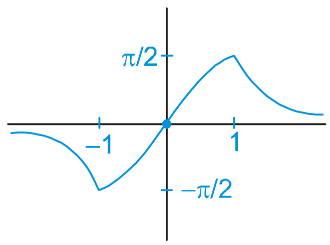
f (x) is said to be differentiable over an open interval if it is differentiable at each point of the interval and f(x) is said to be differentiable over a closed interval [a, b] if:

(i) For the points a and b, $f'(a^+)$ and $f'(b^-)$ exist finitely

(ii) For any point c such that a < c < b, $f'(c^+)$ & $f'(c^-)$ exist finitely and are equal.

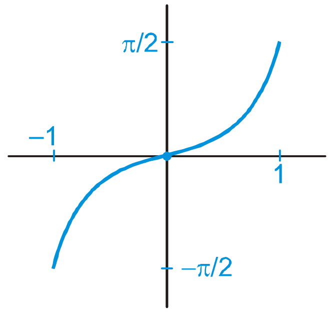
All polynomial, exponential, logarithmic and trigonometric (inverse trigonometric not included) functions are differentiable in their domain.

Graph of $y = \sin^{-1}\frac{2x}{1+x^2}$



Non differentiable at x = 1 & x = – 1

Graph of $y = \sin^{-1} x$.



Non differentiable at x = 1 & x = – 1

Derivability should be checked at following points

(i) At all points where continuity is required to be checked.

(ii) At the critical points of modulus and inverse trigonometric function.

Ex. $f(x) = \begin{cases} \left\{x + \frac{1}{3}\right\}[\sin \pi x] & , \quad 0 \le x < 1 \\ [2x]\,\text{sgn}\left(x - \frac{4}{3}\right) & , \quad 1 \le x \le 2 \end{cases}$; find that points at which continuity and differentiability should be checked.

Also check the continuity and differentiability of f(x) at x = 1, where [ ] denotes greatest integer function & { } denotes fractional part function.

Sol. $f(x) = \begin{cases} \left\{x + \frac{1}{3}\right\}[\sin x\pi]; \ 0 \le x < 1 \\ [2x]\,\text{sgn}\left(x - \frac{4}{3}\right); 1 \le x \le 2 \end{cases}$

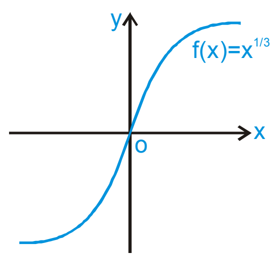
$$f(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ \frac{5}{6}, & x = \frac{1}{2} \\ 0, & \frac{1}{2} < x < 1 \\ -2, & 1 \le x < \frac{4}{3} \\ 0, & x = \frac{4}{3} \\ 2, & \frac{4}{3} < x < \frac{3}{2} \\ 3, & \frac{3}{2} \le x < 2 \\ 4, & x = 2 \end{cases}$$

Hence function is discontinuous & non-derivable at $x = \frac{1}{2}, 1, \frac{4}{3}, \frac{3}{2}$ & 2

CONCEPT OF TANGENT AND ITS ASSOCIATION WITH DERIVABILITY

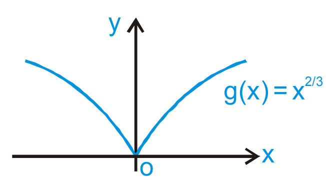
Vertical Tangent

(i) If for $y = f(x)$; $f'(a^+) \to \infty$ and $f'(a^-) \to \infty$ or $f'(a^+) \to -\infty$ and $f'(a^-) \to -\infty$ then at $x = a$, $y = f(x)$ has vertical tangent at $x = a$



E.g. **(1)** $f(x) = x^{1/3}$ has vertical tangent at $x = 0$

since $f'(0^+) \to \infty$ and $f'(0^-) \to \infty$ hence f(x) is not differentiable at $x = 0$



(2) $g(x) = x^{2/3}$ does'nt have vertical tangent at $x = 0$

since $g'(0^+) \to \infty$ and $g'(0^-) \to -\infty$ hence g(x) is not differentiable at $x = 0$.

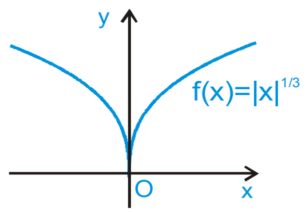
(ii) If a function has vertical tangent at $x = a$ then it is non differentiable at $x = a$.

GEOMETRICAL INTERPRETATION OF DIFFERENTIABILITY

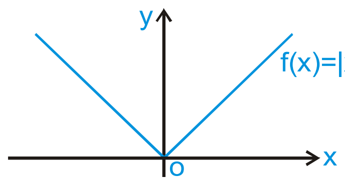
(i) If the function $y = f(x)$ is differentiable at $x = a$, then a unique tangent can be drawn to the curve $y = f(x)$ at the point $P(a, f(a))$ & $f'(a)$ represent the slope of the tangent at point P.

(ii) If a function f(x) does not have a unique tangent ($p \ne q$) but is continuous at $x = a$, it geometrically implies a sharp corner at $x = a$. Note that p and q may not be finite, where $p = f'(a^+)$ and $q = f'(a^-)$

e.g. (1) $f(x) = |x|$ and $|x|^{1/3}$ is continuous but not differentiable at $x = 0$ & there is sharp corner at $x = 0$.

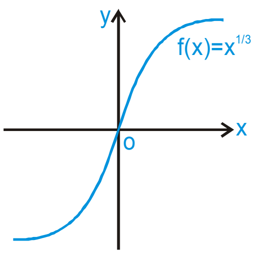


(does not have unique tangent) $p \to \infty$, $q \to -\infty$



(does not have unique tangent) $p = 1$, $q = -1$

(2) $f(x) = x^{1/3}$ is continuous but not differentiable at $x = 0$ because $f'(0^+) \to \infty$ and $f'(0^-) \to \infty$.



(have a unique tangent but does not have sharp corner) $p \to +\infty$, $q \to +\infty$

> non differentiable $\not\Rightarrow$ sharp corner

Ex. Find the equation of tangent to $y = (x)^{1/3}$ at $x = 1$ and $x = 0$.

Sol. At $x = 1$ Here $f(x) = (x)^{1/3}$

$$\text{L.H.D} = f'(1^-) = \lim_{h \to 0^+} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0^+} \frac{(1-h)^{1/3} - 1}{-h} = \frac{1}{3}$$

$$\text{R.H.D.} = f'(1^+) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h)^{1/3} - 1}{h} = \frac{1}{3}$$

As R.H.D. = L.H.D. = $\frac{1}{3}$

$\therefore$ slope of tangent $= \frac{1}{3}$ $\qquad \therefore$ $y - f(1) = \frac{1}{3}(x - 1)$

$$y - 1 = \frac{1}{3}(x - 1)$$

$\Rightarrow$ $3y - x = 2$ is tangent to $y = x^{1/3}$ at $(1, 1)$

At $x = 0$

$$\text{L.H.D.} = f'(0^-) = \lim_{h \to 0^+} \frac{(0-h)^{1/3} - 0}{-h} = +\infty$$

$$\text{R.H.D.} = f'(0^+) = \lim_{h \to 0^+} \frac{(0+h)^{1/3} - 0}{h} = +\infty$$

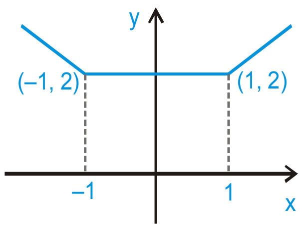
As L.H.D. and R.H.D are infinite. $y = f(x)$ will have a vertical tangent at origin.

$\therefore$ $x = 0$ is the tangent to $y = x^{1/3}$ at origin.

Ex. Let $f(x) = \max\{(1+x), (1-x), 2\}$. Find the number of points where it is not differentiable.

Sol.

$$f(x) = \begin{cases} 1 - x; & x < -1 \\ 2; & -1 \le x \le 1 \\ 1 + x; & x > 1 \end{cases}$$



at $x = -1$

$$q = \text{LHD} = f'(-1^-) = \lim_{h\to 0} \frac{f(-1-h) - f(-1)}{-h} = \lim_{h\to 0} \frac{1-(-1-h)-2}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1$$

$$p = \text{RHD} = f'(-1^+) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{2-2}{h} = 0$$

$\because$ $q \ne p$

$\therefore$ not differentiable but continuous at $x = -1$ and having sharp corner.

Now, at $x = 1$

$$q = \text{LHD} = f'(1^-) = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h\to 0} \frac{2-2}{-h} = 0$$

$$p = \text{RHD} = f'(1^+) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{1+(1+h)-2}{h} = 1$$

$\because$ $q \ne p$

$\therefore$ not differentiable but continuous at $x = 1$ and having sharp corner.

$\Rightarrow$ $f(x)$ is not differentiable at $x = \pm 1$.

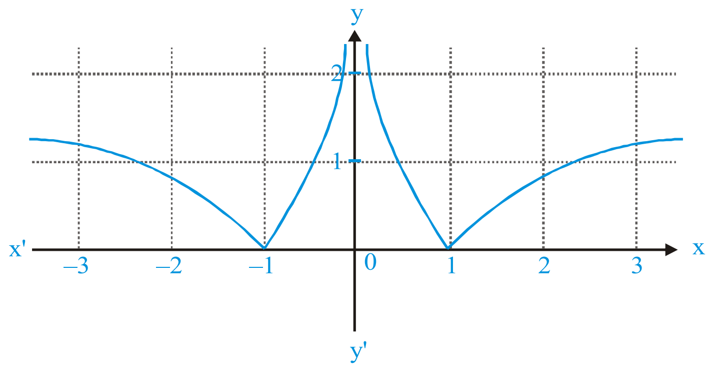
DIFFERENTIABILITY USING GRAPHS

Ex. Discuss the differentiability of

(A) $f(x) = |\log_e |x||$ **(B)** $f(x) = \max\{\sec^{-1} x, \operatorname{cosec}^{-1} x\}$

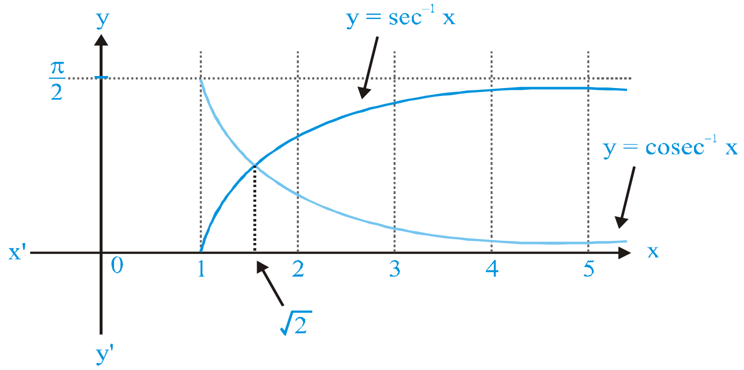
(C) $f(x) = \max\{x^2 - 3x + 2, 2 - |x - 1|\}$

Sol. **(A)** $f(x) = |\log_e |x||$



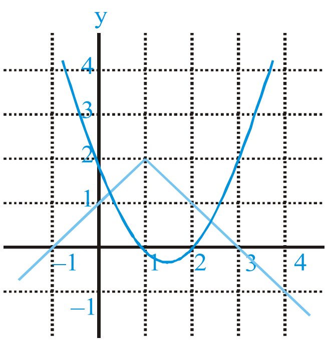
Clearly, from the graph, f(x) is non-differentiable at $x = 0, \pm 1$.

(B) $f(x) = \max\{\sec^{-1} x, \operatorname{cosec}^{-1} x\}$



Clearly, from the graph, f(x) is non-differentiable at $x = \sqrt{2}$.

(C) From the garph, f(x) is non-differentiable



(i) at $x = 1$

(ii) where $x^2 - 3x + 2 = 2 - (1 - x)$, when $x < 1$ **(iii)** where $x^2 - 3x + 2 = 2 - (x - 1)$, where $x > 1$

Hence, f(x) is discontinuous at $x = 1$, $x = 2 - \sqrt{3}$, and $x = 1 + \sqrt{2}$.

DIFFERENTIABILITY BY DIFFERENTIATION

Ex. If $f(x) = \begin{cases} x, & x \le 1 \\ x^2 + bx + c, & x > 1 \end{cases}$, then find the values of b and c if f(x) is differentiable at x = 1.

Sol. $f(x) = \begin{cases} x, & x \le 1 \\ x^2 + bx + c, & x > 1 \end{cases}$

$\therefore$ $f'(x) \begin{cases} 1, & x < 1 \\ 2x + b, & x > 1 \end{cases}$

f(x) is differentiable at x = 1.

Then, it must be continuous at x = 1 for which $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$

or $1 + b + c = 1$ or $b + c = 0$

Also, $f'(1^+) = f'(1^-)$

or $\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^-} f'(x)$

or $2 + b = 1$ or $b = -1$

$\therefore$ $c = 1$

Ex. $f(x) = \begin{cases} ax(x-1) + b, & x < 1 \\ x - 1, & 1 \le x \le 3 \\ px^2 + qx + 2, & x > 3 \end{cases}$.

Find the values of the constants, a, b, p and q so that all the following conditions are satisfied.

(A) f(x) is continuous for all x. **(B)** f'(1) does not exist. **(C)** f'(x) is continuous at x = 3.

Sol. f(x) is continuous $\forall\ x \in R$

Hence, it must be continuous at x = 1, 3

$f(1^-) = \lim_{x \to 1} ax(x-1) + b = b$

$f(1^+) = \lim_{x \to 1} (x-1) = 0$

Now, $f(1^-) = f(1^+)$ (for continuity at x = 1)

or $b = 0$

$f(3^-) = \lim_{x \to 3} (x-1) = 2$

$f(3^+) = \lim_{x \to 3} (px^2 + qx + 2) = 9p + 3q + 2$

Now, $f(3^-) = f(3^+)$ (for continuity at x = 3)

or $9p + 3q = 0$ ..... **(i)**

$f'(x) = \begin{cases} 2ax - a, & x < 1 \\ 1, & 1 < x < 3 \\ 2px + q, & x > 3 \end{cases}$

Now, givne that f'(1) does not exist. Therefore,

$f'(1^+) \ne f'(1^-)$

or $1 \ne 2a - a$ or $a \ne 1$

Also, given that f'(3) exists. Therefore,

$f'(3^-) = f'(3^+)$

or $1 = 6p + q$ ..... **(ii)**

Solving **(i)** and **(ii)** for p and q, we get p = 1/3, q = –1.

Determination of Function which are Differentiable and satisfying the given Functional Rule

STEPS

1. Write down the expression for f ' (x) as $f'(x) = \dfrac{f(x+h)-f(x)}{h}$

2. Manipulate f (x + h) – f (x) in such a way that the given functional rule is applicable. Now apply the functional rule and simplify the RHS to get f ' (x) as a function of x along with constants if any.

3. Integrate f ' (x) to get f (x) as a function of x and a constant of integration. In some cases a Differential Equation in formed which can be solved to get f (x).

4. Apply the boundary value conditions to determine the value of this constant.

Ex. Let f be a differentiable function satisfying $f\left(\dfrac{x}{y}\right) = f(x) - f(y)\ \forall\ x, y > 0$. If f'(1) = 1. Find f(x).

Sol. Put x = y = 1 in given rule $\Rightarrow$ f(1) = f(1) – f(1) = 0

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)}{h} \qquad \{\text{from given functional rule}\}$$

$$\therefore \quad f'(x) = \lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{\frac{h}{x}}\times\frac{1}{x} = \frac{f'(1)}{x}$$

$$\therefore \quad f'(x) = \frac{1}{x} \qquad \{\because f'(1) = 1\}$$

Integrating both sides $\Rightarrow$ $f(x) = \ell nx + c$

putting x = 1 we get c = 0 $\Rightarrow$ $f(x) = \ell nx$

Ex. $f\left(\dfrac{x+y}{2}\right) = \dfrac{f(x)+f(y)}{2}\ \forall\ x, y \in R$ and f(0) = 1 and f'(0) = – 1 and function is differentiable for all x, then find f(x).

Sol.
$$f'(x) = \lim_{h\to 0}\frac{f\left(\frac{2x+2h}{2}\right)-f\left(\frac{2x+0}{2}\right)}{h} = \lim_{h\to 0}\frac{\frac{f(2x)+f(2h)}{2}-\frac{f(2x)+f(0)}{2}}{h}$$

$$= \lim_{h\to 0}\frac{f(2h)-f(0)}{2h} = f'(0) = -1$$

$f'(x) = -1$

integrating both sides, we get $f(x) = -x + c$

$\because$ c = + 1 (as f(0) = 1)

$\therefore$ f(x) = – x + 1 = 1 – x

TIPS & FORMULAS

1. Introduction

The derivative of a function 'f' is a function ; this function is denoted by symbols such as

$$f'(x),\ \frac{df}{dx},\ \frac{d}{dx}f(x) \quad \text{or} \quad \frac{df(x)}{dx}$$

The derivative evaluated at a point a, is written, $f'(a),\ \left.\frac{df(x)}{dx}\right|_{x=a},\ f'(x)_{x=a},$ etc.

2. Right Hand & Left Hand Derivatives

(A) Right Hand Derivative :

The right hand derivative of f(x) at x = a denoted by $f'(a^+)$ is defined as :

$$f'(a^+) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$, provided the limit exists & is finite. $(h > 0)$

(B) Left Hand Derivative :

The left hand derivative of f(x) at x = a denoted by $f'(a^-)$ is defined as :

$$f'(a^-) = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$$, provided the limit exists & is finite. $(h > 0)$

(C) Derivability of Function at a Point :

If $f(a^+) = f(a^-)$ = finite quantity, then f(x) is said to be derivable or differentiable at x = a. In such case $f(a^+) = f(a^-) = f(a)$ and it is called derivative or differential coefficient of f(x) at x = a.

Note

(i) All polynomial, trigonometric, logarithmic and exponential function are continuous and differentiable in their domains, except at end points.

(ii) If f(x) & g(x) are differentiable at x = a then the functions f(x) + g(x), f(x) – g(x), f(x). g(x) will also be derivable at x = a & if $g(a) \neq 0$ then the function f(x)/g(x) will also be derivable at x = a.

3. Important Note

(A) Let $f'(a^+) = p$ & $f'(a^-) = q$ where p & q are finite then :

(i) $p = q \Rightarrow$ f is derivable at x = a $\Rightarrow$ f is continuous at x = a

(ii) $p \neq q \Rightarrow$ f is not derivable at x = a, but f is continuous at x = a.

It is very important to note that 'f' may still continuous at x = a

In short, for a function 'f'.

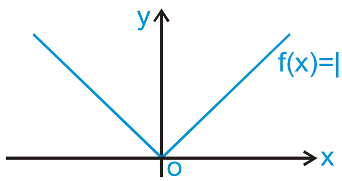
Differentiable	$\Rightarrow$	**Continuous**
Not Differentiable	$\not\Rightarrow$	**Not Continuous**
But Not Continuous	$\Rightarrow$	**Not Differentiable**
Continuous	$\Rightarrow$	**May or may not be Differentiable**

(B) Geometrical Interpretation of Differentiability :

(i) If the function $y = f(x)$ is differentiable at $x = a$, then a unique tangent can be drawn to the curve $y = f(x)$ at the point $P(a, f(a))$ & $f'(a)$ represent the slope of the tangent at point P.

(ii) If LHD and RHD are finite but unequal then it geometrically implies a sharp corner at $x = 0$.

e.g. $f(x) = |x|$ is continuous but not differentiable at $x = 0$.



A sharp corner is seen at $x = 0$. A sharp corner is seen at $x = 0$ in the graph of $f(x) = |x|$.

(iii) If a function has vertical tangent at $x = a$, then also it is nonderivable at $x = a$.

(C) Vertical tangent :

If for $y = f(x)$,

$f'(a^+) \to \infty$ and $f'(a^-) \to \infty$ or $f'(a^+) \to -\infty$ and $f'(a^-) \to -\infty$

then at $x = a$, $y = f(x)$ has vertical tangent $f(x)$ is not differentiable at $x = a$

4. Differentiability over an Interval

(A) $f(x)$ is said to be derivable over an open interval (a, b) if it is derivable at each & every point of the open interval (a, b).

(B) $f(x)$ is said to be derivable over the closed interval $[a, b]$ if :

(i) $f(x)$ is derivable in (a, b) &

(ii) for the points a and b, $f'(a^+)$ & $f'(b^-)$ exist.

Note

(i) If $f(x)$ is differentiable at $x = a$ & $g(x)$ is not differentiable at $x = a$, then the product function $F(x) = f(x).g(x)$ can still be differentiable at $x = a$.

(ii) If $f(x)$ & $g(x)$ both are not differentiable at $x = a$ then the product function; $F(x) = f(x).g(x)$ can still be differentiable at $x = a$.

(iii) If $f(x)$ & $g(x)$ both are non-derivable at $x = a$ then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function.

(iv) If $f(x)$ is derivable at $x = a$ $\not\Rightarrow$ $f'(x)$ is continuous at $x = a$.