

## MATHS FOR JEE MAIN &amp; ADVANCED

## HINTS &amp; SOLUTIONS

## EXERCISE - 1

## Single Choice

$$3. \lim_{x \rightarrow \infty} \sqrt{x+1} - \sqrt{x} = 0 \Rightarrow \cot^{-1}(0) = \pi/2$$

$$\lim_{x \rightarrow \infty} \left( \frac{2x+1}{x-1} \right)^x \rightarrow \infty \Rightarrow \sec^{-1}(\infty) = \pi/2$$

$$\therefore l = 1$$

$$4. \lim_{n \rightarrow \infty} \left( \frac{1}{1.3} + \frac{1}{3.5} + \dots + \text{to } n \text{ terms} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2} \left( \frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \dots + \frac{1}{2n-1} - \frac{1}{2n+1} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2} \left( 1 - \frac{1}{2n+1} \right) = \frac{1}{2}$$

$$6. \text{ Let } \lim_{x \rightarrow a} f(x) = L \text{ \& } \lim_{x \rightarrow a} g(x) = M$$

$$\therefore L + M = 2 \text{ \& } L - M = 1$$

$$\Rightarrow L = \frac{3}{2} \text{ \& } M = \frac{1}{2}$$

$$\text{So } \lim_{x \rightarrow a} f(x)g(x) = L.M = \frac{3}{4}$$

$$7. \lim_{x \rightarrow \alpha} \frac{1 - \cos(ax^2 + bx + c)}{(x - \alpha)^2}$$

$$= \lim_{x \rightarrow \alpha} \left( \frac{1 - \cos a(x - \alpha)(x - \beta)}{(x - \alpha)^2 (x - \beta)^2 a^2} \right) (x - \beta)^2 a^2$$

$$= \frac{1}{2} a^2 (\alpha - \beta)^2$$

$$8. \lim_{x \rightarrow 0} \frac{\sin 3x}{x^3} + \frac{a}{x^2} + b = \lim_{x \rightarrow 0} \frac{\sin 3x + ax + bx^3}{x^3}$$

$$= \lim_{x \rightarrow 0} \frac{3 \frac{\sin 3x}{3x} + a + bx^2}{x^2}$$

$$\text{for existence of limit } 3 + a = 0 \Rightarrow a = -3$$

$$\therefore l = \lim_{x \rightarrow 0} \frac{\sin 3x - 3x + bx^3}{x^3} =$$

$$9. \lim_{x \rightarrow 0} \frac{2 \sin^2 \left( \frac{ax^2 + bx + c}{2} \right) \left( \frac{ax^2 + bx + c}{2} \right)^2}{\left( \frac{ax^2 + bx + c}{2} \right)^2 4(x - \alpha)^2}$$

$$\lim_{x \rightarrow \alpha} \frac{2a^2 \left( x^2 + \frac{b}{a}x + \frac{c}{a} \right) \left( x^2 + \frac{b}{a}x + \frac{c}{a} \right)}{4(x - \alpha) (x - \alpha)}$$

$$\lim_{x \rightarrow \alpha} \frac{a^2 (x - \alpha)^2 (x - \beta)^2}{2(x - \alpha)^2};$$

$$\lim_{x \rightarrow \alpha} \frac{a^2}{2} (\alpha - \beta)^2 \Rightarrow C$$

$$10. l = e^2 \text{ hence } \{e^2\} = e^2 - 7$$

$$11. \lim_{x \rightarrow 1} (1 - x) \tan \left( \frac{\pi x}{2} \right) = \lim_{h \rightarrow 0} (1 - 1 + h) \tan \left( \frac{\pi}{2} - \frac{\pi h}{2} \right)$$

$$= \lim_{h \rightarrow 0} h \cot \frac{\pi}{2} h = \lim_{h \rightarrow 0} \frac{h}{\tan \frac{\pi h}{2}} = \frac{2}{\pi}$$

$$12. \lim_{x \rightarrow 0} \frac{(\tan\{x\} - 1) \sin\{x\}}{\{x\}(\{x\} - 1)}$$

$$\text{LHL} = \lim_{h \rightarrow 0^-} \frac{(\tan((1 - h) - 1)) \sin(1 - h)}{(1 - h)((1 - h) - 1)}$$

$$[\because \{x\} = \{0 - h\} = 1 - h]$$

$$= \lim_{h \rightarrow 0^-} \frac{-\tanh \sin(1 - h)}{(1 - h)(-h)} = \sin 1$$

$$\text{RHL} = \lim_{h \rightarrow 0^+} \frac{\tan(h - 1) \sinh}{h(h - 1)} = \tan 1$$

$$\therefore \text{LHL} \neq \text{RHL}$$

$$\therefore \text{limit does not exist}$$

$$\begin{aligned}
 13. \quad & \lim_{x \rightarrow \infty} \sin \sqrt{x+1} - \sin \sqrt{x} \\
 &= \lim_{x \rightarrow \infty} 2 \cos \left( \frac{\sqrt{x+1} + \sqrt{x}}{2} \right) \sin \left( \frac{\sqrt{x+1} - \sqrt{x}}{2} \right) \\
 &= \lim_{x \rightarrow \infty} \frac{2 \cos \left( \frac{\sqrt{x+1} + \sqrt{x}}{2} \right) \sin \left( \frac{\sqrt{x+1} - \sqrt{x}}{2} \right) \times \left( \frac{\sqrt{x+1} - \sqrt{x}}{2} \right)}{\left( \frac{\sqrt{x+1} - \sqrt{x}}{2} \right)} \\
 &= \lim_{x \rightarrow \infty} 2 \cos \left( \frac{\sqrt{x+1} + \sqrt{x}}{2} \right) \times \frac{1}{2(\sqrt{x+1} + \sqrt{x})} = 0
 \end{aligned}$$

$$\begin{aligned}
 14. \quad & \lim_{x \rightarrow \frac{\pi}{2}} \frac{(1 - \tan x / 2) (1 - \sin x)}{(1 + \tan x / 2) (\pi - 2x)^3} \\
 &= \lim_{x \rightarrow \frac{\pi}{2}} \tan \left( \frac{\pi}{4} - \frac{x}{2} \right) \frac{(1 - \sin x)}{(\pi - 2x)^3} \\
 &= \lim_{h \rightarrow 0} \frac{\tan \left( \frac{\pi}{4} - \frac{\pi}{4} + \frac{h}{2} \right) (1 - \cosh)}{(2h)^3} \\
 &= \lim_{h \rightarrow 0} \frac{2 \tan \frac{h}{2} \sin^2 \frac{h}{2}}{8h^3} = \frac{1}{32}
 \end{aligned}$$

$$\begin{aligned}
 15. \quad & \lim_{x \rightarrow \infty} \left( 1 + \frac{a}{x} + \frac{b}{x^2} \right)^{2x} = e^2 \Rightarrow e^{\lim_{x \rightarrow \infty} 2x \left( 1 + \frac{a}{x} + \frac{b}{x^2} - 1 \right)} = e^2 \\
 & \Rightarrow e^{2a} = e^2 \Rightarrow a = 1 \text{ \& } b \in \mathbb{R}
 \end{aligned}$$

$$\begin{aligned}
 16. \quad & \lim_{x \rightarrow 0} \frac{(1 + 3x)^{1/3} - 1 - x}{(1 + x)^{101} - 1 - 101x} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{3}{3}(1 + 3x)^{-2/3} - 1}{(101)(1 + x)^{100} - 101} \quad (\text{By L' Hospital rule}) \\
 &= \lim_{x \rightarrow 0} \frac{-\frac{2}{3}(1 + 3x)^{-5/3} \cdot 3}{(101)(100)(1 + x)^{99}} = -\frac{1}{5050}
 \end{aligned}$$

$$\begin{aligned}
 17. \quad & \lim_{x \rightarrow 0} \frac{e^{x^3} - \tan x + \sin x - 1}{x^n} \\
 &= \lim_{x \rightarrow 0} \frac{e^{x^3} - 1}{x^n} + \frac{-\tan x(1 - \cos x)}{x^n}
 \end{aligned}$$

Now for existence of limit n should be 3

$$18. \text{ Let } y = \sqrt{\alpha + \sqrt{\alpha + \sqrt{\alpha + \dots \dots \dots \infty}}}$$

where  $\alpha = 1 - \cos x$ ; as  $x \rightarrow 0$ ,  $\alpha \rightarrow 0$

$$y = \sqrt{\alpha + y}; y^2 = \alpha + y \Rightarrow y^2 - y - \alpha = 0$$

$$y = \frac{1 \pm \sqrt{1 + 4\alpha}}{2}$$

(neglecting -ve sign as y can not be -ve)

$$y = \frac{1 + \sqrt{1 + 4\alpha}}{2}$$

$$\text{now } l = \lim_{\substack{x \rightarrow 0 \\ \alpha \rightarrow 0}} \frac{\left[ \frac{1 + \sqrt{1 + 4\alpha} - 1}{2} \right]}{\frac{x^2}{1 - \cos x} \cdot (1 - \cos x)}$$

$$= \lim_{\alpha \rightarrow 0} \frac{\sqrt{1 + 4\alpha} - 1}{2 \cdot 2 \cdot \alpha} \quad (\text{as } \lim_{x \rightarrow 0} \frac{x^2}{1 - \cos x} = 2)$$

$$= \lim_{\alpha \rightarrow 0} \frac{(1 + 4\alpha) - 1}{4\alpha(\sqrt{1 + 4\alpha} + 1)} = 2 = \frac{4}{2}$$

$$(\text{rationalising the D}^r) = \lim_{\alpha \rightarrow 0} \frac{1}{\sqrt{1 + 4\alpha} + 1} = \frac{1}{2}$$

$$19. \quad \lim_{x \rightarrow \infty} [(x + a)(x + b)(x + c)]^{1/3} - x$$

$$(\text{using } a - b = \frac{a^3 - b^3}{a^2 + ab + b^2})$$

$$= \frac{[x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc] - x^3}{[(x + a)(x + b)(x + c)]^{2/3} + x^2 + x[(x + a)(x + b)(x + c)]^{1/3}}$$

$$\lim_{x \rightarrow \infty} x^2 \left[ (a + b + c) + \frac{ab + bc + ca}{x} + \frac{abc}{x^3} \right] /$$

$$x^2 \left[ \left( 1 + \frac{a + b + c}{x} + \frac{ab + bc + ca}{x^2} + \frac{abc}{x^3} \right) + 1 + \left( 1 + \frac{a + b + c}{x} + \frac{ab + bc + ca}{x^2} + \frac{abc}{x^3} \right)^{1/3} \right]^{1/3}$$

$$= \frac{a + b + c}{3}$$

20. Let  $f(n) = \frac{1}{n^2+n+1} + \frac{2}{n^2+n+2} + \dots + \frac{n}{n^2+n+n}$   
consider  $g(n)$

$$= \frac{1}{n^2+n+n} + \frac{2}{n^2+n+n} + \dots + \frac{n}{n^2+n+n}$$

$$= \frac{1+2+3+\dots+n}{n^2+2n} = \frac{n(n+1)}{2(n^2+2n)}$$

$$g(n) < f(n) \quad \dots(1)$$

||ly  $h(n)$

$$= \frac{1}{n^2+n+1} + \frac{2}{n^2+n+1} + \dots + \frac{n}{n^2+n+1}$$

$$= \frac{n(n+1)}{2(n^2+n+1)}$$

$$\therefore f(n) < h(n) \quad \dots(2)$$

from (1) and (2)

$$g(n) < f(n) < h(n)$$

but  $\lim_{n \rightarrow \infty} g(n) = \lim_{n \rightarrow \infty} h(n) = \frac{1}{2}$ ;

Hence using Sandwich theorem

$$\therefore \lim_{n \rightarrow \infty} f(n) = \frac{1}{2}$$

21.  $n < \sqrt{n^2+n+1} < n+1$

Hence  $\left[ \sqrt{n^2+n+1} \right] = n$

$$\therefore l = \lim_{n \rightarrow \infty} \left( \sqrt{n^2+n+1} - n \right) = \frac{1}{2}$$

23.  $\alpha, \beta$  are the roots of the equation  $ax^2 + bx + c = 0$

$$\therefore ax^2 + bx + c = a(x-\alpha)(x-\beta)$$

$$= e^{\lim_{x \rightarrow \alpha} \frac{ax^2+bx+c}{x-\alpha}} = e^{\lim_{x \rightarrow \alpha} \frac{a(x-\alpha)(x-\beta)}{(x-\alpha)}} = e^{a(\alpha-\beta)}$$

24. R.H.L. =  $\lim_{x \rightarrow 0^+} \left( (1-e^x) \frac{\sin x}{x} \right)$

when  $x \in (0, h)$  and  $h \rightarrow 0$  then  $(1-e^x) \in (-1, 0)$

and  $\frac{\sin x}{x} < 1$

So  $-1 < (1-e^x) \frac{\sin x}{x} < 0$ ,

$$\lim_{x \rightarrow 0^+} \left[ (1-e^x) \frac{\sin x}{x} \right] = -1$$

L.H.L. =  $\lim_{x \rightarrow 0^-} \left[ (1-e^x) \frac{\sin x}{-x} \right] =$

$$\lim_{x \rightarrow 0^-} \left[ (e^x - 1) \frac{\sin x}{x} \right]$$

when  $x \in (-h, 0)$  and  $h \rightarrow 0$ , then  $e^x - 1 \in (-1, 0)$

and  $\frac{\sin x}{x} < 1$

So  $-1 < (e^x - 1) \frac{\sin x}{x} < 0$  So

$$\lim_{x \rightarrow 0^-} \left[ (e^x - 1) \frac{\sin x}{x} \right] = -1$$

L.H.L. = R.H.L. = -1

26.

$$\lim_{y \rightarrow 0} \left( \lim_{x \rightarrow \infty} \frac{\exp \left( x \ln \left( 1 + \frac{ay}{x} \right) \right) - \exp \left( x \ln \left( 1 + \frac{by}{x} \right) \right)}{y} \right)$$

$$= \lim_{y \rightarrow 0} \left( \lim_{x \rightarrow \infty} \frac{\left( 1 + \frac{ay}{x} \right)^x - \left( 1 + \frac{by}{x} \right)^x}{y} \right) \text{ by expansion}$$

$$= \lim_{y \rightarrow 0}$$

$$\left( \lim_{x \rightarrow \infty} \frac{\left( 1 + ay + \frac{x(x-1)}{2!} \frac{a^2 y^2}{x^2} + \dots \right) - \left( 1 + by + \frac{x(x-1)}{2!} \frac{b^2 y^2}{x^2} + \dots \right)}{y} \right)$$

$$= \lim_{y \rightarrow 0} \left[ \frac{y(a-b) + \frac{y^2}{2} (a^2 - b^2) + \dots}{y} \right]$$

$$= a - b$$

27.  $\lim_{x \rightarrow \infty} \left( x \sin \left( \frac{1}{x} \right) + \sin \left( \frac{1}{x^2} \right) \right)$

$$\Rightarrow \lim_{x \rightarrow \infty} \left( \frac{\sin \left( \frac{1}{x} \right)}{\frac{1}{x}} + \sin \left( \frac{1}{x^2} \right) \right) = 1 + 0 = 1$$

EXERCISE - 2

Part # I : Multiple Choice

$$1. f(x) = \lim_{n \rightarrow \infty} \frac{2x^{2n} \sin \frac{1}{x} + x}{1 + x^{2n}}$$

$$= \begin{cases} 2 \sin \frac{1}{x} & \text{for } x > 1 \text{ or } x < -1 \\ \frac{2 \sin(1/x) + x}{2} & \text{for } x = 1 \text{ or } -1 \\ x & \text{if } -1 \leq x < 1 \end{cases}$$

$$\lim_{x \rightarrow \infty} x f(x) = \lim_{x \rightarrow \infty} 2x \sin \frac{1}{x} = 2$$

$$\lim_{x \rightarrow 1^+} f(x) = 2 \sin 1 ; \lim_{x \rightarrow 1^-} f(x) = 1$$

$\Rightarrow \lim_{x \rightarrow 1} f(x)$  does not exist.  $\Rightarrow$  (B)

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x = 0$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} 2 \sin \frac{1}{x} = 0 ]$$

$$5. f(x) = \frac{|x + \pi|}{\sin x}$$

$$(A) f(-\pi^+) = \lim_{h \rightarrow 0} \frac{|-\pi + h + \pi|}{\sin(-\pi + h)} = \lim_{h \rightarrow 0} \frac{|h|}{-\sin h} = -1$$

$$(B) f(-\pi^-) = \lim_{h \rightarrow 0} \frac{|-\pi - h + \pi|}{\sin(-\pi - h)} = \lim_{h \rightarrow 0} \frac{|h|}{\sin h} = 1$$

(C)  $f(-\pi^+) \neq f(-\pi^-)$  So  $\lim_{x \rightarrow -\pi} f(x)$  does not exist

(D) for  $\lim_{x \rightarrow \pi} f(x)$

$$\text{LHL} = \lim_{x \rightarrow \pi^-} \frac{|x + \pi|}{\sin x} = \lim_{h \rightarrow 0} \frac{2\pi - h}{\sinh} = \frac{2\pi}{0} = \infty$$

$$\text{RHL} = \lim_{x \rightarrow \pi^+} \frac{|x + \pi|}{\sin x} = \lim_{h \rightarrow 0} \frac{2\pi + h}{-\sinh} = -\frac{2\pi}{0} = -\infty$$

LHL  $\neq$  RHL

So  $\lim_{x \rightarrow \pi} f(x)$  does not exist.

$$9. \text{ put } \theta = -1 ; \frac{1-1-2}{2} \leq f(-1) \leq \frac{1-2-1}{2}$$

$$-1 \leq f(-1) \leq -1$$

$$\Rightarrow f(-1) = -1$$

$$\lim_{\theta \rightarrow -1} \frac{\theta^2 + \theta - 2}{\theta + 3} = -1 = \lim_{\theta \rightarrow -1} \frac{\theta^2 + 2\theta - 1}{\theta + 3}$$

using squeeze play theorem

$$\lim_{\theta \rightarrow -1} \frac{f(\theta)}{\theta^2} = -1 ; \lim_{\theta \rightarrow -1} f(\theta) = -1$$

$$12. \text{ Put } x = \frac{\pi}{2} - h$$

$$l = \lim_{h \rightarrow 0} \frac{a^{\tanh} - a^{\sinh}}{\tanh - \sinh}$$

$$= \lim_{h \rightarrow 0} a^{\sinh} \left( \frac{a^{\tanh - \sinh} - 1}{\tanh - \sinh} \right)$$

$$= \log_e a$$

$$m = \lim_{x \rightarrow -\infty} \frac{x^2 + ax - x^2 + ax}{\sqrt{x^2 + ax} + \sqrt{x^2 - ax}}$$

$$= \lim_{x \rightarrow -\infty} \frac{2ax}{|x| \left( \sqrt{1 + \frac{a}{x}} + \sqrt{1 - \frac{a}{x}} \right)}$$

$$= \frac{2ax}{-x(2)} = -a$$

Part # II : Assertion & Reason

1. Statement - 1

$$\lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos 2x}}{x} = \lim_{x \rightarrow 0} \frac{|\sin x|}{x}$$

$\Rightarrow$  L.H.L. = -1 & R.H.L. = 1

Statement - 2 is true

2. For all  $x$ ,  $x - 1 < [x] \leq x$ , where  $[ \cdot ]$  denotes greatest integer function.

$$\Rightarrow x^n - 1 < [x^n] \leq x^n \Rightarrow \frac{1}{x^n} \leq \frac{1}{[x^n]} < \frac{1}{x^n - 1}$$

Multiplying the inequation by  $x^n + nx^{n-1} + 1$  and taking the limit as  $x \rightarrow \infty$ ,

We get,

$$\lim_{x \rightarrow \infty} \frac{x^n + nx^{n-1} + 1}{x^n} \leq \lim_{x \rightarrow \infty} \frac{x^n + nx^{n-1} + 1}{[x^n]} < \lim_{x \rightarrow \infty} \frac{x^n + nx^{n-1} + 1}{x^n - 1}$$

Evaluating the limits on the left and right side of the inequality, we obtain

$$\lim_{x \rightarrow \infty} \frac{x^n + nx^{n-1} + 1}{x^n} = \lim_{x \rightarrow \infty} \frac{x^n + nx^{n-1} + 1}{x^n - 1} = 1$$

And hence by sandwich theorem,

$$\lim_{x \rightarrow \infty} \frac{x^n + nx^{n-1} + 1}{[x^n]} = 1$$

$\Rightarrow$  Statement 1 is false.

EXERCISE - 3

Part # I : Matrix Match Type

3. (A)  $\lim_{x \rightarrow 0} \frac{\tan[e^2]x^2 - \tan[-e^2]x^2}{\sin^2 x}$

$$= \lim_{x \rightarrow 0} \frac{[e^2]x^2 \frac{\tan[e^2]x^2}{[e^2]x^2} - [-e^2]x^2 \frac{\tan[-e^2]x^2}{[-e^2]x^2}}{x^2 \frac{\sin^2 x}{x^2}}$$

$$= [e^2] - [-e^2] = 15$$

(B)  $\lim_{x \rightarrow 0} \left[ (\min(t^2 + 4t + 6)) \frac{\sin x}{x} \right] = \lim_{x \rightarrow 0} \left[ \frac{2 \sin x}{x} \right]$

$\therefore \sin x < x$

$$\Rightarrow \frac{2 \sin x}{x} < 2 \Rightarrow \left[ \frac{2 \sin x}{x} \right] = 1$$

So  $\lim_{x \rightarrow 0} \left[ \frac{2 \sin x}{x} \right] = 1$

(C)  $\lim_{x \rightarrow 0} \frac{(1+x^2)^{1/3} - (1-2x)^{1/4}}{x+x^2}$

$$= \lim_{x \rightarrow 0} \frac{\left( 1 + \frac{x^2}{3} + \dots \right) - \left( 1 - \frac{x}{2} + \frac{1}{4} \left( \frac{1-1}{4} \right) \frac{1}{2!} 4x^2 + \dots \right)}{x+x^2}$$

$$= \frac{1}{2}$$

(D)  $\lim_{x \rightarrow 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x} = \lim_{x \rightarrow 0} \frac{2\sqrt{2} \sin^2 \frac{x}{4}}{\sin^2 x}$

$$= \lim_{x \rightarrow 0} \frac{2\sqrt{2} \sin^2 \frac{x}{4}}{16 \cdot \frac{x^2}{16} \cdot \frac{\sin^2 x}{x^2}}$$

$$= \frac{\sqrt{2}}{8}$$

$$4. \text{ (A) } L = \lim_{h \rightarrow 0} \frac{\cos \left( \tan^{-1} \left( \tan \left( \frac{\pi}{2} + h \right) \right) \right)}{\frac{\pi}{2} + h - \frac{\pi}{2}}$$

$$= \frac{\cos \left( -\frac{\pi}{2} + h \right)}{h} = \frac{\sinh}{h} = 1$$

Now  $\cos [2\pi (1)] = 1$

$$\text{(B) } k = \lim_{n \rightarrow \infty} \prod_{r=2}^n \frac{r^3 - 1}{r^3 + 1} = \lim_{n \rightarrow \infty} \prod_{r=2}^n \frac{(r-1)(r^2+r+1)}{(r+1)(r^2-r+1)}$$

$$= \lim_{n \rightarrow \infty} \frac{2}{n(n+1)} \times \frac{n^2+n+1}{3} = \frac{2}{3}$$

so  $\operatorname{cosec} \theta = \frac{2}{3} \Rightarrow$  No. of solution is zero

$$\text{(C) } \lim_{x \rightarrow \infty} \left( \frac{x+c}{x-c} \right)^x = 4$$

$$\Rightarrow e^{\lim_{x \rightarrow \infty} x \left( \frac{x+c-x+c}{x-c} \right)} = 4 \Rightarrow e^{\lim_{x \rightarrow \infty} x \left( \frac{2c}{x-c} \right)} = 4$$

$$\Rightarrow e^{2c} = 4 \Rightarrow e^c = 2$$

(only positive value)

$$\Rightarrow \frac{-e^c}{2} = -1$$

$$\text{(D) } \lim_{x \rightarrow -\infty} \frac{(3x^4 + 2x^2) \sin(1/x) - x^3 + 5}{(-x)^3 + (-x)^2 - x + 1} = k$$

$$\Rightarrow \lim_{x \rightarrow -\infty} \frac{\left( 3 + \frac{2}{x^2} \right) \frac{\sin(1/x)}{(1/x)} - 1 + \frac{5}{x^3}}{-1 + \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}} = k$$

$$\Rightarrow \frac{3-1}{-1} = k \Rightarrow \frac{k}{2} = -1$$

$$5. \text{ (A) put } x = 1/y \quad \lim_{y \rightarrow 0} \left( \frac{1}{1+y} \right)^{1/y}$$

$$e^{\lim_{y \rightarrow 0} \frac{1-y-1}{y(1+y)}} = e^{-1} \quad z \Rightarrow \text{(S)}$$

$$\text{(B) } \lim_{y \rightarrow 0} (\sin y + \cos y)^{1/y} = e^{\lim_{y \rightarrow 0} \frac{\sin y + \cos y - 1}{y}} = e$$

$\Rightarrow \text{(R)}$

$$\text{(C) } e^{\lim_{x \rightarrow 0} \frac{\cos x - 1}{\tan^2 x \cdot x^2}} = e^{-\frac{1}{2}} \Rightarrow \text{(Q)}$$

$$\text{(D) } \lim_{x \rightarrow 0} \frac{\tan((\pi/4)+x) - \tan(\pi/4)}{x}$$

$$= \lim_{x \rightarrow 0} \frac{\tan x [1 + \tan((\pi/4)+x) \cdot 1]}{x} = e^2 \Rightarrow \text{(P)}$$

$$6. \text{ (A) } l = \lim_{x \rightarrow 0} \frac{e^{x^2} - 1 + x - e^x + 1}{2 \frac{\sin^2 x}{x^2} \cdot x^2}$$

$$= \frac{1}{2} \left[ \lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{x^2} + \lim_{x \rightarrow 0} \frac{x - e^x + 1}{x^2} \right]$$

$$= \frac{1}{2} \left[ 1 - \lim_{x \rightarrow 0} \frac{e^x - x - 1}{x^2} \right]$$

$$= \frac{1}{2} \left[ 1 - \frac{1}{2} \right] = \frac{1}{4} \Rightarrow \frac{1}{1} = 4$$

$$= e^{\lim_{x \rightarrow 0} \frac{1}{x} \left( \frac{3+x}{3-x} - 1 \right)} = e^{\lim_{x \rightarrow 0} \frac{2x}{x(3-x)}} = e^{2/3}$$

$$\Rightarrow 2+3=5$$

$$\text{(C) } \lim_{x \rightarrow 0} \frac{(\tan^3 x - x^3) - (\tan x^3 - x^3)}{x^5}$$

$$= \lim_{x \rightarrow 0} \frac{\tan^3 x - x^3}{x^5} - \lim_{x \rightarrow 0} \frac{\tan x^3 - x^3}{x^5}$$

zero (by expansion)

$$= \lim_{x \rightarrow 0} \frac{(\tan x - x) \cdot (\tan^2 x + x \tan x + x^2)}{x^3}$$

$$= \frac{1}{3} \times 3 = 1$$

(D) rationalising gives

$$\lim_{x \rightarrow 0} \frac{(x + 2 \sin x) \left[ \sqrt{(x^2 + 2 \sin x + 1)} + \sqrt{\sin^2 x - x + 1} \right]}{(x^2 + 2 \sin x + 1) - (\sin^2 x - x + 1)}$$

$$2 \cdot \lim_{x \rightarrow 0} \frac{x + \sin 2x}{x^2 - \sin^2 x + 2 \sin x + x}$$

$$= 2 \cdot \lim_{x \rightarrow 0} \frac{1 + \frac{\sin 2x}{x}}{x - \frac{\sin^2 x}{x} + 2 + 1} = 2 \left( \frac{1+2}{3} \right) = 2$$

Part # II : Comprehension

Comprehension-3

1 to 3

$$f(x) = \lim_{n \rightarrow \infty} \left( \cos \frac{x}{\sqrt{n}} \right)^n = e^{\lim_{n \rightarrow \infty} \left( \cos \frac{x}{\sqrt{n}} - 1 \right) \cdot n}$$

Substituting,  $n = \frac{1}{t^2}$

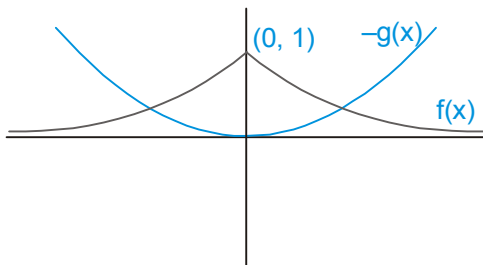
$$f(x) = e^{\lim_{t \rightarrow 0} \left( \frac{\cos tx - 1}{t^2} \right)} = e^{\lim_{t \rightarrow 0} -x^2 \left( \frac{1 - \cos tx}{t^2 x^2} \right)} = e^{-\frac{1}{2}x^2}$$

$$g(x) = -x^{4b}$$

$$b = \lim_{x \rightarrow \infty} \left( \sqrt{x^2 + x + 1} - \sqrt{x^2 + 1} \right)$$

$$= \lim_{x \rightarrow \infty} \left( \frac{x^2 + x + 1 - x^2 - 1}{\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1}} \right) = \frac{1}{2}$$

$$\therefore g(x) = -x^{4 \cdot \frac{1}{2}} = -x^2$$



By observation, graphs of  $f(x)$  and  $-g(x)$  intersect each other at two points

$\therefore$  Number of solutions is 2.

EXERCISE - 4

Subjective Type

1. Using L'Hospital rule

$$\lim_{x \rightarrow \frac{3\pi}{4}} \frac{1 + (\tan x)^{\frac{1}{3}}}{1 - 2 \cos^2 x} = \lim_{x \rightarrow \frac{3\pi}{4}} \frac{\frac{1}{3} \tan x^{-\frac{2}{3}} \sec^2 x}{4 \cos x \sin x} = -\frac{1}{3}$$

3.  $\lim_{x \rightarrow \infty} x^2 \sin \left( \ln \sqrt{\cos \frac{\pi}{x}} \right)$

$$= \lim_{x \rightarrow \infty} \frac{x^2 \sin \left( \ln \sqrt{\cos \left( \frac{\pi}{x} \right)} \right)}{\left( \ln \sqrt{\cos \left( \frac{\pi}{x} \right)} \right)} \times \left( \ln \sqrt{\cos \left( \frac{\pi}{x} \right)} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{x^2}{2} \ln \left( 1 + \cos \left( \frac{\pi}{x} \right) - 1 \right)$$

$$= \lim_{x \rightarrow \infty} \frac{x^2}{2} \frac{\ln \left( 1 + \left( \cos \left( \frac{\pi}{x} \right) - 1 \right) \right)}{\left( \cos \left( \frac{\pi}{x} \right) - 1 \right)} \times \left( \cos \left( \frac{\pi}{x} \right) - 1 \right)$$

5. Let  $f(x) = \left( \frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2}} + \dots + \frac{1}{\sqrt{n^2}} \right)$

$$= \frac{2n+1}{\sqrt{n^2}} \quad \dots (2n+1) \text{ terms}$$

$$\lim_{n \rightarrow \infty} f(x) = 2$$

$$g(x) = \left( \frac{1}{\sqrt{n^2+2n}} + \frac{1}{\sqrt{n^2+2n}} + \dots + \frac{1}{\sqrt{n^2+2n}} \right)$$

$\dots (2n+1) \text{ terms}$

$$\lim_{n \rightarrow \infty} g(x) = 2$$

$$\therefore \lim_{n \rightarrow \infty} \left( \frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+2n}} \right) = 2$$

6.  $\lim_{x \rightarrow 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2$

$$= \lim_{x \rightarrow 0} \frac{ae^x + ce^{-x} - b \cos x}{x^2} = 2$$

$$= \lim_{x \rightarrow 0} \frac{a \left(1 + x + \frac{x^2}{2!} + \dots\right) + c \left(1 - x + \frac{x^2}{2!} + \dots\right) - b + b - b \cos x}{x^2}$$

$$= 2$$

$$\Rightarrow a + c - b = 0, a - c = 0 \text{ \& } \frac{a+c}{2} + \frac{b}{2} = 2$$

$$\Rightarrow a = c = 1, b = 2$$

$$= \lim_{x \rightarrow \infty} \frac{x^2}{2} \left( \cos\left(\frac{\pi}{x}\right) - 1 \right)$$

Now put  $x = 1/y \Rightarrow \lim_{y \rightarrow 0} \frac{\cos(\pi y) - 1}{2y^2} = \frac{-\pi^2}{4}$

7.  $\lim_{x \rightarrow \pi/4} \frac{\sqrt{1 - \sqrt{\sin 2x}}}{\pi - 4x}$

$$\text{LHL} = \lim_{h \rightarrow 0} \frac{\sqrt{1 - \sqrt{\sin\left(\frac{\pi}{2} - 2h\right)}}}{\pi - \pi + 4h}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{1 - \sqrt{\cos 2h}}}{4h} \times \frac{\sqrt{1 + \sqrt{\cos 2h}}}{\sqrt{1 + \sqrt{\cos 2h}}}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{1 - \cos 2h}}{4h} \times \frac{1}{\sqrt{1 + \sqrt{\cos 2h}}}$$

$$= \frac{\sqrt{2} |\sin h|}{4h \times \sqrt{2}} = \frac{1}{4}$$

Similarly  $\text{RHL} = \frac{-1}{4}$

Hence  $\text{LHL} \neq \text{RHL} \therefore$  limit does not exist.

10.  $\lim_{x \rightarrow 1} \frac{\ln\left(\frac{1+x}{2}\right) \times 3 \cdot (4^{x-1} - x)}{\left[(7+x)^{\frac{1}{3}} - (1+3x)^{\frac{1}{2}}\right] \sin(x-1)}$

$$= \lim_{h \rightarrow 0} \frac{\ln\left(\frac{2+h}{2}\right) \times 3(4^h - (1+h))}{\left[(8+h)^{\frac{1}{3}} - (4+3h)^{1/2}\right] \cdot \sin h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{\ln\left(1 + \frac{h}{2}\right)}{2(h/2)} \cdot 3 \cdot (4^h - (1+h))}{2 \left\{ \left[1 + \frac{h}{8} \times \frac{1}{3} + \dots\right] - \left[1 + \frac{3h}{4} \times \frac{1}{2} + \dots\right] \right\} \frac{\sin h}{h}}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{3}{4} \left[ \frac{4^h - 1}{h} - 1 \right]}{\frac{1}{h} \left[ \frac{h}{24} - \frac{3h}{8} \right]} = \frac{-9}{4} \ln \frac{4}{e}$$

11. (i)  $\lim_{x \rightarrow 0} \left( \frac{(1+x)^{\frac{1}{x}}}{e} \right)^{\frac{1}{x}}$

$$= \lim_{x \rightarrow 0} \left( \frac{e \left(1 - \frac{x}{2} + \frac{11}{24}x^2 - \dots\right)}{e} \right)^{\frac{1}{x}}$$

$$= \lim_{x \rightarrow 0} \left( 1 - \frac{x}{2} + \frac{11}{24}x^2 - \dots \right)^{\frac{1}{x}}$$

$$= e^{\lim_{x \rightarrow 0} \left( -\frac{x}{2} + \frac{11}{24}x^2 - \dots \right) \cdot \frac{1}{x}} = e^{-\frac{1}{2}}$$

(ii)  $\lim_{x \rightarrow 0} \left[ \sin^2\left(\frac{\pi}{2-ax}\right) \right]^{\sec^2\left(\frac{\pi}{2-bx}\right)}$  ( $1^\infty$  form)

$$= e^{\lim_{x \rightarrow 0} \left( \sin^2\left(\frac{\pi}{2-ax}\right) - 1 \right) \cdot \sec^2\left(\frac{\pi}{2-bx}\right)}$$

$$= e^{\lim_{x \rightarrow 0} \frac{-\cos^2\left(\frac{\pi}{2-ax}\right)}{\cos^2\left(\frac{\pi}{2-bx}\right)}} \quad \left( \frac{0}{0} \text{ form} \right)$$

Applying L.Hospital rule

$$= \lim_{x \rightarrow 0} \frac{\frac{-\sin\left(\frac{2\pi}{2-ax}\right)}{\sin\left(\frac{2\pi}{2-bx}\right)} \cdot \frac{a\pi(2-bx)^2}{b\pi(2-ax)^2}}{\frac{-\sin\left(\frac{2\pi}{2-ax}\right)}{\sin\left(\frac{2\pi}{2-bx}\right)}} \cdot \frac{a\pi}{b\pi}$$

$$= \lim_{x \rightarrow 0} \frac{-\cos\left(\frac{2\pi}{2-ax}\right) \cdot \frac{2a^2\pi^2}{(2-ax)^2}}{\cos\left(\frac{2\pi}{2-bx}\right) \cdot \frac{2b^2\pi^2}{(2-bx)^2}} = e^{-\frac{a^2}{b^2}}$$



$$\begin{aligned}
 12. \lim_{x \rightarrow 1} \frac{\sum_{k=1}^{100} x^k - 100}{x-1} \\
 &= \lim_{x \rightarrow 1} \left[ \frac{x + x^2 + x^3 + \dots + x^{100} - 100}{x-1} \right] \\
 &= \lim_{x \rightarrow 1} \left[ \frac{1 + 2x + 3x^2 + \dots + 100x^{99}}{1} \right] \\
 &\text{(using L'Hospital)} \\
 &= 5050.
 \end{aligned}$$

$$\begin{aligned}
 13. \text{Case (i)} \quad \lim_{x \rightarrow 0^+} f(x) \\
 &= \lim_{x \rightarrow 0^+} \frac{\sin^{-1}(1-\{x\}) \cdot \cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}(1-\{x\})} \\
 &= \lim_{h \rightarrow 0} \frac{\sin^{-1}(1-h) \cdot \cos^{-1}(1-h)}{\sqrt{2h}(1-h)} \\
 &= \lim_{h \rightarrow 0} \frac{\sin^{-1}(1-h) \sin^{-1} \sqrt{h(2-h)}}{(1-h) \sqrt{2h}} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{\pi}{2}}{1} \frac{\sin^{-1} \sqrt{h(2-h)}}{\sqrt{2h-h^2} \sqrt{2h}} \sqrt{2h-h^2} \\
 &= \lim_{h \rightarrow 0} \frac{\pi}{2} \cdot 1 \cdot \sqrt{1-\frac{h}{2}} = \frac{\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Case (ii)} \quad \lim_{x \rightarrow 0^-} f(x) \\
 &= \lim_{x \rightarrow 0^-} \frac{\sin^{-1}(1-\{x\}) \cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}(1-\{x\})} \\
 &= \lim_{h \rightarrow 0} \frac{\sin^{-1} h \cdot \cos^{-1} h}{\sqrt{2(1-h)} h} = \frac{\pi}{2\sqrt{2}} = \frac{\pi}{2\sqrt{2}}
 \end{aligned}$$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

$$\begin{aligned}
 1. \quad y &= \lim_{x \rightarrow 1} \frac{\sqrt{f(x)} - 1}{\sqrt{x} - 1} \\
 \Rightarrow y &= \lim_{x \rightarrow 1} \frac{(\sqrt{f(x)} - 1)(\sqrt{f(x)} + 1)(\sqrt{x} + 1)}{(\sqrt{x} - 1)(\sqrt{x} + 1)(\sqrt{f(x)} + 1)} \\
 \Rightarrow y &= \lim_{x \rightarrow 1} \frac{f(x) - 1}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{f(x)} + 1}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow y &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} \cdot \lim_{x \rightarrow 1} \frac{\sqrt{x} + 1}{\sqrt{f(x)} + 1} \\
 \Rightarrow y &= f'(1) \cdot \frac{2}{\sqrt{f(1)} + 1} \Rightarrow y = 2 \cdot \frac{2}{2} = 2
 \end{aligned}$$

Applying L-Hospital's rule.

3. We have,

$$\begin{aligned}
 \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \left( \frac{x^2 + 5x + 3}{x^2 + x + 2} \right)^x \\
 &= \lim_{x \rightarrow \infty} \left( 1 + \frac{4x + 1}{x^2 + x + 2} \right)^x = e^{\lim_{x \rightarrow \infty} \frac{x(4x+1)}{x^2+x+2}} = e^4
 \end{aligned}$$

$$4. \quad \lim_{x \rightarrow \infty} \frac{\log x^n - [x]}{[x]} = \lim_{x \rightarrow \infty} \frac{\log x^n}{[x]} - \lim_{x \rightarrow \infty} \frac{[x]}{[x]} = 0 - 1 = -1$$

$$5. \quad \lim_{x \rightarrow 0} \frac{\log(3+x) - \log(3-x)}{x} = k$$

By L-Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{\frac{1}{3+x} + \frac{1}{3-x}}{1} = k \Rightarrow \frac{2}{3} = k$$

$$6. \quad \text{We have, } \lim_{x \rightarrow a} \frac{f(a)g(x) - f(x)g(a)}{g(x) - f(x)} = 4$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{f(a)g'(x) - 0 - g(a)f'(x) + 0}{g'(x) - f'(x)} = 4$$

$$\Rightarrow \frac{f(a)g'(a) - g(a)f'(a)}{g'(a) - f'(a)} = 4$$

$$\Rightarrow \frac{k\{g'(a) - f'(a)\}}{\{g'(a) - f'(a)\}} = 4 \quad [\because f(a) = g(a) = k]$$

$$\Rightarrow k = 4$$

$$7. \lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan\left(\frac{\pi-x}{4}\right)(1-\sin x)}{(\pi-2x)^3}$$

Let  $x = \frac{\pi}{2} + y : y \rightarrow 0$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{\tan\left(-\frac{y}{2}\right)(1-\cos y)}{(-2y)^3}$$

$$= \lim_{y \rightarrow 0} \frac{-\tan \frac{y}{2} \cdot 2 \sin^2 \frac{y}{2}}{(-8)y^3} = \lim_{y \rightarrow 0} \frac{1}{32} \tan \frac{y}{2} \cdot \left[\frac{\sin \frac{y}{2}}{\frac{y}{2}}\right]^2 = \frac{1}{32}$$

8. Since,  $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} + \frac{b}{x^2}\right)^{2x} = e^2$

$$\therefore \lim_{x \rightarrow \infty} \left[ \left(1 + \frac{ax+b}{x^2}\right)^{\frac{x^2}{ax+b}} \right]^{\frac{2(ax+b)}{x}} = e^2$$

$$\Rightarrow \lim_{x \rightarrow \infty} e^{\frac{2(ax+b)}{x}} = e^2$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{2(ax+b)}{x} = 2 \Rightarrow 2a = 2 \Rightarrow a = 1$$

Thus  $a = 1$  and  $b \in \mathbb{R}$

12.  $\lim_{x \rightarrow 5} \frac{(f(x))^2 - 9}{\sqrt{|x-5|}} = 0$

$\therefore$  Question must be in  $\frac{0}{0}$  form

$$\therefore (f(5))^2 - 9 = 0$$

$$\Rightarrow f(5) = 3$$

13.  $\lim_{x \rightarrow a} \frac{x^2 f(a) - a^2 f(x)}{x-a}$   $\left[\frac{0}{0} \text{ form}\right]$

Use L'Hospital rule

$$= \lim_{x \rightarrow a} \frac{2x f(a) - a^2 f'(x)}{1} = 2af(a) - a^2 f'(a)$$

14.  $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{4x^2} \cdot (4x^2) \cdot \frac{(3+1)}{x \tan 4x} \cdot \frac{1}{4x}$

$$\frac{1}{2} \cdot 4 = 2$$

18.  $p = \lim_{x \rightarrow 0^+} (1 + \tan^2 \sqrt{x})^{\frac{1}{2x}} = \lim_{x \rightarrow 0^+} e^{\frac{1}{2x}(\tan^2 \sqrt{x})} = e^{\frac{1}{2}}$

$$\log_e p = \frac{1}{2}$$

19.  $\ell = \left(\frac{(n+1)}{n} \cdot \frac{n+2}{n} \cdot \frac{n+3}{n} \cdot \dots \cdot \frac{n+2n}{n}\right)^{1/n}$

$$\log \ell = \frac{1}{n} \left[ \log\left(\frac{n+1}{n}\right) + \log\left(\frac{n+2}{n}\right) + \dots + \log\left(\frac{n+2n}{n}\right) \right]$$

$$\log \ell = \int_0^2 \log(1+x) dx$$

$$\log \ell = \int_1^3 \log t dt$$

$$\log \ell = t \log t - \int \frac{1}{t} \cdot t dt$$

$$\log \ell = t(\log t - 1)$$

$$\log \ell = 3(\log 3 - 1) - 1(\log 1 - 1)$$

$$= 3 \log 3 - 2$$

$$= \log 27 - \log e^2$$

$$= \log \ell = \log \frac{27}{e^2}$$

$$\ell = \frac{27}{e^2}$$

Part # II : IIT-JEE ADVANCED

6.  $\frac{a - a \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} - \frac{x^2}{4}}{x^4} = \frac{a - a \left(1 - \frac{x^2}{2a^2} - \frac{1}{8} \frac{x^4}{a^4}\right) - \frac{x^2}{4}}{x^4}$

$a = 2$ , (coefficient of  $x^2 = 0$ )

$$\therefore L = \frac{1}{64}$$

$$7. \lim_{x \rightarrow \infty} \left( \frac{x^2 + x + 1}{x + 1} - ax - b \right) = 4$$

$$\Rightarrow \lim_{x \rightarrow \infty} \left( \frac{(1-a)x^2 + x(1-a-b) + 1-b}{x+1} \right) = 4$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{\left( (1-a) \cdot x + 1 - a - b + \left( \frac{1-b}{x} \right) \right)}{1 + \frac{1}{x}} = 4$$

for limit to exist finitely

$$1 - a = 0 \text{ and } 1 - a - b = 4$$

$$\Rightarrow a = 1 \text{ and } b = -4.$$

$$8. \left( \left( 1 + \frac{a}{3} \right) - 1 \right) x^2 + \left( \left( 1 + \frac{a}{2} \right) - 1 \right) x + \left( 1 + \frac{a}{6} - 1 \right) = 0$$

$$a \left( \frac{x^2}{3} + \frac{x}{2} + \frac{1}{6} \right) = 0 \Rightarrow 2x^2 + 3x + 1 = 0$$

$$\Rightarrow x = -\frac{1}{2}, -1$$

$$\Rightarrow \lim_{a \rightarrow 0^+} \alpha(a) \text{ and } \lim_{a \rightarrow 0^+} \beta(a) \text{ are } -\frac{1}{2} \text{ and } -1$$

$$10. \lim_{x \rightarrow 0} \frac{x^2 \left\{ \beta x - \frac{(\beta x)^3}{3!} + \dots \right\}}{\alpha x - \left( x - \frac{x^3}{3!} + \dots \right)} = 1$$

$$\lim_{x \rightarrow 0} \frac{x^3 \left( \beta - \frac{\beta^3 x^2}{3!} + \dots \right)}{(\alpha - 1)x + x - \frac{x^3}{3!} - \frac{x^5}{5!} + \dots} = 1$$

As limit 1  $\Rightarrow \alpha = 1$

$$\lim_{x \rightarrow 0} \frac{\beta - \frac{\beta^3}{3!} x^2 + \dots}{\frac{1}{3!} - \frac{x^2}{5!} + \dots} = 1$$

$$\therefore \beta = \frac{1}{3!} = \frac{1}{6}$$

$$\therefore 6(\alpha + \beta) = 6 \left( 1 + \frac{1}{6} \right) = 7$$

$$11. \ln f(x) = \lim_{n \rightarrow \infty} \frac{x}{n} \ln \left[ \frac{\prod_{r=1}^n \left( x + \frac{n}{r} \right)}{\prod_{r=1}^n \left( x^2 + \frac{n^2}{r^2} \right)} \cdot \frac{1}{\prod_{r=1}^n \left( \frac{r}{n} \right)} \right]$$

$$\ln f(x) = \lim_{n \rightarrow \infty} \frac{x}{n} \ln \left[ \frac{\prod_{r=1}^n \left( x + \frac{1}{\frac{r}{n}} \right)}{\prod_{r=1}^n \left( x^2 + \frac{1}{\left( \frac{r}{n} \right)^2} \right)} \cdot \frac{1}{\prod_{r=1}^n \left( \frac{r}{n} \right)} \right]$$

$$= x \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \ln \left[ \frac{x \left( \frac{r}{n} \right) + 1}{x \left( \frac{r}{n} \right)^2 + 1} \right]$$

$$= x \int_0^1 \ln \left( \frac{1+tx}{1+t^2x^2} \right) dt$$

Put,  $tx = p$ , we get

$$\ln f(x) = \int_0^x \ln \left( \frac{1+p}{1+p^2} \right) dp$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \ln \left( \frac{1+x}{1+x^2} \right)$$

sign scheme of  $f'(x)$

Also  $f'(1) = 0$ .

$$\Rightarrow f\left(\frac{1}{2}\right) < f(1), f\left(\frac{1}{3}\right) < f\left(\frac{2}{3}\right), f'(2) < 0$$

$$\text{Also, } \frac{f'(3)}{f(3)} - \frac{f'(2)}{f(2)} = \ln \left( \frac{4}{10} \right) - \ln \left( \frac{3}{5} \right)$$

$$= \ln \left( \frac{4}{6} \right) < 0 \Rightarrow \frac{f'(3)}{f(3)} < \frac{f'(2)}{f(2)}$$

MOCK TEST

2. (D)

$$\begin{aligned} \lim_{x \rightarrow 1} \sec \frac{\pi}{2^x} \cdot \ln x &= \lim_{x \rightarrow 1} \frac{\ln x}{\cos \frac{\pi}{2^x}} \\ &= \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{\pi \sin \frac{\pi}{2^x} \cdot \frac{1}{2^x} \cdot \ln 2} = \frac{1}{\pi \ln 2 \cdot \frac{1}{2}} = \frac{2}{\pi \ln 2} \end{aligned}$$

3. (A)

$$\text{RHL} = \lim_{h \rightarrow 0^+} |h(h-1)|^{\cos 2h} = \lim_{h \rightarrow 0^+} (h(1-h))^0 = 1$$

$$\text{LHL} = \lim_{h \rightarrow 0^-} |-h(-h-1)|^{\cos 2h} = \lim_{h \rightarrow 0^-} (h(1+h))^0 = 1$$

∴ Required limit = 1

7. (C)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x^4 - x^4 \cos x^4 + x^{20}}{x^4(e^{2x^4} - 1 - 2x^4)} \\ &= \lim_{t \rightarrow 0} \frac{\sin t - t \cos t + t^5}{t(e^{2t} - 1 - 2t)} \quad \{\text{Let } x^4 = t\} \\ &= \lim_{t \rightarrow 0} \frac{\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots\right) - t\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots\right) + t^5}{t\left(1 + 2t + \frac{4t^2}{2!} + \frac{8t^3}{3!} + \frac{16t^4}{4!} + \dots - 1 - 2t\right)} \\ &= \lim_{t \rightarrow 0} \frac{-\frac{t^3}{6} + \frac{t^3}{2} + \frac{t^5}{5!} - \frac{t^5}{4!} + \dots + t^5}{2t^3 + \frac{8t^4}{3!} + \dots} \\ &= \frac{-\frac{1}{6} + \frac{1}{2}}{2} = \frac{-1+3}{12} = \frac{1}{6} \end{aligned}$$

9. (C)

$$\lim_{x \rightarrow 0^+} (x)^{\frac{1}{\ln \sin x}} = \lim_{x \rightarrow 0^+} e^{\frac{\ln x}{\ln x + \ln \frac{\sin x}{x}}} = e$$

10. (A)

S<sub>1</sub> : true

$$\begin{aligned} S_2 : \lim_{x \rightarrow 0^+} \sqrt{x} (\log x)^2 &= \lim_{x \rightarrow 0^+} \frac{(\log x)^2}{x^{\frac{1}{2}}} \\ &= \lim_{x \rightarrow 0^+} \frac{2 \log x \cdot \frac{1}{x}}{-\frac{1}{2} x^{-3/2}} = \lim_{x \rightarrow 0^+} (-4) \frac{\log x}{x^{-1/2}} \\ &= \lim_{x \rightarrow 0^+} (-4) \frac{1/x}{-\frac{1}{2} x^{-3/2}} = 8 \lim_{x \rightarrow 0^+} \sqrt{x} = 0 \end{aligned}$$

S<sub>3</sub> : Since  $0 < \frac{\sin x}{x} < 1$  for all x near to 0

$$\therefore \left\{ \frac{\sin x}{x} \right\} = \frac{\sin x}{x}$$

$$\therefore \lim_{x \rightarrow 0} \left\{ \frac{\sin x}{x} \right\} = 1$$

S<sub>4</sub> : When  $x \rightarrow \infty$ , through the values  $n\pi$ ,  $n \in \mathbb{N}$

$$\text{then } \frac{\ln \cos^2 x}{x^2} = 0 \quad \therefore S_4 \text{ is false}$$

11. (A, B)

$$(A) \lim_{x \rightarrow 0^-} \left[ x + \frac{1}{2} \right] = 0, \lim_{x \rightarrow 0^+} \left[ 2x + \frac{1}{3} \right] = 0$$

$$\therefore \lim_{x \rightarrow 0} [f(x)] = 0$$

$$(B) \lim_{x \rightarrow 0^-} \left( x + \frac{1}{2} \right) = \frac{1}{2}, \lim_{x \rightarrow 0^+} \left( 2x + \frac{1}{3} \right) = \frac{1}{3}$$

$$\therefore f(0^-) \neq f(0^+) \quad \therefore \lim_{x \rightarrow 0} f(x) \text{ does not exist}$$

(C)  $\lim_{x \rightarrow 0} f(x)$  does not exist so  $\left[ \lim_{x \rightarrow 0} f(x) \right]$  does not exist.

$$(D) \lim_{x \rightarrow 0^-} \left[ \frac{x + \frac{1}{2}}{x} \right] = 0, \lim_{x \rightarrow 0^+} \left[ \frac{2x + \frac{1}{3}}{x} \right] = 0$$

$$\therefore \lim_{x \rightarrow 0} \frac{[f(x)]}{x} = 0$$

12. (A, B)

$$\lim_{t \rightarrow 0} \frac{\sin(\tan t)}{\sin t} = \lim_{t \rightarrow 0} \frac{\tan t}{t} = 1$$

$$\lim_{t \rightarrow \pi/2} \frac{\sin(\cos x)}{\cos x} = 1$$

$$\lim_{t \rightarrow 0} \frac{|x|}{x} = \text{DNE}$$

$$\lim_{t \rightarrow \pi/2} \frac{1 - \cos x}{x^2} = 0$$

13. (B, C)

By standard results

14. (C)

$$f(x) = \left(\frac{x}{2+x}\right)^{2x}$$

$$\Rightarrow \ell \lim_{x \rightarrow \infty} f(x) = \ell \lim_{x \rightarrow \infty} \left(\frac{x}{2+x}\right)^{2x}$$

$$= \ell \lim_{x \rightarrow \infty} \left(1 + \frac{x}{2+x} - 1\right)^{2x}$$

$$= e^{x \lim_{x \rightarrow \infty} 2x \left(\frac{-2}{2+x}\right)} = e^{x \lim_{x \rightarrow \infty} -4 \left(\frac{x}{2+x}\right)} = e^{-4}$$

Also  $\ell \lim_{x \rightarrow 1} f(x) = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

15. (A, D)

If  $x \in \mathbb{Q}$

$n! \pi x \rightarrow$  multiple of  $\pi$

$\cos(n! \pi x) \rightarrow \pm 1$

$1 + 1 = 2$

if  $x \notin \mathbb{Q}$   $\cos((n! \pi x))$  be any number between  $-1$  &  $1$

$$\lim_{m \rightarrow \infty} [1 + [\text{any no between } -1 \text{ \& } 1]^{2m}]$$

$$[1] = 1$$

16. (B)

$$\ell \lim_{x \rightarrow 0^-} \sin^{-1} \{x\} = \ell \lim_{h \rightarrow 0} \sin^{-1} \{0 - h\}$$

$$= \ell \lim_{h \rightarrow 0} \sin^{-1} (1 - \{h\}) = \frac{\pi}{2}$$

$$\ell \lim_{x \rightarrow 0^+} \sin^{-1} \{x\} = \ell \lim_{h \rightarrow 0} \sin^{-1} \{0 + h\} = \ell \lim_{h \rightarrow 0} \sin^{-1} \{h\} = 0$$

$f(0^-) \neq f(0^+)$  so limit does not exist

Statement-I is true, Statement-II is true but not explaining statement -I.

17. (D)

Obvious

18. (A)

Statement-I :

$$\lim_{x \rightarrow \infty} \left\{ 1 + \left( \frac{a_1^{1/x} + a_2^{1/x} + a_3^{1/x} + \dots + a_n^{1/x}}{n} - 1 \right) \right\}^{nx}$$

$$= \lim_{x \rightarrow \infty} e^{\frac{a_1^{1/x} - 1}{1/x}} \lim_{x \rightarrow \infty} e^{\frac{a_2^{1/x} - 1}{1/x}} \dots \lim_{x \rightarrow \infty} e^{\frac{a_n^{1/x} - 1}{1/x}}$$

$$= e^{\ell na_1} e^{\ell na_2} \dots e^{\ell na_n}$$

$$= a_1 \cdot a_2 \dots a_n$$

Statement-II : Similarly

19. (A)

Statement-II is true. Also  $\ell \lim_{x \rightarrow 1^+} \frac{3x^2 + 1}{(x-1)(x-2)} = +\infty$

and  $\ell \lim_{x \rightarrow 2^-} \frac{3x^2 + 1}{(x-1)(x-2)} = -\infty$

$$\Rightarrow f(1^+) = 0 \text{ \& } f(2^-) = 2$$

20. (C)

$$\lim_{x \rightarrow \infty} \left( \frac{1^2}{x^3} + \frac{2^2}{x^3} + \frac{3^2}{x^3} + \dots + \frac{x^2}{x^3} \right)$$

$$\Rightarrow S_n = \frac{1^2 + 2^2 + 3^2 + \dots + x^2}{x^3}$$

$$S_n = \lim_{x \rightarrow \infty} \frac{x(x+1)(2x+1)}{x^3} \Rightarrow S_n = \frac{\left(1 + \frac{1}{x}\right) \left(2 + \frac{1}{x}\right)}{6}$$

$$S_n = \frac{1}{3}$$

Statement-II is correct only when R.H.S. does not take any indeterminate form.

21. (A) → (t), (B) → (r), (C) → (q), (D) → (p)

$$\begin{aligned} \text{(A)} \quad \ell \lim_{x \rightarrow 1} \int_1^{x^2} \frac{f(t) - t}{(x-1)^2} dt &= \ell \lim_{x \rightarrow 1} \frac{\int_1^{x^2} (f(t) - t) dt}{(x-1)^2} \\ &= \ell \lim_{x \rightarrow 1} \frac{2x(f(x^2) - x^2)}{2(x-1)} \\ &= \ell \lim_{x \rightarrow 1} \frac{f(x^2) - x^2 + 2x^2 f'(x^2) - 2x^2}{1} = 4 \end{aligned}$$

$$\begin{aligned} \text{(B)} \quad \ell \lim_{n \rightarrow \infty} \left( \frac{1 + \sqrt[n]{4}}{2} \right)^n &= \ell \lim_{n \rightarrow \infty} \left( 1 + \frac{\sqrt[n]{4} - 1}{2} \right)^n \\ &= e^{n \lim_{n \rightarrow \infty} \frac{4^{1/n} - 1}{2} \cdot n} = e^{\ell \lim_{n \rightarrow \infty} \frac{1}{2} \frac{4^{1/n} - 1}{1/n}} = e^{\frac{1}{2} \ell n 4} = 2 \end{aligned}$$

$$\begin{aligned} \text{(C)} \quad f(x) &= \ell \lim_{n \rightarrow \infty} \frac{2x}{\pi} \tan^{-1}(nx) = x, \quad x > 0 \\ \therefore \ell \lim_{x \rightarrow 0^+} [f(x) - 1] &= \ell \lim_{x \rightarrow 0^+} [x - 1] = -1 \end{aligned}$$

$$\text{(D)} \quad \ell \lim_{n \rightarrow \infty} \left[ \sum_{r=1}^n \frac{1}{2^r} \right] = \ell \lim_{n \rightarrow \infty} \left[ 1 - \frac{1}{2^n} \right] = 0$$

22. (A) → (r), (B) → (q), (C) → (p), (D) → (q)

$$\begin{aligned} \text{(A)} \quad \ell \lim_{x \rightarrow 0} \left( \frac{f^2(a+x)}{f(a)} \right)^{1/x} &= e^{\ell \lim_{x \rightarrow 0} \left( \frac{f^2(a+x) - f^2(a)}{x f^2(a)} \right)} \\ &= e^{\ell \lim_{x \rightarrow 0} \frac{f^2(a+x) - (f(a))^2}{x}} = e^4 \\ \therefore k &= 4 \end{aligned}$$

$$\begin{aligned} \text{(B)} \quad \ell \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{\cos(\tan^{-1}(\tan x))}{x - \frac{\pi}{2}} &= \ell \lim_{h \rightarrow 0^+} \frac{\cos(\tan^{-1}(\tan(\frac{\pi}{2} + h)))}{h} \\ &= \ell \lim_{h \rightarrow 0^+} \frac{\cos\left(h - \frac{\pi}{2}\right)}{h} = \ell \lim_{h \rightarrow 0^+} \frac{\sinh h}{h} = 1 \end{aligned}$$

$$\text{(C)} \quad \ell \lim_{x \rightarrow \pi} \frac{\sin(\cos x + 1)}{\cos\left(\frac{x}{2}\right)} = \ell \lim_{x \rightarrow \pi} \frac{\sin\left(2 \cos^2\left(\frac{x}{2}\right)\right)}{\left(2 \cos^2 \frac{x}{2}\right)}$$

$$\left(2 \cos \frac{x}{2}\right) = 1 \times 0 = 0$$

$$\begin{aligned} \text{(D)} \quad \ell \lim_{x \rightarrow 0} \frac{x e^{\sin x} - e^x \sin^{-1}(\sin x)}{\sin^2 x - x \sin x} \\ = \ell \lim_{x \rightarrow 0} \frac{x e^x}{\sin x} \cdot \frac{e^{\sin x} - 1}{\sin x - x} = 1 \end{aligned}$$

23.  $\lim_{x \rightarrow 0^+} f(x) = \text{finite}$

$$\Rightarrow \lim_{h \rightarrow 0} f(0 + h) = \text{finite}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{\sinh + a e^h + b e^{-h} + c \ln(1+h)}{h^3} = \text{finite}$$

at  $h \rightarrow 0$ , Numerator  $\rightarrow a + b$  and Denominator  $\rightarrow 0$

RHS is finite

$$\therefore a + b = 0$$

$$\Rightarrow b = -a \quad \dots\dots\text{(i)}$$

$$\text{Now, } \lim_{h \rightarrow 0} \frac{\sinh + a e^h - a e^{-h} + c \ln(1+h)}{h^3} = \text{finite}$$

$$\begin{aligned} \Rightarrow \lim_{h \rightarrow 0} \frac{\frac{\sinh}{h} + \frac{a(e^h - 1)}{h} - a \frac{(e^{-h} - 1)}{h} + \frac{c \ln(1+h)}{h}}{h^2} \\ = \text{finite} \end{aligned}$$

at  $h \rightarrow 0$ , Numerator  $\rightarrow 1 + a + a + c$  and

Denominator  $\rightarrow 0$

RHS is finite

$$\therefore 1 + 2a + c = 0$$

$$\Rightarrow c = -1 - 2a \quad \dots\dots\text{(ii)}$$

$$\therefore \lim_{h \rightarrow 0} \frac{\sinh + a e^h - a e^{-h} - (1 + 2a) \ln(1+h)}{h^3} = \text{finite}$$

(by L'Hospital's rule)  $\left( \text{from} \left( \frac{0}{0} \right) \right)$

$$\lim_{h \rightarrow 0} \frac{\cosh + a e^h + a e^{-h} - \frac{(1 + 2a)}{(1+h)}}{3h^2}$$

$$\lim_{h \rightarrow 0} \frac{-\sinh + ae^h - ae^{-h} + \frac{(1+2a)}{(1+h^2)}}{6h}$$

Numerator must be 0

$$0 + a - a + \frac{1+2a}{(1)^2} = 0$$

$$\therefore a = -\frac{1}{2}$$

23.

1. (A)

$$\text{From eq. (i), } b = \frac{1}{2}$$

2. (B)

$$\text{and from eq. (ii), } c = -1 + 1 = 0$$

3. (C)

24. 1. (B) 2. (C) 3. (D)

$$f(x) = \lim_{n \rightarrow \infty} \left( \cos \sqrt{\frac{x}{n}} \right)^n = \lim_{n \rightarrow \infty} \left( 1 + \left( \cos \sqrt{\frac{x}{n}} - 1 \right) \right)^n$$

$$= e^{\lim_{n \rightarrow \infty} \left( \cos \sqrt{\frac{x}{n}} - 1 \right) n} = e^{-\lim_{n \rightarrow \infty} 2 \sin^2 \left( \frac{1}{2} \sqrt{\frac{x}{n}} \right) n}$$

$$= e^{-2 \lim_{n \rightarrow \infty} \frac{\left( \frac{1}{2} \sqrt{\frac{x}{n}} \right)^2}{\frac{1}{n}}} = e^{-2 \lim_{n \rightarrow \infty} \frac{\frac{x}{4n}}{\frac{1}{n}}} = e^{-\frac{x}{2}}$$

$$y = f(x) = e^{-x/2}, x \geq 0 \quad \text{range} = (0, 1]$$

$$g(x) = \lim_{n \rightarrow \infty} (1 - x + x \sqrt[n]{e})^n$$

$$= e^{\lim_{n \rightarrow \infty} x \frac{(e^n - 1)}{1/n}} = e^x \quad \forall x \in \mathbb{R}$$

$$h(x) = \tan^{-1} (g^{-1} (f^{-1}(x)))$$

$$-\frac{x}{2} = \ln y \quad \Rightarrow \quad x = 2 \ln \frac{1}{y}$$

$$\Rightarrow f^{-1}(x) = 2 \ln \frac{1}{x}; \quad 0 < x \leq 1$$

$$y = g(x) = e^x$$

$$x = \ln y \quad \Rightarrow \quad g^{-1}(x) = \ln x$$

$$\therefore g^{-1} \left( 2 \ln \frac{1}{x} \right) = \ln \left( 2 \ln \left( \frac{1}{x} \right) \right) \quad \text{for } 0 < x < 1$$

$$\therefore h(x) = \tan^{-1} \left( \ln \left( \ln \frac{1}{x^2} \right) \right) \quad \text{for } 0 < x < 1$$

$$1. \quad \lim_{x \rightarrow 0^+} \frac{\ln f(x)}{\ln g(x)} = \lim_{x \rightarrow 0^+} \frac{-x/2}{x} = -\frac{1}{2}$$

2. domain of h(x) is (0, 1)

$$3. h(x) = \tan^{-1} (\ln (\ln 1/x^2)) \quad 0 < x < 1$$

$$1 < \frac{1}{x^2} < \infty \quad \Rightarrow \quad 0 < \ln \frac{1}{x^2} < \infty$$

$$\therefore -\infty < \ln (\ln (1/x^2)) < \infty$$

$$\therefore \text{range of } h(x) \text{ is } (-\pi/2, \pi/2)$$

$$25. a = \lim_{n \rightarrow \infty} \lim_{\alpha \rightarrow 1^+} \frac{\alpha^n |\sin x| + \alpha^{-n} |\cos x|}{\alpha^n + \alpha^{-n}}$$

$$= \lim_{n \rightarrow \infty} \lim_{\alpha \rightarrow 1^+} \frac{|\sin x| + \alpha^{-2n} |\cos x|}{1 + \alpha^{-2n}} = |\sin x|$$

$$b = \lim_{n \rightarrow \infty} \lim_{\alpha \rightarrow 1^-} \frac{\alpha^n |\sin x| + \alpha^{-n} |\cos x|}{\alpha^n + \alpha^{-n}}$$

$$= \lim_{n \rightarrow \infty} \lim_{\alpha \rightarrow 1^-} \frac{\alpha^{2n} |\sin x| + |\cos x|}{\alpha^{2n} + 1} = |\cos x|$$

$$c = \lim_{n \rightarrow \infty} \frac{\pi}{4n} \left[ 1 + \cos \frac{\pi}{2n} + \cos \frac{2\pi}{2n} + \dots + \cos \frac{(n-1)\pi}{2n} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{\pi}{4n} \left( \frac{\sin \frac{\pi}{4} \cos \frac{(n-1)\pi}{4n}}{\sin \frac{\pi}{4n}} \right) = \sin \frac{\pi}{4} \cos \frac{\pi}{4} = \frac{1}{2}$$

$$\therefore b + c - \frac{1}{2} = |\cos x|$$

$$f(x) = \max \{ |\sin x|, |\cos x|, \frac{1}{2} \}$$

$$\therefore \text{range of } f(x) \text{ is } \left[ \frac{1}{\sqrt{2}}, 1 \right]$$

26. (7)

$$\lim_{x \rightarrow \infty} (\sqrt{x^4 + ax^3 + 3x^2 + bx + 2} - \sqrt{x^4 + 2x^3 - cx^2 + 3x - d}) = 4$$

rationalizing

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{(a-2)x^3 + (3+c)x^2 + (b-3)x + (2+d)}{\sqrt{x^4 + ax^3 + 3x^2 + bx + c} + \sqrt{x^4 + 2x^3 - cx^2 + 3x - d}} = 4$$

Limit is finite so degree of  $N_r$  must be 2

So,  $a - 2 = 0 \Rightarrow a = 2$

then divide  $N_r$  &  $D_r$  both by  $x^2$

$$\lim_{x \rightarrow \infty} \frac{(3+c) + (b-3)/x + (2+d)/x^2}{\sqrt{1 + \frac{a}{x} + \frac{3}{x^2} + \frac{b}{x^3} + \frac{2}{x^4}} + \sqrt{1 + \frac{2}{x} - \frac{c}{x^2} + \frac{3}{x^3} - \frac{d}{x^4}}} = 4$$

$$\Rightarrow \frac{3+c}{2} = 4 \Rightarrow c = 5, a = 2 \text{ and } b, d \in \mathbb{R}$$

27.  $\lim_{x \rightarrow 0} \frac{\sin 3x}{x^3} + \frac{a}{x^2} + b = \lim_{x \rightarrow 0} \frac{\sin 3x + ax + bx^3}{x^3}$

$$= \lim_{x \rightarrow 0} \frac{3 \frac{\sin 3x}{3x} + a + bx^2}{x^2}$$

for existence of limit  $3 + a = 0 \Rightarrow a = -3$

$$\therefore l = \lim_{x \rightarrow 0} \frac{\sin 3x - 3x + bx^3}{x^3}$$

$$= 27 \cdot \frac{\sin t - t}{t^3} + b = 0 \quad (3x = t)$$

$$= -\frac{27}{6} + b = 0 \Rightarrow b = \frac{9}{2}$$

$$a + 2b = 6$$

28. (3)

$$\lim_{x \rightarrow 0} ([f(x)] + x^2)^{\frac{1}{f(x)}} = \lim_{x \rightarrow 0} (1 + x^2)^{\frac{1}{\tan x - 1}}$$

$$= \lim_{x \rightarrow 0} (1 + x^2)^{\frac{x}{\tan x - x}}$$

$$= e^{\lim_{x \rightarrow 0} \frac{x^3}{\tan x - x}} = e^{\lim_{x \rightarrow 0} \frac{3x^2}{\sec^2 x - 1}}$$

$$= e^{\lim_{x \rightarrow 0} \frac{6x}{2\sec^2 x \tan x}} = e^3$$

29. Limit  $\frac{\cot^{-1}\left(\frac{\log_a x}{x^a}\right)}{\sec^{-1}\left(\frac{a^x}{\log_a x}\right)}$  ; as  $\left(\frac{\log_a x}{x^a}\right) \rightarrow 0$

and  $\left(\frac{a^x}{\log_a x}\right) \rightarrow \infty$  (using L'opital rule)

$$\therefore l = \frac{\pi/2}{\pi/2} = 1$$

30. (2)

$$P = \frac{\prod_{r=1}^n \left( (2r-1)^4 + \frac{1}{4} \right)}{\prod_{r=1}^n \left( (2r)^4 + \frac{1}{4} \right)}$$

$$\therefore (2r-1)^4 + \frac{1}{4} = (2r-1)^4 + \frac{1}{4} + (2r-1)^2 - (2r-1)^2$$

$$= \left( (2r-1)^2 + \frac{1}{2} \right)^2 - (2r-1)^2$$

$$= \left( (2r-1)^2 + (2r-1) + \frac{1}{2} \right) \left( (2r-1)^2 - (2r-1) + \frac{1}{2} \right)$$

$$\Rightarrow (2r-1)^4 + \frac{1}{4}$$

$$= \left( (2r-1)2r + \frac{1}{2} \right) \left( (2r-1)(2r-2) + \frac{1}{2} \right) \dots\dots(i)$$

Now  $(2r)^4 + \frac{1}{4} = (2r)^4 + \frac{1}{4} + (2r)^2 - (2r)^2$

$$= \left( (2r)^2 + \frac{1}{2} \right)^2 - (2r)^2$$

$$= \left( (2r)^2 + 2r + \frac{1}{2} \right) \left( (2r)^2 - 2r + \frac{1}{2} \right)$$



$$\Rightarrow (2r)^4 + \frac{1}{4} = \left(2r(2r+1) + \frac{1}{2}\right) \left(2r(2r-1) + \frac{1}{2}\right)$$

.....(ii)

Now by equation (i) and (ii)

$$\therefore P = \frac{\prod_{r=1}^n \left( (2r-1)(2r-2) + \frac{1}{2} \right)}{\prod_{r=1}^n \left( 2r(2r+1) + \frac{1}{2} \right)} =$$

$$\frac{\frac{1}{2} \cdot \left(3 \cdot 2 + \frac{1}{2}\right) \left(5 \cdot 4 + \frac{1}{2}\right) \dots \dots \left( (2n-1)(2n-2) + \frac{1}{2} \right)}{\left(2 \cdot 3 + \frac{1}{2}\right) \left(4 \cdot 5 + \frac{1}{2}\right) \dots \dots \left( (2n-2)(2n-1) + \frac{1}{2} \right) \left( 2n(2n+1) + \frac{1}{2} \right)}$$

$$\Rightarrow P = \frac{1}{2 \cdot \left( 2n(2n+1) + \frac{1}{2} \right)}$$

.....(iii)

as  $\lim_{n \rightarrow \infty} n^\alpha P$  exists

$$\Rightarrow \lim_{n \rightarrow \infty} n^\alpha \frac{1}{2 \cdot \left( 2n(2n+1) + \frac{1}{2} \right)}$$
 exists

$$\Rightarrow \alpha = 2 \quad \text{and} \quad \lim_{n \rightarrow \infty} n^\alpha P = \frac{1}{8}$$