LIMIT

DEFINITION

The value to which f(x) approaches, as x tends to 'a' from the left hand side ($x \to a^-$) is called left hand limit of f(x) at x = a. Symbolically, LHL = $\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$.

The value to which f(x) approaches, as x tends to 'a' from the right hand side ($x \to a^+$) is called right hand limit of f(x) at x = a. Symbolically, RHL = $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$.

Limit of a function f(x) is said to exist as, $x \to a$ when $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ = Finite quantity.

FUNDAMENTAL THEOREMS ON LIMITS

Let $\lim_{x \to a} f(x) = l$ & $\lim_{x \to a} g(x) = m$. If l & m exists finitely then :

(A) Sum rule : $\lim_{x \to a} \{f(x) + g(x)\} = l + m$

(B) Difference rule : $\lim_{x \to a} \{f(x) - g(x)\} = l - m$

(C) Product rule : $\lim_{x \to a} f(x) . g(x) = l . m$

(D) Quotient rule : $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m}$ provided $m \neq 0$

(E) Constant multiple rule : $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$; where k is constant.

(F) Power rule : If m and n are integers then $\lim_{x \to a} [f(x)]^{m/n} = l^{m/n}$ provided $l^{m/n}$ is a real number.

(G) $\lim_{x \to a} f[g(x)] = f\left(\lim_{x \to a} g(x)\right) = f(m)$; provided f(x) is continuous at x = m.

For example : $\lim_{x \to a} \ell n(g(x)) = \ell n[\lim_{x \to a} g(x)] = \ell n\,(m)$; provided ℓnx is continuous at x = m, m = $\lim_{x \to a} g(x)$.

Ex. If $\lim_{x \to a} [f(x) + g(x)] = 2$ and $\lim_{x \to a} [f(x) - g(x)] = 1$, then find the value of $\lim_{x \to a} (f)x\, g(x)$.

Sol. $\lim_{x \to a} [f(x) + g(x)] = 2$

or $\lim_{x \to a} (f)x + \lim_{x \to a} g(x) = 2$ ....**(i)**

$\lim_{x \to a} [f(x) - g(x)] = 1$

or $\lim_{x \to a} f(x) - \lim_{x \to a} g(x) = 1$ ....**(ii)**

Adding **(i)** and **(ii)**,

$2 \lim_{x \to a} f(x) = 3$ or $\lim_{x \to a} f(x) = \frac{2}{3}$

Subtracting **(ii)** from **(i)**,

$2 \lim_{x \to a} g(x) = 1$ or $\lim_{x \to a} g(x) = \frac{1}{2}$

or $\lim_{x \to a} f(x)\, g(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x) = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4}$

INDETERMINATE FORMS

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, 1^{\infty}, 0^0, \infty^0.$

Initially we will deal with first five forms only.

(i) Here 0, 1 are not exact, infact both are aproaching to their corresponding values.

(ii) We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number It does not obey the laws of elementary algebra,

(A) $\infty + \infty \to \infty$ **(B)** $\infty \times \infty \to \infty$ **(C)** $\infty^{\infty} \to \infty$ **(D)** $0^{\infty} \to 0$

(iii) $\frac{a}{\infty} = 0$, if a is finite.

(iv) $\frac{a}{0}$ is not defined for any $a \in R$.

(v) $\lim_{x \to 0} \frac{x}{x}$ is an indeterminate form whereas $\lim_{x \to 0} \frac{[x^2]}{x^2}$ is not an indeterminate form (where [.] represents greatest integer function)

Please remember these forms alongwith the prefix 'tending to'

i.e. $\frac{\text{tending to zero}}{\text{tending to zero}}$ is an indeterminate form whereas $\frac{\text{exactly zero}}{\text{tending to zero}}$ is not an indeterminate form, its value is zero.

Similarly $(\text{tending to one})^{\text{tending to } \infty}$ is indeterminate form whereas $(\text{exactly one})^{\text{tending to } \infty}$ is not an indeterminate form, its value is one.

EVALUATION OF ALGEBRIC LIMIT

Factorisation Method

We can cancel out the factors which are leading to indeterminancy and find the limit of the remaining expression.

Important factors

(i) $x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + ... + a^{n-1}), n \in N$

(ii) $x^n + a^n = (x + a)(x^{n-1} - ax^{n-2} + ... + a^{n-1})$, n is an odd natural number.

Ex. Evaluate $\lim\limits_{x\to 2}\dfrac{x^2-5x+6}{x^2-4}$

Sol. When x = 2, the expression $\dfrac{x^2-5x+6}{x^2-4}$ assumes the indeterminate from $\dfrac{0}{0}$. Here, (x – 2) is a common factor in numerator and denominator. Factorizing the numerator and denominator, we have

$$\lim_{x\to 2}\frac{x^2-5x+6}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(x-3)}{(x+2)(x-2)}$$

$$=\lim_{x\to 2}\frac{x-3}{x+2}=\frac{2-3}{2+2}=\frac{1}{4}$$

Ex. Evaluate $\lim\limits_{x\to 2}\dfrac{2^x+2^{3-x}-6}{2^{-x/2}-2^{1-x}}$.

Sol. $$\lim_{x\to 2}\frac{2^x+2^{3-x}-6}{2^{-x/2}-2^{1-x}}=\lim_{x\to 2}\frac{\dfrac{\left(2^{2x}+2^3-6.2^x\right)}{2^x}}{\dfrac{1}{2^{x/2}}-\dfrac{2}{2^x}}=\lim_{x\to 2}\frac{2^{2x}-6.2^x+8}{2^{x/2}-2}=\lim_{x\to 2}\frac{\left(2^x-4\right)\left(2^x-2\right)}{\left(2^{x/2}-2\right)}$$

$$=\lim_{x\to 2}\left(2^{\frac{x}{2}}+2\right)\left(2^x-2\right)=\left(2+2\right).\left(4-2\right)=8$$

Rationalisation Method

We can rationalise the irrational expression in numerator or denominator or in both to remove the indeterminancy.

Ex. The value of $\lim\limits_{x\to 0}\dfrac{\sin x+\log_e\left(\sqrt{1+\sin^2 x}-\sin x\right)}{\sin^3 x}$

Sol. Let sin x = t. Therefore, $\lim\limits_{x\to 0}\dfrac{\sin x+\log_e\left(\sqrt{1+\sin^2 x}-\sin x\right)}{\sin^3 x}$

$$=\lim_{t\to 0}\frac{t+\log_e\left(\sqrt{1+t^2}-t\right)}{t^3}$$

$$=\lim_{t\to 0}\frac{1+\dfrac{1}{\sqrt{1+t^2}-t}\left(\dfrac{t}{\sqrt{1+t^2}}-1\right)}{3t^2}$$ [L'Hospital's rule]

$$=\lim_{t\to 0}\frac{1-\dfrac{1}{\sqrt{1+t^2}}}{3t^2}\quad\Rightarrow\quad\lim_{t\to 0}\frac{\sqrt{1+t^2}-1}{3t^2\sqrt{1+t^2}}$$

$$=\lim_{t\to 0}\frac{(1+t^2)-1}{3t^2}\times\frac{1}{\left(\sqrt{1+t^2}+1\right)}\quad\Rightarrow\quad\lim_{t\to 0}\frac{1}{3}\frac{1}{\left(\sqrt{1+t^2}+1\right)}=\frac{1}{6}$$

Ex. Evaluate : $\lim_{x\to 1}\left(\frac{\sqrt{x^2+8}-\sqrt{10-x^2}}{\sqrt{x^2+3}-\sqrt{5-x^2}}\right)$

Sol. This is of the form $\frac{3-3}{2-2}=\frac{0}{0}$ if we put x = 1

To eliminate the $\frac{0}{0}$ factor, multiply by the conjugate of numerator and the conjugate of the denominator

$$\therefore \quad \text{Limit} = \lim_{x\to 1}\left(\sqrt{x^2+8}-\sqrt{10-x^2}\right)\frac{(\sqrt{x^2+8}+\sqrt{10-x^2})}{(\sqrt{x^2+8}+\sqrt{10-x^2})}\times\frac{(\sqrt{x^2+3}+\sqrt{5-x^2})}{(\sqrt{x^2+3}+\sqrt{5-x^2})(\sqrt{x^2+3}-\sqrt{5-x^2})}$$

$$=\lim_{x\to 1}\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\times\frac{(x^2+8)-(10-x^2)}{(x^2+3)-(5-x^2)}=\lim_{x\to 1}\left(\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\right)\times 1=\frac{2+2}{3+3}=\frac{2}{3}$$

STANDARD LIMITS

(A) **(i)** $\ell im_{x\to 0}\frac{\sin x}{x}=\ell im_{x\to 0}\frac{\tan x}{x}=1$ [Where x is measured in radians]

(ii) $\ell im_{x\to 0}\frac{\tan^{-1}x}{x}=\ell im_{x\to 0}\frac{\sin^{-1}x}{x}=1$

(iii) $\ell im_{x\to 0}(1+x)^{\frac{1}{x}}=e$; $\ell im_{x\to 0}(1+ax)^{\frac{1}{x}}=e^a$

(iv) $\ell im_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$; $\ell im_{x\to\infty}\left(1+\frac{a}{x}\right)^x=e^a$

(v) $\ell im_{x\to 0}\frac{e^x-1}{x}=1$; $\ell im_{x\to 0}\frac{a^x-1}{x}=\log_e a=\ell na$, $a>0$

(vi) $\ell im_{x\to 0}\frac{\ell n(1+x)}{x}=1$ **(vii)** $\ell im_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$

(B) If $f(x)\to 0$, when $x\to a$, then

(i) $\ell im_{x\to a}\frac{\sin f(x)}{f(x)}=1$ **(ii)** $\ell im_{x\to a}\cos f(x)=1$

(iii) $\ell im_{x\to a}\frac{\tan f(x)}{f(x)}=1$ **(iv)** $\ell im_{x\to a}\frac{e^{f(x)}-1}{f(x)}=1$

(v) $\ell im_{x\to a}\frac{b^{f(x)}-1}{f(x)}=\ell n\, b,\ (b>0)$ **(vi)** $\ell im_{x\to a}\frac{\ell n(1+f(x))}{f(x)}=1$

(vii) $\ell im_{x\to a}(1+f(x))^{\frac{1}{f(x)}}=e$

(C) $\ell im_{x\to a} f(x)=A>0$ and $\ell im_{x\to a}\phi(x)=B$(a finite quantity), then $\ell im_{x\to a}[f(x)]^{\phi(x)}=A^B$.

Ex. Evaluate : $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$

Sol. $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3} = \lim\limits_{x\to 0} \dfrac{\tan x(1-\cos x)}{x^3} = \lim\limits_{x\to 0} \dfrac{\tan x \,.\, 2\sin^2 \frac{x}{2}}{x^3}$

$$= \lim_{x\to 0} \frac{1}{2} \,.\, \frac{\tan x}{x} \,.\, \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}.$$

Ex. Evaluate : $\lim\limits_{n\to\infty} \dfrac{\sin \frac{a}{n}}{\tan \frac{b}{n+1}}$

Sol. As $n \to \infty, \frac{1}{n} \to 0$ and $\frac{a}{n}$ also tends to zero

$\sin \frac{a}{n}$ should be written as $\dfrac{\sin \frac{a}{n}}{\frac{a}{n}}$ so that it looks like $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta}$

The given limit $= \lim\limits_{n\to\infty} \left(\dfrac{\sin \frac{a}{n}}{\frac{a}{n}}\right)\left(\dfrac{\frac{b}{n+1}}{\tan \frac{b}{n+1}}\right) . \dfrac{a(n+1)}{n.b}$

$$= \lim_{n\to\infty} \left(\frac{\sin \frac{a}{n}}{\frac{a}{n}}\right)\left(\frac{\frac{b}{n+1}}{\tan \frac{b}{n+1}}\right) . \frac{a}{b}\left(1+\frac{1}{n}\right) = 1 \times 1 \times \frac{a}{b} \times 1 = \frac{a}{b}$$

Ex. Evaluate $\lim\limits_{x\to 1} \dfrac{\sqrt{x} + \sqrt{\sqrt{x}} + \sqrt{\sqrt{\sqrt{x}}} + \sqrt{\sqrt{\sqrt{\sqrt{x}}}} - 4}{x-1}$

Sol. $\lim\limits_{x\to 1} \dfrac{\sqrt{x} + \sqrt{\sqrt{x}} + \sqrt{\sqrt{\sqrt{x}}} + \sqrt{\sqrt{\sqrt{\sqrt{x}}}} - 4}{x-1}$

$$= \lim_{x\to 1} \frac{x^{1/2} + x^{1/4} + x^{1/8} + x^{1/16} - 4}{x-1}$$

$$= \lim_{x\to 1} \left(\frac{x^{1/2}-1}{x-1} + \frac{x^{1/4}-1}{x-1} + \frac{x^{1/8}-1}{x-1} + \frac{x^{1/16}-1}{x-1}\right)$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \quad \Rightarrow \quad \frac{8+4+2+1}{16}$$

$$= \frac{15}{16}$$

Ex. Evaluate : $\lim\limits_{x\to 0}\dfrac{e^{\tan x}-e^{x}}{\tan x - x}$

Sol. $\lim\limits_{x\to 0}\dfrac{e^{\tan x}-e^{x}}{\tan x - x}=\lim\limits_{x\to 0}\dfrac{e^{x}\times e^{(\tan x - x)}-e^{x}}{\tan x - x}$

$$=\lim_{x\to 0}\frac{e^{x}(e^{\tan x - x}-1)}{\tan x - x} = \lim_{x\to 0\, y\to 0}\frac{e^{x}(e^{y}-1)}{y}$$

where $y = \tan x - x$ and $\lim\limits_{y\to 0}\dfrac{e^{y}-1}{y}=1$

$= e^{0}\times 1$ [as $x\to 0$, $\tan x - x \to 0$]

$= 1\times 1 = 1$

Ex. Let $P_n = a^{P_{n-1}}-1$, $\forall$ n = 2, 3, ..., and let $P_1 = a^{x}-1$, where $a \in R^{+}$. Then evaluate $\lim\limits_{x\to 0}\dfrac{P_n}{x}$.

Sol. Clearly, if $P_k \to 0$, then $P_{k+1}\to 0$.
Now, as $x \to 0$, we get $P_1 \to 0$ or $P_2, P_3, P_4, \ldots, P_n \to 0$.
Therefore,

$$\lim_{x\to 0}\frac{P_n}{x}=\lim_{x\to 0}\frac{P_n}{P_{n-1}}\frac{P_{n-1}}{P_{n-2}}\ldots\ldots\frac{P_1}{x}$$

Now, $\lim\limits_{x\to 0}\dfrac{P_k}{P_{k-1}}=\lim\limits_{x\to 0}\dfrac{a^{P_{k-1}}-1}{P_{k-1}}=\ln a$

$\therefore$ Required limit $= (\ln a)^{n}$

Using Substitution

$\lim\limits_{x\to a} f(x)=\lim\limits_{h\to 0} f(a-h)$ or $\lim\limits_{h\to 0} f(a+h)$ i.e. by substituting x by $a-h$ or $a+h$

Ex. Evaluate : $\lim\limits_{x\to\frac{\pi}{2}}(\sec x - \tan x)$

Sol. $\lim\limits_{x\to\frac{\pi}{2}}(\sec x - \tan x)$; ($\infty - \infty$ form)

$\lim\limits_{x\to\frac{\pi}{2}}\left(\dfrac{1-\sin x}{\cos x}\right)$; $\left(\text{now in } \dfrac{0}{0} \text{ form}\right)$

Put $x = \left(\dfrac{\pi}{2}+h\right)$

$$\therefore \quad \text{Limit} = \lim_{h\to 0}\left[\frac{1-\sin\left(\frac{\pi}{2}+h\right)}{\cos\left(\frac{\pi}{2}+h\right)}\right] = \lim_{h\to 0}\left[\frac{1-\cos h}{-\sin h}\right]$$

$$= \lim_{h\to 0}\left[\frac{2\sin^2\frac{h}{2}}{-2\sin\frac{h}{2}\cos\frac{h}{2}}\right] = \lim_{h\to 0}\left[\frac{\sin\frac{h}{2}}{-\cos\frac{h}{2}}\right] = 0$$

Limit when $x \to \infty$

(i) Divide by greatest power of x in numerator and denominator.

(ii) Put $x = 1/y$ and apply $y \to 0$

Ex. Evaluate : $\lim_{x\to\infty} \dfrac{x^2+x+1}{3x^2+2x-5}$

Sol. $\lim_{x\to\infty} \dfrac{x^2+x+1}{3x^2+2x-5}$, $\quad \left(\dfrac{\infty}{\infty} \text{ form}\right)$

Put $x = \dfrac{1}{y}$

$$\text{Limit} = \lim_{y\to 0} \frac{1+y+y^2}{3+2y-5y^2} = \frac{1}{3}$$

Ex. Evaluate $\lim_{x\to -\infty} \dfrac{\sqrt{3x^2+2}}{x-2}$

Sol. $\lim_{x\to -\infty} \dfrac{\sqrt{3x^2+2}}{x-2}$ $\quad$ (Put $x = -\dfrac{1}{t}$, $\quad x \to -\infty \Rightarrow t \to 0^+$)

$$= \lim_{t\to 0^+} \frac{\sqrt{3+2t^2}\,.\,\frac{1}{\sqrt{t^2}}}{\frac{(-1-2t)}{t}} = \lim_{t\to 0^+} \frac{\sqrt{3+2t^2}}{-(1+2t)} \frac{t}{|t|} = \frac{\sqrt{3}}{-1} = -\sqrt{3}\,.$$

(i) $\lim_{x\to\infty} \dfrac{\ell nx}{x} = 0$ $\quad$ **(ii)** $\lim_{x\to\infty} \dfrac{x}{e^x} = 0$ $\quad$ **(iii)** $\lim_{x\to\infty} \dfrac{x^n}{e^x} = 0$

(iv) $\lim_{x\to\infty} \dfrac{(\ell nx)^n}{x} = 0$ $\quad$ **(v)** $\lim_{x\to 0^+} x(\ell nx)^n = 0$

As $x \to \infty$, $\ell n\, x$ increases much slower than any (positive) power of x where as e^x increases much faster than any (positive) power of x.

(vi) $\lim_{n\to\infty} (1-h)^n = 0$ and $\lim_{n\to\infty} (1+h)^n \to \infty$, where $h \to 0^+$.

LIMITS USING EXPANSION

(i) $a^x = 1 + \frac{x\,\ell n\, a}{1!} + \frac{x^2\,\ell n^2 a}{2!} + \frac{x^3\,\ell n^3 a}{3!} + ..., a > 0$

(ii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

(iii) $\ell n\,(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ..., \text{ for } -1 < x \le 1$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$

(v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

(vi) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$

(vii) $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

(viii) $\sin^{-1}x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + ...$

(ix) $\sec^{-1}x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + ...$

(x) for $|x| < 1, n \in R$; $(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + ...\infty$

(xi) $(1+x)^{\frac{1}{x}} = e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 - ....\right)$

Ex. Evaluate $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$

Sol. $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x \to 0} \frac{\left(x + \frac{x^3}{3} + ...\right) - \left(x - \frac{x^3}{3!} + ...\right)}{x^3} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.$

Ex. Find the values of a,b and c so that $\lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$

Sol. $\lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$ .....**(i)**

at $x \to 0$ numerator must be equal to zero

$\therefore$ $a - b + c = 0$ $\Rightarrow$ $b = a + c$ .....**(ii)**

From (i) & (ii), $$\lim_{x\to 0}\frac{ae^{x}-(a+c)\cos x+ce^{-x}}{x\sin x}=2$$

$$\Rightarrow \lim_{x\to 0}\frac{a\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+......\right)-(a+c)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-......\right)+c\left(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+......\right)}{x\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-.....\right)}=2$$

$$\Rightarrow \lim_{x\to 0}\frac{\frac{(a-c)}{x}+(a+c)+\frac{x}{3!}(a-c)+.....}{\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-......\right)}=2$$

Since R.H.S is finite,

$\therefore$ $a-c=0$

$\therefore$ $a=c$, then $\frac{0+2a+0+....}{1}=2$

$\therefore$ $a=1$ then $c=1$

From (ii),

$b=a+c=1+1=2$

Ex. $$\lim_{x\to 0}\frac{e^{x}-e^{-x}-2x}{x-\sin x}$$

Sol. $$\lim_{x\to 0}\frac{e^{x}-e^{-x}-2x}{x-\sin x}$$

$$\Rightarrow \lim_{x\to 0}\frac{1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+......-\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+.....\right)-2x}{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}.....\right)}$$

$$\Rightarrow \lim_{x\to 0}\frac{2.\frac{x^3}{6}+2.\frac{x^5}{5!}+......}{\frac{x^3}{6}+\frac{x^5}{5!}.....} \Rightarrow \lim_{x\to 0}\frac{x^3\left(\frac{1}{3}+\frac{1}{60}x^2+.....\right)}{x^3\left(\frac{1}{6}+\frac{1}{120}x^2+.....\right)}=\frac{1/3}{1/6}=2$$

Limit of the form $\lim_{x\to a}(f(x))^{g(x)}$: Form $0^0, \infty^0$

Let $L=\lim_{x\to a}(f(x))^{g(x)}$. Then,

$$\log_e L=\log_e\left[\lim_{x\to a}\{f(x)\}^{g(x)}\right]$$

$$=\lim_{x\to a}\left[\log_e\{f(x)\}^{g(x)}\right] \Rightarrow \lim_{x\to a} g(x)\log_e[f(x)]$$

or $$L=e^{\lim_{x\to a} g(x)\log_e f(x)}$$

Ex. Evaluate $\lim_{x\to\infty} x^{1/x}$.

Sol. $\lim_{x\to\infty} x^{1/x} = e^{\log \lim_{x\to\infty} x^{1/x}}$

$= e^{\lim_{x\to\infty} \log x^{1/x}}$

$= e^{\lim_{x\to\infty} \frac{\log x}{x}}$ ($\because$ x increasing faster than $\log_e x$ when $x \to \infty$)

$= e^0$

$= 1$

Form - 1^∞

$(1)^\infty$ type of problems can be solved by the following method

(i) $\ell im_{x\to 0} (1+x)^{\frac{1}{x}} = e$

(ii) $\ell im_{x\to a} [f(x)]^{g(x)}$; where $f(x) \to 1$; $g(x) \to \infty$ as $x \to a$

$$= \ell im_{x\to a} [1+f(x)-1]^{\frac{1}{f(x)-1}\{f(x)-1\}.g(x)} = \ell im_{x\to a} \left([1+(f(x)-1)]^{\frac{1}{f(x)-1}}\right)^{(f(x)-1)g(x)} = e^{\ell im_{x\to a}[f(x)-1]g(x)}$$

Ex. Evaluate $\lim_{x\to 0}(1+x)^{\operatorname{cosec} x}$.

Sol. $\lim_{x\to 0}(1+x)^{\operatorname{cosec} x} = \lim_{x\to 0}\left[(1+x)^{\frac{1}{x}}\right]^{\frac{x}{\sin x}}$

$$= \left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right]^{\lim_{x\to 0}\frac{x}{\sin x}} = e^1$$

Ex. Evaluate $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{2/x}$; (a, b, c > 0)

Sol. We have, $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{2/x} = e^{\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}-1\right).\frac{2}{x}}$

$$= e^{\frac{2}{3}\lim_{x\to 0}\left(\frac{a^x+b^x+c^x-3}{x}\right)} \Rightarrow e^{\frac{2}{3}\lim_{x\to 0}\left(\frac{a^x-1}{x}+\frac{b^x-1}{x}+\frac{c^x-1}{x}\right)}$$

$$= e^{\frac{2}{3}\left\{\lim_{x\to 0}\frac{a^x-1}{x}+\lim_{x\to 0}\frac{b^x-1}{x}+\lim_{x\to 0}\frac{c^x-1}{x}\right\}}$$

$$= e^{(2/3)\{\ln a+\ln b+\ln c\}} = e^{(2/3)\ln(abc)} = e^{\ln(abc)^{2/3}} = (abc)^{2/3}$$

Ex. Evaluate $\lim\limits_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x}$

Sol. Since it is in the form of 1^∞ so $\lim\limits_{x\to\frac{\pi}{4}} (\tan x)^{\tan 2x} = e^{\lim\limits_{x\to\frac{\pi}{4}} (\tan x - 1)\tan 2x} = e^{\lim\limits_{x\to\frac{\pi}{4}} (\tan x - 1)\frac{2\tan x}{1-\tan^2 x}}$

$$= e^{2\times\frac{\tan \pi/4}{-1(1+\tan\pi/4)}} = e^{-1} = \frac{1}{e}$$

L'HOSPITAL'S RULE FOR EVALUATING LIMITS

Rule If $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ takes $\frac{0}{0}$ or $\frac{\infty}{\infty}$ from, then,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

where $f'(x) = \frac{df(x)}{dx}$ and $g'(x) = \frac{dg(x)}{dx}$

Ex. Evaluate $\lim\limits_{x\to 0} \log_{\tan^2 x}(\tan^2 2x)$.

Sol. $L = \lim\limits_{x\to 0}\frac{\log(\tan^2 2x)}{\log(\tan^2 x)}$ $\left(\frac{\infty}{\infty}\text{form}\right)$

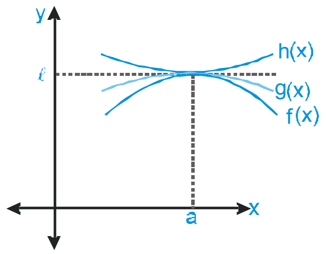
Using L'Hospital's rule, we have

$$L = \lim_{x\to 0}\frac{\left(\frac{1}{\tan^2 2x}2\tan 2x\sec^2 2x\right)\times 2}{\frac{1}{\tan^2 x}2\tan x\,.\,\sec^2 x}$$

$$= \lim_{x\to 0}\frac{2\left(\frac{1}{\sin 2x\cos 2x}\right)}{\left(\frac{1}{\sin x\cos x}\right)} = \lim_{x\to 0}\frac{\left(\frac{1}{\sin 2x\cos 2x}\right)}{\left(\frac{1}{\sin 2x}\right)}$$

$$= \lim_{x\to 0}\frac{1}{\cos 2x} = 1$$

Sandwitch Theorem or Squeeze Play Theorem



Suppose that $f(x) \le g(x) \le h(x)$ for all x in some open interval containing a, except possibly at x = a itself. Suppose also that

$$\lim_{x \to a} f(x) = \ell = \lim_{x \to a} h(x),$$

Then $\lim_{x \to a} g(x) = \ell$.

Ex. If $3 - \left(\frac{x^2}{12}\right) \le f(x) \le 3 + \left(\frac{x^3}{9}\right)$ for all $x \ne 0$, then find the value of $\lim_{x\to 0} f(x)$.

Sol. According to question.

$$\lim_{x\to 0}\left(3 - \frac{x^2}{12}\right) \le \lim_{x\to 0} f(x) \le \lim_{x\to 0}\left(3 + \frac{x^3}{9}\right)$$

or $\quad (3 - 0) \le \lim_{x\to 0} f(x) \le (3 + 0)$

Hence, $\lim_{x\to 0} f(x) = 3$ (Using sandwich theorem)

Ex. Evaluate : $\lim_{n\to\infty} \frac{[x] + [2x] + [3x] + \ldots..[nx]}{n^2}$ Where [.] denotes the greatest integer function.

Sol. We know that $x - 1 < [x] \le x$

$$\Rightarrow \quad x + 2x + \ldots..nx - n < \sum_{r=1}^{n} [rx] \le x + 2x + \ldots\ldots.. + nx$$

$$\Rightarrow \quad \frac{xn}{2}(n+1) - n < \sum_{r=1}^{n} [rx] \le \frac{x.n(n+1)}{2}$$

$$\Rightarrow \quad \frac{x}{2}\left(1 + \frac{1}{n}\right) - \frac{1}{n} < \frac{1}{n^2}\sum_{r=1}^{n} [rx] \le \frac{x}{2}\left(1 + \frac{1}{n}\right)$$

Now, $\lim_{n\to\infty} \frac{x}{2}\left(1 + \frac{1}{n}\right) = \frac{x}{2}$ and $\lim_{n\to\infty} \frac{x}{2}\left(1 + \frac{1}{n}\right) - \frac{1}{n} = \frac{x}{2}$

Thus, $\lim_{n\to\infty} \frac{[x] + [2x] + \ldots\ldots + [nx]}{n^2} = \frac{x}{2}$

TIPS & FORMULAS

1. Definition

Let f(x) be defined on an open interval about 'a' except possibly at 'a' itself. If f(x) gets arbitrarily close to L (a finite number) for all x sufficiently close to 'a' we say that f(x) approaches the limit L as x approaches 'a' and we write $\lim_{x\to a} f(x) = L$ and say "the limit of f(x), as x approaches a, equals L".

2. Left Hand Limit and Right Hand Limit of a Function

Left hand limit (LHL) = $\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)$.

Right han limit (RHL) = $\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$.

Limit of a function f(x) is said to exist as, $x \to$ a when $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ = Finite quantity.

Important note :

In $\lim_{x\to a} f(x)$, $x \to$ a necessarily implies $x \neq a$. That is while evaluating limit at x = a, we are not concerned with the value of the function at x = a. In fact the function may or may not be defined at x = a.

Also it is necessary to note that if f(x) is defined only on one side of **'x = a'**, one sided limits are good enough to establish the existence of limits, & if f(x) is defined on either side of **'a'** both sided limits are to be considered.

3. Fundamental Theorems on Limits

Let $\lim_{x\to a} f(x) = l$ & $\lim_{x\to a} g(x) = m$. If l & m exists finitely then :

(A) Sum rule : $\lim_{x\to a}\{f(x)+g(x)\} = l + m$

(B) Difference rule : $\lim_{x\to a}\{f(x)-g(x)\} = l - m$

(C) Product rule : $\lim_{x\to a} f(x) . g(x) = l . m$

(D) Quotient rule : $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{l}{m}$ provided $m \neq 0$

(E) Constant multiple rule : $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x)$; where k is constant.

(F) Power rule : If m and n are integers then $\lim_{x\to a}[f(x)]^{m/n} = l^{m/n}$ provided $l^{m/n}$ is a real number.

(G) $\lim_{x\to a} f[g(x)] = f\left(\lim_{x\to a} g(x)\right) = f(m)$; provided f(x) is continuous at x = m.

For example : $\lim_{x\to a} \ell n(f(x)) = \ell n[\lim_{x\to a} f(x)]$; provided ℓnx is defined at $x = \lim_{t\to a} f(t)$.

4. Indeterminate Forms

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, 1^{\infty}, 0^0, \infty^0.$

Note :

(i) We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number It does not obey the laws of elementary algebra,

(ii) $\infty + \infty \to \infty$ **(iii)** $\infty \times \infty \to \infty$ **(iv)** $(a/\infty) = 0$, if a is finite.

(v) $\frac{a}{0}$ is not defined. **(vi)** $ab = 0$, if and only if $a = 0$ o $b = 0$ and a & b are finite.

5. General Methods to be used to Evaluate Limits

(A) Factorization :

Important Factors :

(i) $x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + ... + a^{n-1}),\ n \in N$

(ii) $x^n + a^n = (x + a)(x^{n-1} - ax^{n-2} + ... + a^{n-1})$, n is an odd natural number.

Note : $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(B) Rationalization or Double Rationalization :

In this method we rationalise the factor containing the square root and simply.

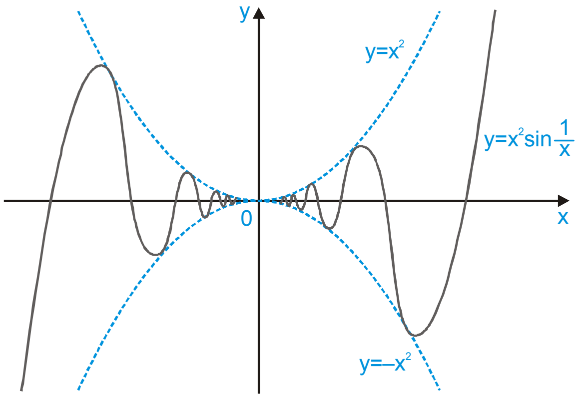
(C) Limit when $x \to \infty$:

(i) Divide by greatest power of x in numerator and denominator.

(ii) Put $x = 1/y$ and apply $y \to 0$

(D) Squeeze Play Theorem (Sandwich Theorem) :

If $f(x) \le g(x) \le h(x)$; $\forall$ x & $\lim_{x \to a} f(x) = \ell = \lim_{x \to a} h(x)$ then $\lim_{x \to a} g(x) = \ell$



for example : $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$, as illustrated by the graph given.

6. Limit of Trigonometric Functions

$\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1 = \lim\limits_{x\to 0}\dfrac{\tan x}{x} = \lim\limits_{x\to 0}\dfrac{\tan^{-1} x}{x} = \lim\limits_{x\to 0}\dfrac{\sin^{-1} x}{x}$ [where x is measured in radians]

(A) If $\lim\limits_{x\to a} f(x) = 0$, then $\lim\limits_{x\to a}\dfrac{\sin f(x)}{f(x)} = 1$, e.g. $\lim\limits_{x\to 1}\dfrac{\sin(\ell nx)}{(\ell nx)} = 1$

(B) Using substitution

$\lim\limits_{x\to a} f(x) = \lim\limits_{h\to 0} f(a-h)$ or $\lim\limits_{h\to 0} f(a+h)$ i.e. by substituting x by a – h or a + h

7. Limit of Exponential Functions

(A) $\lim\limits_{x\to 0}\dfrac{a^x - 1}{x} = \ell n\, a\,(a > 0)$ In particular $\lim\limits_{x\to 0}\dfrac{e^x - 1}{x} = 1$.

In general if $\lim\limits_{x\to a} f(x) = 0$, then $\lim\limits_{x\to a}\dfrac{a^{f(x)} - 1}{f(x)} = \ell na,\ a > 0$

(B) $\lim\limits_{x\to 0}\dfrac{\ell n(1+x)}{x} = 1$

(C) $\lim\limits_{x\to 0}(1+x)^{1/x} = e = \lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$ (Note : The base and exponent depends on the same variable.)

In general, if $\lim\limits_{x\to a} f(x) = 0$, then $\lim\limits_{x\to a}(1+f(x))^{1/f(x)} = e$

(D) If $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a}\phi(x) = \infty$, then ; $\lim\limits_{x\to a}[f(x)]^{\phi(x)} = e^k$ where $k = \lim\limits_{x\to a}\phi(x)\,[f(x) - 1]$

(E) If $\lim\limits_{x\to a} f(x) = A > 0$ & $\lim\limits_{x\to a}\phi(x) = B$ (a finite quantity) then ; $\lim\limits_{x\to a}[f(x)]^{\phi(x)} = e^{B\, ln\, A} = A^B$

8. Limit Using Series Expansion

Expansion of function like binomial expansion, exponential & logarithmic expansion, expansion of sinx, cosx, tanx should be remembered by heart which are given below :

(i) $a^x = 1 + \dfrac{x\ell na}{1!} + \dfrac{x^2\ell n^2 a}{2!} + \dfrac{x^3\ell n^3 a}{3!} + ...\ a > 0$

(ii) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ...$

(iii) $\ell n(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + ...$ for $-1 < x \le 1$

(iv) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + ...$

(v) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + ...$

(vi) $\tan x = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + ...$

(vii) $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + ...$

(viii) $\sin^{-1} x = x + \dfrac{1^2}{3!} x^3 + \dfrac{1^2 . 3^2}{5!} x^5 + \dfrac{1^2 . 3^2 . 5^2}{7!} x^7 + ...$

(ix) $\sec^{-1} x = 1 + \dfrac{x^2}{2!} + \dfrac{5x^4}{4!} + \dfrac{61x^6}{6!} + ...$

(x) for $|x| < 1$, $n \in R$; $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{1.2} x^2 + \dfrac{n(n-1)(n-2)}{1.2.3} x^3 + ...\infty$

(xi) $(1+x)^{\frac{1}{x}} = e\left(1 - \dfrac{x}{2} + \dfrac{11}{24} x^2 - ....\right)$