HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

1. $u(x) = 7v(x) \Rightarrow u'(x) = 7v'(x) \Rightarrow p = 7$ (given)

again $\dfrac{u(x)}{v(x)} = 7 \Rightarrow \left(\dfrac{u(x)}{v(x)}\right)' = 0 \Rightarrow q = 0;$

now $\dfrac{p+q}{p-q} = \dfrac{7+0}{7-0} = 1$

4. multiply Numerator and Denominator by x^m, x^n and x^p respectively

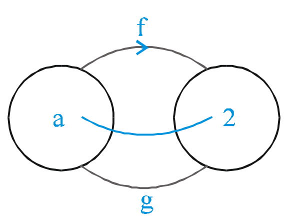
5. Integrating both sides we gets $f^3(x) = x^2 + C$; $f(2) = 1$
$\Rightarrow C = -3$
$f^3(x) = x^2 - 3 \Rightarrow f^3(3) = 6 \Rightarrow f(3) = \sqrt[3]{6}$

6. $f[g(x)] = x \Rightarrow f'[g(x)].[g'(x)] = 1$
$\Rightarrow f'(A).g'(2) = 1$ [putting $x = 2$]

given, $f'(A) = \dfrac{a^{10}}{1+a^2}$

$\Rightarrow g'(2) = \dfrac{1+a^2}{a^{10}}$]

Alternative $g[f(x)] = x$
$g'[f(x)].f'(x) = 1$
now $g(2) = a \Rightarrow f(A) = 2$
$\therefore$ g and f are inverse of each other



now $f(x) = 2 \Rightarrow g(2) = x = a$
$\therefore g'(2).f'(A) = 1$

$g'(2) = \dfrac{1}{f'(a)} = \dfrac{1+a^2}{a^{10}}$

8. $f'(x)\cdot f'(x) - f(x)\cdot f''(x) = 0$

or $\dfrac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = 0$

$\dfrac{d}{dx}\left[\dfrac{f(x)}{f'(x)}\right] = 0$

Integrating, $\dfrac{f(x)}{f'(x)} = C$ ....(i)

put $x = 0$, $\dfrac{f(0)}{f'(0)} = C \Rightarrow C = \dfrac{1}{2}$,

Hence $\dfrac{f(x)}{f'(x)} = \dfrac{1}{2}$

from (1) $2f(x) = f'(x)$

$\therefore \dfrac{f'(x)}{f(x)} = 2$

again Integrating $\ln[f(x)] = 2x + k$
put $x = 0$ to get $k = 0$
$f(x) = e^{2x}$

9. $g(x) = f(-x + f(f(x)))$; $f(0) = 0$; $f'(0) = 2$
$g'(x) = f'(-x + f(f(x)))\cdot[-1 + f'(f(x))\cdot f'(x)]$
$g'(0) = f'(f(0))\cdot[-1 + f'(0)\cdot f'(0)]$
$= f'(0)[-1 + (2)(2)]$
$= (2)(3) = 6$

10. $f(x) = (x-1)^{100}.(x-2)^{2.99}.(x-3)^{3.98}.....(x-100)^{100.1}$
Take log & than differentiate we get

Now $\dfrac{f'(x)}{f(x)} = \dfrac{1.100}{x-1} + \dfrac{2.99}{x-2} + \dfrac{3.98}{x-3} + ... \dfrac{100.1}{x-100}$

$\dfrac{f'(101)}{f(101)} = 1 + 2 + 3 ..... + 100 = 5050$

$\therefore \dfrac{f(101)}{f'(101)} = \dfrac{1}{5050}$

11. $f'(0) = \lim\limits_{h\to 0} \dfrac{g(0+h)\cos(1/h) - 0}{h} = \lim\limits_{h\to 0} \dfrac{g(h)\cos(1/h)}{h}$

$= \lim\limits_{h\to 0} \dfrac{g(h)}{h} \cdot \lim\limits_{h\to 0} \cos\dfrac{1}{h} = g'(0) \lim\limits_{h\to 0} \cos\dfrac{1}{h} = 0$

$g'(x) = -g'(x) \Rightarrow g'(0) = 0$

12. $h''(x) = 0$

13. $x^{\left(\frac{\ell+m}{(m-n)(n-\ell)}+\frac{m+n}{(n-\ell)(\ell-m)}+\frac{n+\ell}{(\ell-m)(m-n)}\right)} = x^{\left(\frac{\ell^2-m^2+m^2-n^2+n^2-\ell^2}{(m-n)(n-\ell)(\ell-m)}\right)}$

$= x^0 = 1 \qquad \therefore \frac{d}{dx}(1) = 0$

14. $f'(x) = \frac{1}{x} + \pi\cos(\pi x) + C$

$f'(2) = \frac{1}{2} + \pi + C = \frac{1}{2} + \pi \Rightarrow C = 0$

$f(x) = \ln|x| + \sin(\pi x) + C'$

$f(1) = C' = 0$

$f(x) = \ln|x| + \sin(\pi x)$

15. Put $\cos\phi = \frac{2}{\sqrt{13}}$; $\sin\phi = \frac{3}{\sqrt{13}}$; $\tan\phi = \frac{3}{2}$

$y = \cos^{-1}\{\cos(x+\phi)\} + \sin^{-1}\{\cos(x-\phi)\}$

$= \cos^{-1}\{\cos(x+\phi)\} + \frac{\pi}{2} - \cos^{-1}\{\cos(\phi - x)\}$ (think !)

$= x + \phi + \frac{\pi}{2} - \phi + x$

$y = 2x + \frac{\pi}{2}$; $z = \sqrt{1+x^2}$

now compute $\frac{dy}{dz}$

18. $D^*f(x) = 2f(x).f'(x)$

$D^*(x\ln x) = 2x\ln x(1+\ln x)$

19. between two consecutive roots of $f'(x) = 0$ the curve can cut the axis of x atmost once

i.e. may cut or may not cut

21. $f(x) = \sqrt{x+2\sqrt{2x-4}} + \sqrt{x-2\sqrt{2x-4}}$

$\therefore f(x) = \sqrt{\left(\sqrt{x-2}+\sqrt{2}\right)^2} + \sqrt{\left(\sqrt{x-2}-\sqrt{2}\right)^2}$

$= \left|\sqrt{x-2}+\sqrt{2}\right| + \left|\sqrt{x-2}-\sqrt{2}\right|$

for $\sqrt{x-2}$ to exist $x \geq 2$

Also, $\sqrt{x-2} + \sqrt{2} > 0$ (always true, think ! why ?)

but $\sqrt{x-2} - \sqrt{2} \geq 0$ only if $x \geq 4$

< 0 only if $x < 4$

$\therefore$ now f(x) becomes

$f(x) = \sqrt{x-2} + \sqrt{2} - \sqrt{x-2} + \sqrt{2}$

for $2 \leq x < 4$

$= \sqrt{x-2} + \sqrt{2} + \sqrt{x-2} - \sqrt{2}$

for $x \geq 4$

$\therefore f(x) = 2\sqrt{2}$, for $2 \leq x < 4$

$= 2\sqrt{x-2}$, for $4 \leq x < \infty$

$\because$ f is continuous $[2, 4) \cup [4, \infty)$ (verify)

$\therefore f'(x) = 0, 2 \leq x < 4$

$= \frac{1}{\sqrt{x-2}}, 4 \leq x < \infty$

$\therefore f'(102_+) = \frac{1}{\sqrt{102-2}} = \frac{1}{10}$

$\therefore 10f'(102_+) = 1$

22. Let $f(x) = px^2 + qx + r$

$f(1) = f(-1)$ gives $p + q + r = p - q + r$

hence $q = 0$

Hence $f(x) = px^2 + r$

$f'(x) = 2px$ ....(i)

Given a, b, c are in A.P.

hence 2pa, 2pb, 2pc will also be in A.P.

or f'**(A)**, f'**(B)**, f'**(C)** will also be in A.P. $\Rightarrow$ **(D)**

23. $2x + 2yy' = 0$

$x + yy' = 0 \Rightarrow y' = -\frac{x}{y}$ ....(i)

$1 + yy'' + (y')^2 = 0$

$y'' = -\frac{1+(y')^2}{y}$

now $k = \frac{y''}{\left(1+(y')^2\right)^{3/2}} = -\frac{1+(y')^2}{y\left(1+(y')^2\right)^{3/2}}$

$= -\frac{1}{y\sqrt{1+(y')^2}} = -\frac{1}{y\sqrt{1+\frac{x^2}{y^2}}} = -\frac{1}{\sqrt{y^2+x^2}} = -\frac{1}{R}$

24. $y = \frac{x}{a + \frac{x}{b+y}} \Rightarrow y = \frac{x(b+y)}{ab+ay+x}$

$\Rightarrow aby + ay^2 + xy = xb + xy$

$\Rightarrow ab\frac{dy}{dx} + 2ay\frac{dy}{dx} = b$

$\Rightarrow \frac{dy}{dx} = \frac{b}{ab+2ay}$

25. n = 2 or 0 only

26. Degree of $f(x) = n$; degree of $f'(x) = n-1$

degree of $f''(x) = (n-2)$

hence $\quad n = (n-1) + (n-2) = 2n-3$

$\therefore \quad n = 3$

hence $\quad f(x) = ax^3 + bx^2 + cx + d, (a \neq 0)$

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$\therefore \quad ax^3 + bx^2 + cx + d = (3ax^2 + 2bx + c)(6ax + 2b)$

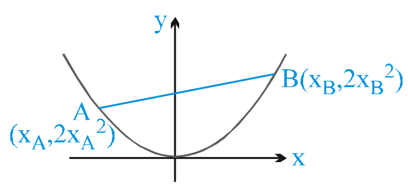
$\therefore \quad 18a^2 = a \quad \Rightarrow \quad a = \dfrac{1}{18}$

28. We have $y = 2x^2$

$(AB)^2 = (x_B - x_A)^2 + (2x_B^2 - 2x_A^2)^2 = 5$

or $\quad (x_B - x_A)^2 + 4(x_B^2 - x_A^2)^2 = 5$

differentiating w.r.t. x_A and denoting $\dfrac{dx_B}{dx_A} = D$



$2(x_B - x_A)(D-1) + 8(x_B^2 - x_A^2)(2x_BD - 2x_A) = 0$

put $x_A = 0;\ x_B = 1$

$2(1-0)(D-1) + 8(1-0)(2D-0) = 0$

$2D - 2 + 16D = 0 \quad \Rightarrow \quad D = 1/9$

30. for objective note that in y highest degree of x is 4 and therefore $\dfrac{d^3y}{dx^3}$ is a linear function of x.

Which satisfies only in **(D)**

31. $y = f(x) - f(2x)$

$y' = f'(x) - 2f'(2x)$

$y'(1) = f'(1) - 2f'(2) = 5 \qquad$ ....**(i)**

and $y'(2) = f'(2) - 2f'(4) = 7 \qquad$ ....**(ii)**

now let $y = f(x) - f(4x)$

$y' = f'(x) - 4f'(4x)$

$y'(1) = f'(1) - 4f'(4) \qquad$ ....**(iii)**

Substituting the value of $f'(2) = 7 + 2f'(4)$ in **(i)**

$f'(1) - 2[7 + 2f'(4)] = 5$

$f'(1) - 4f'(4) = 19 \quad \Rightarrow \quad$ **(A)**

34. $y = 3e^2 - x$

let $\quad x^y = x^{3e^2 - x}$

$f(x) = x^{3e^2 - x}$

$\ln(f(x)) = (3e^2 - x)\ln x$

$\dfrac{1}{f(x)} \cdot f'(x) = \dfrac{3e^2 - x}{x} - \ln x$

$\therefore \quad f'(x) = 0$

$\Rightarrow \quad 3e^2 - x = x \ln x$

$\Rightarrow \quad 3e^2 = x(1 + \ln x)$

$\Rightarrow \quad x = e^2$ (by verification)

35. $f(x) = x^n$

$\therefore \quad f'(x) = nx^{n-1},\ f''(x) = n(n-1)x^{n-2}$

....

$f^{''''.......n\text{ times}}(x) = n!$

Now $f(1) - \dfrac{f'(1)}{1!} + \dfrac{f''(1)}{2!} - .... \dfrac{f^{''''.........n\text{ times}}(1)}{n!}$

$= 1 - \dfrac{n}{1!} + \dfrac{n(n-1)}{2!} - .... \dfrac{n!}{n!} = (1-1)^n = 0$

EXERCISE - 2

Part # I : Multiple Choice

1. $y = e^{-x}$ & $y = e^{-x}\sin x$

$y' = -e^{-x}$ ...(i) & $y' = -e^{-x}(\sin x - \cos x)$ ...(ii)

equating (i) & (ii)

$e^{-x}(1 - \sin x + \cos x) = 0$

$e^{-x} \neq 0$

$\Rightarrow 1 - \sin x + \cos x = 0$

$\Rightarrow 2\cos^2\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$\Rightarrow 2\cos\frac{x}{2}\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) = 0$

$\Rightarrow x = \frac{\pi}{2}, \pi$

slope can be $-e^{-\pi/2}$ & $-e^{-\pi}$.

2. $x = \cos t, y = \ell n\, t$

$\frac{dy}{dx} = \frac{1}{t}\cdot\frac{1}{-\sin t}$

at $t = \frac{\pi}{2}$: $\frac{dy}{dx} = \frac{-2}{\pi}$ | at $t = \frac{\pi}{6}$: $\frac{dy}{dx} = -\frac{12}{\pi}$

3. square both sides, differentiate and rationalise

5. $\because$ $t = \frac{1}{2}\ell n(x^2 + y^2)$

$\Rightarrow \frac{1}{2}\ell n(x^2 + y^2) = \sin^{-1}\left(\frac{y}{\sqrt{x^2 + y^2}}\right)$

Case-I : When $x \geq 0$

$\Rightarrow \ell n(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$

$\Rightarrow \frac{1}{x^2 + y^2}(2x + 2yy') = \frac{2}{1 + \frac{y^2}{x^2}}\left(\frac{xy' - y}{x^2}\right)$

$\Rightarrow xy' - yy' = x + y$ $\Rightarrow y' = \frac{x + y}{x - y}$

let $y = x\tan\theta$; $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow \sin^{-1}\left(\frac{y}{\sqrt{x^2 + y^2}}\right) = \sin^{-1}\left(\frac{x\tan\theta}{|x\sec\theta|}\right) = \sin^{-1}\left(\frac{x}{|x|}\sin\theta\right) = \begin{cases} \theta = \tan^{-1}\left(\frac{y}{x}\right), & x \geq 0 \\ -\theta = -\tan^{-1}\left(\frac{y}{x}\right), & x < 0 \end{cases}$

Case-II : When $x < 0$

$\ell n(x^2 + y^2) = -2\tan^{-1}\left(\frac{y}{x}\right)$

$\Rightarrow \frac{(2x + 2yy')}{x^2 + y^2} = \frac{-2}{1 + \frac{y^2}{x^2}}\left(\frac{xy' - y}{x^2}\right)$

$y'(x + y) = y - x$

$y' = \frac{y - x}{y + x}$

9. $f^{-1}(x) = g(x)$ $\Rightarrow$ $x = f(g(x))$

Differentiating both sides,

$1 = f'(g(x))\,g'(x)$ $\Rightarrow$ $g'(x) = \frac{1}{f'(g(x))}$

Now $f'(x) = 2x + 3$

So $g'(x) = \frac{1}{2g(x) + 3}$ $\Rightarrow$ $g'(1) = \frac{1}{2(g(1)) + 3}$

$gof(x) = x$ $\Rightarrow$ $g'(f(x))\,f'(x) = 1$

$f(x) = 1$ at $x = 1$ & $f'(1) = 5$

$g'(1)f'(1) = 1$ $\Rightarrow$ $g'(1) = 1/5$

11. $u = e^x\sin x, v = e^x\cos x$

$v\frac{du}{dx} - u\frac{dv}{dx}$

$= v(e^x\cos x + e^x\sin x) - u(e^x\cos x - e^x\sin x)$

$= e^x\sin x(v + u) + e^x\cos x(v - u)$

$= u(v + u) + v(v - u)$

$= v^2 + u^2$

again $\frac{du}{dx} = e^x\sin x + e^x\cos x$

$\frac{d^2u}{dx^2} = e^x\sin x + e^x\cos x + e^x\cos x - e^x\sin x$

$\frac{d^2u}{dx^2} = 2v$

Similarly other options can be checked.

12. $f(x) = (x^2 + bx + c)e^x$

$\therefore \quad f'(x) = (x^2 + (b+2)x + (b+c))e^x$

$f(x) > 0 \quad \text{iff } D = b^2 - 4c < 0$

now $f'(x) > 0$ iff $D' = (b+2)^2 - 4(b+c) = D + 4 < 0$

Thus for $f'(x) > 0$ $D + 4 < 0$ holds. $\Rightarrow \quad D < 0$

$\Rightarrow \quad f(x) > 0$

Note that the converse need not be true, e.g. $b = c = 1$,

$f(x) > 0$ but $f'(-1) = 0$

14. $f''(x) = -f(x)$ .... **(i)**

$f'(x) = g(x)$ .... **(ii)**

$h'(x) = (f(x))^2 + (g(x))^2$ .... **(iii)**

$h(0) = 2, h(1) = 4$

Differentiating equation **(ii)** w.r.t. x

$f''(x) = g'(x) = -f(x)$

Differentiating equation **(iii)** w.r.t. x

$h''(x) = 2f(x) . f'(x) + 2\, g(x) . g'(x)$

$= 2f(x) . f'(x) - 2f'(x) . f(x) = 0 \qquad \{\because \; g'(x) = -f(x)\}$

$\Rightarrow$ $h'(x)$ is constant

$\Rightarrow$ $h(x)$ is linear function

$\because$ $h(0) = 2$ $\Rightarrow$ $h(x)$ not passing through (0, 0)

Let $y = h(x) = ax + b$

at $x = 0$

$y = 2 = b \quad \Rightarrow \quad y = ax + 2$

at $x = 1$

$a + 2 = 4$

$a = 2$

$\Rightarrow$ curve is $y = 2x + 2$

15. $$f(x) = \frac{\sqrt{\left(\sqrt{x-1}\right)^2 + 1 - 2\sqrt{x-1}}}{\sqrt{x-1} - 1} . x$$

$$= \frac{\left|\sqrt{x-1} - 1\right|}{\sqrt{x-1} - 1} . x = \begin{cases} -x & \text{if } x \in [1, 2) \\ x & \text{if } x \in (2, \infty) \end{cases}$$

17. $$f(x) = \begin{cases} x^3 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

18. $$y = \cos^{-1}\sqrt{\frac{\sqrt{1+x^2}+1}{2\sqrt{1+x^2}}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1 - \left(\frac{1}{2} + \frac{1}{2\sqrt{1+x^2}}\right)}} \cdot \frac{1}{2\sqrt{\frac{1}{2} + \frac{1}{2\sqrt{1+x^2}}}} \cdot \frac{1}{2}$$

$$\frac{2x}{(-2)(1+x^2)^{3/2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{\frac{1}{4} - \frac{1}{4(1+x^2)}}} \cdot \frac{x}{4(1+x^2)^{3/2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sqrt{1+x^2}}{\sqrt{1+x^2-1}} \cdot \frac{x}{4\sqrt{1+x^2}(1+x^2)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{2|x|(1+x^2)}$$

when $x < 0$

$$\frac{dy}{dx} = \frac{-1}{2(1+x^2)}$$

when $x > 0$

$$\frac{dy}{dx} = \frac{1}{2(1+x^2)}$$

19. $y = x^{(\ell n x)^{\ell n (\ell n x)}}$

$\ell ny = (\ell n x)^{\ell n(\ell nx)} . \ell nx$ .....**(1)**

$\ell n\,(\ell ny) = \ell n(\ell n x) . \ell n(\ell nx) + \ell n(\ell nx)$

$$\frac{1}{\ell ny} . \frac{1}{y}\frac{dy}{dx} = \frac{2\,\ell n(\ell nx)}{\ell n x} . \frac{1}{x} + \frac{1}{x\,\ell nx}$$

$$= \frac{2\,\ell n(\ell nx) + 1}{x\,\ell nx}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x} . \frac{\ell ny}{\ell nx}(2\,\ell n(\ell nx) + 1) \Rightarrow \textbf{D}$$

Substituting the value of $\ell n\, y$ from (1)

$$\frac{dy}{dx} = \frac{y}{x}(\ell nx)^{\ell n(\ell nx)}(2\,\ell n(\ell nx) + 1) \Rightarrow \textbf{B}$$

Part # II : Assertion & Reason

1. **Hint :** Statement I : f(x) is constant function

Statement II : It is true

2. $y = x^2$

$$\frac{dy}{dx} = 2x$$

$$\frac{d^2y}{dx^2} = 2$$

again $\quad 2x\frac{dx}{dy} = 1$

$$2\left(\frac{dx}{dy}\right)^2 + 2x\frac{d^2x}{dy^2} = 0$$

$$\Rightarrow \quad x\frac{d^2x}{dy^2} = -\left(\frac{dx}{dy}\right)^2 \qquad \Rightarrow \quad \frac{d^2x}{dy^2} = -\frac{1}{4x^2}$$

$$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right) \neq 1$$

Statement-2 :

$$\because \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{1}{\left(\frac{dx}{dy}\right)^2}.\frac{d^2x}{dy^2}.\frac{dy}{dx} = -\left(\frac{dy}{dx}\right)^3.\frac{d^2x}{dy^2}$$

4. Consider $f(x) = x + \sin x$ which is aperiodic but $f'(x) = 1 + \cos x$ which is periodic with period 2π.

EXERCISE - 3

Part # I : Matrix Match Type

3. **(A)** $f'(x) = 3x^2 + 1 \quad \Rightarrow \quad f'(x^2+1) = 3(x^2+1)^2 + 1$

$f'(x^2+1)$ at $x = 0$ is 4

(B) $f(x) = \log_{x^2}\log(x) = \frac{1}{2}\log_x(\log x) = \frac{1}{2}\frac{\log(\log x)}{\log x}$

$$f'(x) = \frac{1}{2}\left(\frac{\log x.\frac{1}{\log x}.\frac{1}{x} - \frac{\log(\log(x))}{x}}{(\log x)^2}\right)$$

$$= \frac{1}{2}\left(\frac{1 - \log.(\log(x))}{x(\log x)^2}\right) \Rightarrow f'(e^e) = 0$$

(C) $y = \ell n\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$

$$\frac{dy}{dx} = \frac{1}{2}\frac{\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)} = \frac{1}{\sin\left(\frac{\pi}{2}+x\right)} = \frac{1}{\cos x}$$

$= \sec x$

Hence $p = 0$

(D) $f(x) = |x^3 - x^2 + x - 1|\sin x$

$f(x) = |(x^2+1)(x-1)|\sin x$

$= (x^2+1)(x-1)\sin x \quad$ when $x \geq 1$

$= -(x^2+1)(x-1)\sin x \quad$ when $x < 1$

Now $28f(f(\pi)) = 0$

$\therefore$ At $x = 0$

$f'(x) = -[2x(x-1)\sin x + (x^2+1)$

$\sin x + (x^2+1)(x-1)\cos x]$

$4f'(0) = 4$

4. **(A)** $y = f(x^3)$

$$\therefore \quad \frac{dy}{dx} = f'(x^3).3x^2$$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{x=1} = f'(1).3 = 9$$

(B) $f(xy) = f(x) + f(y)$

$f(1) = f(1) + f(1)$

$\therefore \quad f(1) = 0$

$$\because \quad f(1) = f(e) + f\left(\frac{1}{e}\right)$$

$$\therefore \quad f(e) + f\left(\frac{1}{e}\right) = 0$$

(C) $f''(x) = -f(x),\ f'(x) = g(x)$

$\therefore\ g'(x) = f''(x) = -f(x)$

$h(x) = (f(x))^2 + (g(x))^2$

$\therefore\ h'(x) = 2f(x).f'(x) + 2g(x).g'(x)$

$= 2f(x).g(x) + 2g(x)(-f(x)) = 0$

$\therefore\ h(x) = c,\ x \in R$

$\therefore\ h(10) = h(5) = 9.$

(D) $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x),\ \frac{\pi}{2} < x < \pi$

$$\frac{dy}{dx} = \frac{-\operatorname{cosec}^2 x}{1+\cot^2 x} + \frac{-1}{1+\tan^2 x}.\sec^2 x$$

$= -1 - 1 = -2$

Part # II : Comprehension

Comprehension # 1

$f(x+y) - f(x) = f(y) - 1 + 2xy$

$\Rightarrow\ f(0+0) - f(0) = f(0) - 1 + 2(0)(0) \Rightarrow f(0) = 1$

and $f'(0) = 1$ (given)

Also $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}\left[\frac{f(h)-1}{h} + \frac{2xh}{h}\right]$

$= \lim\limits_{h\to 0}\frac{f(h)-f(0)}{h} + 2x = f'(0) + 2x$

$f'(x) = 1 + 2x$

Integrate it

$f(x) = x^2 + x + c$

$f(x) = x^2 + x + 1 \quad [f(0) = 1 \Rightarrow c = 1]$

1. $\ell n(x^2+x+1) \to$ Domain R

2. $y = \log_{3/4}(x^2+x+1)$

Now $x^2 + x + 1 \geq \frac{3}{4}$

hence range is $(-\infty, 1]$

3. $g(0) = \frac{g(0)+g(0)}{k} \Rightarrow 2g(0) = kg(0)$

$\Rightarrow\ g(0) = 0$ (as $k \neq 2$)

$$g'(x) = \lim_{h\to 0}\frac{g\left(\frac{x+h}{1}\right) - g(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{g(x)+g(h)}{1} - g(x)}{h}$$

$$= \lim_{h\to 0}\frac{g(h)-g(0)}{h} = g'(0) = \lambda$$

$g(x) = \lambda x + c$

$\Rightarrow\ g(x) = \lambda x\ [g(0) = 0]$

Now $x^2 + x + 1 = \lambda x \quad \Rightarrow\ x^2 + (1-\lambda)x + 1 = 0$

For concident pt. D = 0

$(1-\lambda)^2 - 4 = 0$

$\Rightarrow\ \lambda = 3, -1$

Comprehension # 3

1. LHD $= \lim\limits_{h\to 0^-}\frac{f(-a+h)-f(-a)}{h}$

$= \lim\limits_{h\to 0^-}\frac{-f(a-h)+f(a)}{h} = \lim\limits_{h\to 0^-}\frac{f(a-h)-f(a)}{-h}$

2. RHD $= \lim\limits_{h\to 0^-}\frac{f'(a)-f'(a-h)}{h} = \frac{f'(a)+f'(h-a)}{h}$

Since derivative of even function is odd & vice versa.

3. $\lim\limits_{h\to 0}\frac{f(-x)-f(-x-h)}{h} = \lim\limits_{h\to 0}\frac{f(-x-h)-f(-x)}{-h}$

$= f'(-x)$ ... **(i)**

and $\lim\limits_{h\to 0}\frac{f(x)-f(x-h)}{-h} = -f'(x)$ ... **(ii)**

from **(i)** and **(ii)** f'(x) is odd function and hence f(x) is even function.

Comprehension # 4

(i) $y = e^t \sin t \Rightarrow \frac{dy}{dt} = e^t[\cos t + \sin t]$

$x = e^t \cos t \Rightarrow \frac{dx}{dt} = e^t[\cos t - \sin t]$

$\therefore\ \frac{dy}{dx} = \frac{\cos t + \sin t}{\cos t - \sin t} = \tan\alpha$

$\therefore\ \tan\left(\frac{\pi}{4} + t\right) = \tan\alpha$

$\left(\frac{\pi}{4} + t\right) = \alpha$

$t = \alpha - \frac{\pi}{4}$

(ii) $\frac{d^2y}{dx^2} = \frac{\sec^2\left(\frac{\pi}{4}+t\right)}{e^t(\cos t - \sin t)}$

$\left.\frac{d^2y}{dx^2}\right|_{t=0} = 2$

(iii) $F(t) = \int e^t(\cos t + \sin t)\,dt = e^t \sin t + C$

$F\left(\frac{\pi}{2}\right) - F(0) = (e^{\pi/2} + C) - 0 = e^{\pi/2}$

Comprehension # 5

Sol. $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{2x+5y-2}{5x+2y+1} = \frac{-5}{8}$ at (1, 1)

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$= \frac{(5x+2y+1)(2+5\frac{dy}{dx}) - (2x+5y-2)(5+2\frac{dy}{dx})}{(5x+2y+1)^2}$

$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(1,1)} = \frac{111}{256}$

For question 6

Slope of normal at (1, 1) $= -\frac{dx}{dy} = \frac{8}{5}$

Equation of normal

$y - 1 = \frac{8}{5}(x-1) \Rightarrow 5y - 5 = 8x - 8$

$\Rightarrow 8x - 5y - 3 = 0$

EXERCISE - 4

Subjective Type

1. $\sqrt{5}$

2. $\frac{(n+1)x^{n+2} - (n+2)x^{n+1} + 1}{(1-x)^2}$

3. 2

5. $f_1(x) = e^{f_0(x)} = e^x$

$f_2(x) = e^{f_1(x)} = e^{e^x}$

$f_3(x) = e^{e^{e^x}}$

similarly $f_n(x) = e^{e^{e^{\cdots(n-1)\text{times}(x)}}}$

Now $\frac{d}{dx}[f_n(x)] = e^{f_{n-1}(x)}.\frac{d}{dx}(e^{f_{n-1}(x)})$

On differentiating it completely we get

$\frac{d}{dx}[f_n(x)] = e^{f_{n-1}(x)}.e^{f_{(n-2)}(x)}.e^{f_{(n-3)}(x)}.......e^{f_0(x)}$

$= f_n(x) . f_{(n-2)}x..............f_1(x)$

7. $x = \frac{1}{z} \Rightarrow \frac{dx}{dz} = -\frac{1}{z^2}$

Now $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{dy}{dz}(-z^2)$

$\frac{d}{dx}(y') = \frac{dy'}{dz}.\frac{dz}{dx} + \frac{dy}{dz}.\frac{d^2z}{dx^2}$

$\frac{d^2y}{dx^2} = 2z^3.\frac{dy}{dz} + z^4\frac{d^2y}{dz^2}$

$\Rightarrow \frac{d^2f}{dx^2} = 2z^3\frac{dy}{dz} + z^4\frac{d^2y}{dz^2}$

10. 2^n

11. $y = 1 + \frac{c_1}{x-c_1} + \frac{c_2x}{(x-c_1)(x-c_2)}$

$+ \frac{c_3x^2}{(x-c_1)(x-c_2)(x-c_3)}$

$= \frac{x}{x-c_1} + \frac{c_2x}{(x-c_1)(x-c_2)} + \frac{c_3x^2}{(x-c_1)(x-c_2)(x-c_3)}$

$= \frac{x^2}{(x-c_1)(x-c_2)} + \frac{c_3x^2}{(x-c_1)(x-c_2)(x-c_3)}$

$$= \frac{x^3}{(x-c_1)(x-c_2)(x-c_3)} = \frac{x}{x-c_1} \cdot \frac{x}{x-c_2} \cdot \frac{x}{x-c_3}$$

$$\Rightarrow \ \ell ny = \ell nx - \ell n(x-c_1) + \ell nx - \ell n(x-c_2) + \ell nx - \ell n(x-c_3)$$

Differentiating both sides w.r.t. x , we get

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \left(\frac{1}{x} - \frac{1}{x-c_1}\right) + \left(\frac{1}{x} - \frac{1}{x-c_2}\right) + \left(\frac{1}{x} - \frac{1}{x-c_3}\right)$$

$$= -\frac{c_1}{x(x-c_1)} - \frac{c_2}{x(x-c_2)} - \frac{c_3}{x(x-c_3)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left[\frac{c_1}{c_1-x} + \frac{c_2}{c_2-x} + \frac{c_3}{c_3-x}\right]$$

12. 100

13. $f(2x) = f'(x)f''(x)$

Let the degree of 'f' be n.

Comparing highest power on both sides

$n = n-1+n-2 \Rightarrow n = 3$

Let $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$

$f(2x) = f'(x)f''(x)$

$\therefore \ (8a_0x^3 + 4a_1x^2 + 2a_2x + a_3)$

$= (3a_0x^2 + 2a_1x + a_2)(6a_0x + 2a_1)$

Comparing coefficient of x^3

$8a_0 = 18a_0^2 \ \Rightarrow \ a_0 = \frac{4}{9}$

Rest all are zero

$\therefore \ f(x) = \frac{4}{9}x^3$

14. $\frac{b}{ab + 2ay}$

15. $f(x) = x^2 - 4x - 3$ & $f(x) = 9$

For $x = 6, -2$

$\Rightarrow x = 6 \quad (x > 2)$

Now $y = f(x) \quad \Rightarrow \quad f^{-1}(y) = x$

$\Rightarrow g(y) = x$

$\Rightarrow g'(y) = \frac{dx}{dy} = \frac{1}{2x-4} = \frac{1}{8}$

16. $\text{cosec}^2 x - (1/x^2)$

17. $z = \ell n\left(\tan\frac{x}{2}\right)$

$$\frac{dz}{dx} = \frac{1}{\sin x}$$

Now $\frac{dy}{dz} = \frac{dy}{dx}.\frac{dx}{dz} = \frac{dy}{dx}\sin x$

$$\frac{d^2y}{dz^2} = \frac{d}{dx}\left(\frac{dy}{dx}\sin x\right).\frac{dx}{dz}$$

$$\frac{d^2y}{dz^2} = \left(\sin x\frac{d^2y}{dx^2} + \cos x\frac{dy}{dx}\right)\sin x \qquad \text{......(i)}$$

$$\frac{d^2y}{dz^2} = \sin^2 x\frac{d^2y}{dx^2} + \sin x \cos x\frac{dy}{dx}$$

$$\text{cosec}^2 x\frac{d^2y}{dz^2} = \frac{d^2y}{dx^2} + \cot x\frac{dy}{dx}$$

Now put value from given equation

$$\text{cosec}^2 x\frac{d^2y}{dz^2} + 4y\,\text{cosec}^2 x = 0$$

$$\frac{d^2y}{dz^2} + 4y = 0$$

18. $\lim\limits_{x\to 0}\frac{\log|\tan 2x|}{\log|\tan x|} = \lim\limits_{x\to 0} 2\left(\frac{\sec^2 2x}{\sec^2 x}.\frac{\tan x}{\tan 2x}\right) = 1$

19. $H'(1) = 1, g'(1) = 2, H(1) = 1, g(1) = 2$

$$\lim_{x\to 1}\frac{H(x).g(1) - g(x)\,H(1)}{\sin(x-1)} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 1}\frac{g(1)\cdot H'(x) - H(1).g'(x)}{\cos(x-1)} \qquad \text{By L.H. Rule}$$

$$= \frac{2\times 1 - 1\times 2}{1} = 0$$

20. $\begin{vmatrix} X & sX & tX \\ X_1 & sX_1 + s_1X & t_1X + tX_1 \\ X_2 & sX_2 + 2s_1X_1 + Xs_2 & tX_2 + 2X_1t_1 + Xt_2 \end{vmatrix}$

$[C_2 \to C_2 - sC_1,\ C_3 \to C_3 - tC_1]$

$$= \begin{vmatrix} X & 0 & 0 \\ X_1 & Xs_1 & Xt_1 \\ X_2 & Xs_2 + 2s_1X_1 & Xt_2 + 2t_1X_1 \end{vmatrix}$$

$$= X\begin{vmatrix} Xs_1 & Xt_1 \\ Xs_2 + 2s_1X_1 & Xt_2 + 2t_1X_1 \end{vmatrix}$$

$$= X^2\begin{vmatrix} s_1 & t_1 \\ Xs_2 + 2s_1X_1 & Xt_2 + 2t_1X_1 \end{vmatrix}$$

$(R_2 \rightarrow R_2 - 2\,X_1R_1)$

$$= X^2\begin{vmatrix} s_1 & t_1 \\ Xs_2 & Xt_2 \end{vmatrix} = X^3\begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$$

21. $x = at^3$ and $y = bt^2$

$$\frac{dx}{dt} = 3at^2,\ \frac{dy}{dt} = 2bt$$

$$\frac{dy}{dx} = \frac{2b}{3a}\cdot\frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{2b}{3a}\left(-\frac{1}{t^2}\right)\cdot\frac{dt}{dx} = -\frac{2b}{3a}\cdot\frac{1}{t^2}\cdot\frac{1}{3at^2}$$

$$\frac{d^3y}{dx^3} = -\frac{2b}{9a^2}\cdot\left(\frac{-4}{t^5}\right)\cdot\frac{dt}{dx} = \frac{8b}{9a^2}\cdot\frac{1}{t^5}\cdot\left(\frac{1}{3at^2}\right)$$

$$= \frac{8b}{27a^3}\cdot\frac{1}{t^7}$$

22. (i) $y = \tan^{-1}\left(\frac{1+2x}{1-2x}\right),\ z = \sqrt{1+4x^2}$

$y = \tan^{-1}(1) + \tan^{-1}(2x)$

$$\frac{dy}{dx} = \frac{2}{1+4x^2}$$

$$\frac{dz}{dx} = \frac{1}{2\sqrt{1+4x^2}}.8x$$

$$\Rightarrow \frac{dy}{dz} = \frac{2}{1+4x^2}\cdot\frac{2\sqrt{1+4x^2}}{8x} = \frac{1}{2x\sqrt{1+4x^2}}$$

(ii) $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, $z = \tan^{-1}x$,

$z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \Rightarrow \quad x = \tan z$

$$\Rightarrow y = \tan^{-1}\left(\frac{\sec z - 1}{\sec z}\right)$$

$y = \tan^{-1}(\tan z/2)$

$y = \frac{z}{2} \qquad \because\ -\frac{\pi}{4} < \frac{z}{2} < \frac{\pi}{4}$

$$\frac{dy}{dz} = \frac{1}{2}$$

26. $\dfrac{1}{1+(x+n)^2} - \dfrac{1}{1+x^2}$

27. (A) Not differentiable at $x = 0$

(B) Not derivable at $x = \pm 1/2$

30. 3

33. $\because$ $F(x) = f(x)\,.\,g(x)$

Differentiating both sides w.r.t. x

$F'(x) = f'(x)\,g(x) + g'(x)\,f(x)$

Again differentiating both sides w.r.t.x

$F''(x) = f''(x)\,g(x) + 2f'(x)\,g'(x) + g''(x)\,f(x)$

dividing both sides by $f(x)\,.\,g(x)$

$$\frac{F''(x)}{f(x).g(x)} = \frac{f''(x)}{f(x)} + \frac{g''(x)}{g(x)} + \frac{2f'(x).g'(x)}{f(x)g(x)}$$

$$\Rightarrow \frac{F''(x)}{f(x).g(x)} = \frac{f''(x)}{f(x)} + \frac{g''(x)}{g(x)} + \frac{2c}{f(x)g(x)}$$

$\because$ $F''(x) = f''(x)\,g(x) + g''(x)\,f(x) + 2c$

Differentiating both sides w.r.t.x

$F'''(x) = f'''(x)\,g(x) + f''(x)\,g'(x) + g'''(x)\,f(x) + g''(x)\,f'(x)$ ... **(i)**

$\because$ $f'(x)\,.\,g'(x) = c \quad \Rightarrow \quad f''(x)\,g'(x) + g''(x)\,f'(x) = 0$

$\therefore$ from **(i)**, we get $\dfrac{F'''(x)}{F(x)} = \dfrac{f'''(x)}{f(x)} + \dfrac{g'''(x)}{g(x)}$

34. $$f(x) = \begin{cases} -\frac{2}{3}\left[\frac{1}{6} + \ell n\frac{3}{2}\right]x & \text{if} \quad x \le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x} & \text{if} \quad x > 0 \end{cases}$$

35. $a = 6, b = 6, c = 0\,;\ 3/40$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

3. $y = \log_y x$ $\qquad y = \frac{\log x}{\log y}$

$y \times \log y = \log x$

$\frac{dy}{dx}\log y + y \times \frac{1}{y} \times \frac{dy}{dx} = \frac{1}{x}$

$\frac{dy}{dx}(\log y + 1) = \frac{1}{x}$ $\qquad \boxed{\frac{dy}{dx} = \frac{1}{x(1+\log y)}}$

4. $x = 3\cos\theta - 2\cos^3\theta$ $\qquad y = 3\sin\theta - 2\sin^3\theta$

$\frac{dx}{d\theta} = -3\sin\theta + 6\cos^2\theta \times \sin\theta$

$\frac{dy}{d\theta} = 3\cos\theta - 6\sin^2\theta\cos\theta$

$\frac{dy}{dx} = \frac{3\cos\theta - 6\sin^2\theta\cos\theta}{-3\sin\theta + 6\cos^2\theta\sin\theta}$

$= \frac{\cos\theta - 2\sin^2\theta\cos\theta}{-\sin\theta + 2\cos^2\theta\sin\theta} = \cot\theta$

6. $f(x) = x^n$

$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} \ldots \frac{(-1)^x f^n(1)}{n!}$

put $n = 1$

$f(x) = x$ $\quad$ for series $f(1) - \frac{f'(1)}{1} = 0$

$f(1) = 1$ $\qquad$ put $n = 2$

$f(x) = x^2$ $\qquad f'(x) = 2x \quad f''(x) = 2$

$f(1) = 1$

so series $= f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} = 1 - 2 + \frac{2}{2} = 0$

Put $x = 3$ $\qquad f(x) = x^3$

$f'(x) = 3x^2$ $\qquad f''(x) = 6x$

$f(1) = 1$ $\qquad f'''(x) = 6$

series $f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!}$

$= 1 - 3 + \frac{6}{2} - \frac{6}{6} = 0$

7. f(x) is a polynomial function

$f(x) = ax^2 + bx + c$ $\qquad f(1) = f(-1)$

$a + b + c = a - b + c$ $\qquad b = 0$

a, b, c in A.P. $\qquad b = \frac{a+c}{2}$ $\qquad \boxed{a = -c}$

$f(x) = ax^2 + bx + c$ $\qquad f'(x) = 2ax + b$

$f'(a) = 2a^2 + b$ $\qquad f'(c) = 2ac + b$

$f'(b) = 2ab + b$ $\qquad$ then $f'(a),\ f'(b),\ f'(c)$

$f'(b) = 0 \quad f'(a) = 2a^2$ $\qquad f'(c) = -2a^2$

so that $\boxed{f'(a), f'(b), f'(c) \text{ are in A.P.}}$

8. $x = e^{y + e^{y + \ldots\ldots\infty}}$ $\qquad x > 0$ $\qquad \frac{dy}{dx} = ?$

$x = e^{y+x}$ $\qquad 1 = e^{y+x}\left(1 + \frac{dy}{dx}\right)$

$\frac{1}{x} = 1 + \frac{dy}{dx}$ $\qquad \frac{1-x}{x} = \frac{dy}{dx}$

10. $(x^x)^2 - 2\cot y\, x^x - 1 = 0$

$x^x = \frac{2\cot y \pm \sqrt{4\cot^2 y + 4}}{2}$ $\qquad \left\{\begin{array}{l} \text{at } x = 1, \\ 1 = \cot y + \operatorname{cosec} y \\ \Rightarrow y = \frac{\pi}{2} \end{array}\right.$

$= \cot y \pm \operatorname{cosec} y$

$x^x = \cot y + \operatorname{cosec} y$

diff. w.r. to x

$x^x(1 + \log x) = [-\operatorname{cosec}^2 y - \operatorname{cosec} y \cot y]\frac{dy}{dx}$

$1 = -\operatorname{cosec} y[\operatorname{cosec} y + \cot y]\frac{dy}{dx}$

$\frac{dy}{dx} = -1$

11. $g(x) = [f(2f(x) + 2)]^2$

$g'(x) = 2f(2f(x) + 2)\, f'(2f(x) + 2)\, 2f'(x)$

Put $x = 0$

$g'(0) = 2f(2f(0) + 2)\, f'(2f(0) + 2)\, 2f'(0)$

$= 2f(2(-1) + 2)\, f'(2(-1) + 2)\, 2f'(0)$

$= 2f(0)\, f'(0)\, 2f'(0)$

$= 4(-1)(1)(1) = -4$

12. $\frac{d}{dy}\left(\left(\frac{dy}{dx}\right)^{-1}\right) = \frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^{-1}\right) \cdot \frac{dx}{dy}$

$= -\left(\frac{dy}{dx}\right)^{-2} \cdot \frac{d^2x}{dx^2} \cdot \frac{dx}{dy} = -\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

13. $y = \sec(\tan^{-1} x) = \sqrt{1 + x^2}$

$\frac{dy}{dx} = \frac{x}{\sqrt{1 + x^2}} \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{\sqrt{2}}$

Part # II : IIT-JEE ADVANCED

2. **(B)** Let $P(x) = ax^2 + bx + c$

$P(0) = 0 \Rightarrow c = 0$

$P(1) = 1 \Rightarrow a + b = 1$

$\therefore P(x) = (1-a)x^2 + ax$

$P'(x) = 2(1-a)x + a > 0$

put $x = 0$, $a > 0$

$x = 1$, $a < 2$

$S = \{(1-a)x^2 + ax ; 0 < a < 2\}$.

5. **(A)** $g(x+1) = \log(f(x+1)) = \log x + \log f(x)$

$\Rightarrow g(x+1) = \log x + g(x) \Rightarrow g(x+1) - g(x) = \log x$

$\Rightarrow g'(x+1) - g'(x) = \frac{1}{x} \Rightarrow g''(x+1) - g''(x) = -\frac{1}{x^2}$

$\Rightarrow g''\left(1+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) = -4$

$\Rightarrow g''\left(2+\frac{1}{2}\right) - g''\left(1+\frac{1}{2}\right) = -\frac{4}{9}$

....

$g''\left(N+\frac{1}{2}\right) - g''\left(N-\frac{1}{2}\right) = \frac{-4}{(2N-1)^2}$

By adding

Hence $g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right)$

$= -4\left(1+\frac{1}{9}+......+\frac{1}{(2N-1)^2}\right)$

(B) $\lim_{x\to 0}\frac{[g(x)\cos x - g(0)]}{\sin x}$

$= \lim_{x\to 0}\frac{g'(x)\cos x - g(x)\sin x}{\cos x} = 0$

Now $f(x) = g(x)\sin x$

$f'(x) = g'(x)\sin x + g(x)\cos x$

$\therefore f'(0) = 0$

$f''(x) = g''(x)\sin x + g'(x)\cos x - g(x)\sin x + g'(x)\cos x$

$f''(0) = 0$

$\therefore$ Given limit = $f''(0)$ & also $f'(0) = g(0)$

So S(**I**) & S(**II**) both are correct but S(**II**) is not correct explaination of S(**I**)

6. $f(x) = x^3 + e^{x/2}$, $g(x) = f^{-1}(x)$

$\Rightarrow g'(f(x)).f'(x) = 1$

Put $f(x) = 1 \Rightarrow x^3 + e^{x/2} = 1$

$\Rightarrow x = 0$

$\Rightarrow g'(1).f'(0) = 1$, $f'(x) = 3x^2 + e^{x/2}.\frac{1}{2}$

$\Rightarrow g'(1) = 2$

7. Let $f(\theta) = \sin\alpha$ where $\alpha = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)$

$\Rightarrow \tan\alpha = \frac{\sin\theta}{\sqrt{\cos 2\theta}}$

$\Rightarrow \sin\alpha = \frac{\sin\theta}{\cos\theta} = \tan\theta \quad \left(\because \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)\right)$

$\Rightarrow f(\theta) = \tan\theta$

$\Rightarrow \frac{d(f(\theta))}{d(\tan\theta)} = 1$

MOCK TEST

1. $y = x^{(\ell nx)^{\ell n\,\ell n\,x}}$ ... **(i)**

taking $\log_e$ both sides, we get $\ell ny = (\ell n\, x)^{\ell n\,(\ell n\,x)} . \ell nx$

Again taking $\log_e$, we get $\ell n\,(\ell n\, y) = \ell n\,(\ell n\, x) . \ell n\,(\ell n\, x)$

$+ \ell n\,(\ell n\, x) = \{\ell n\,(\ell n\, x)\}\ [\ell n\,(\ell n\, x) + 1]$

Diff. w.r.t. x,

$$\left(\frac{1}{\ell n\ y}\right)\left(\frac{1}{y}\cdot\frac{dy}{dx}\right) = \left(\frac{1}{x\ell nx}\right)[\ell n\,(\ell n\, x) + 1] + \{\ell n\,(\ell n\, x)\}\left[\frac{1}{x\ell nx}\right]$$

$$\Rightarrow \frac{dy}{dx} = y\ \ell ny\left[\frac{\ell n(\ell nx)}{x\ell nx} + \frac{1}{x\ell nx} + \frac{\ell n(\ell nx)}{x\ell nx}\right]$$

$$\Rightarrow \frac{dy}{dx} = y\ \ell ny\left[\frac{2\ell n(\ell nx)+1}{x\ell nx}\right]$$

3. $\therefore\ x = \text{cosec}\theta - \sin\theta$

$\Rightarrow\ x^2 + 4 = (\text{cosec}\theta + \sin\theta)^2$

and $y^2 + 4 = (\text{cosec}^n\theta + \sin^n\theta)^2$

Now $$\frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{n\cot\theta(\text{cosec}^n\theta + \sin^n\theta)}{\cot\theta(\text{cosec}\theta + \sin\theta)}$$

$$= \frac{n\sqrt{y^2+4}}{\sqrt{x^2+4}}$$

Squaring both sides, we get $$\left(\frac{dy}{dx}\right)^2 = \frac{n^2(y^2+4)}{x^2+4}$$

or $$(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$$

4. **(C)**

$y^2 = P(x)$

$$\Rightarrow\ 2y\frac{dy}{dx} = P'(x)$$

or $$2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = P''(x)$$

or $$2y\frac{d^2y}{dx^2} = P'' - 2\left(\frac{dy}{dx}\right)^2 = P'' - \frac{P'^2}{2y^2}$$

$$\because\ 2y^3\frac{d^2y}{dx^2} = y^2P'' - \frac{1}{2}P'^2 = PP'' - \frac{1}{2}P'^2$$

$$\therefore\ 2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right) = P'\,P'' + PP''' - P'\,P'' = PP'''$$

5. Given that $g^{-1}(x) = f(x)$

$\Rightarrow\ x = g(f(x))$ or $g'(f(x))\,f'(x) = 1$

$$\Rightarrow\ g'(f(x)) = \frac{1}{f'(x)}$$

$$\Rightarrow\ g''(f(x))\cdot f'(x) = \frac{-f''(x)}{(f'(x))^2}$$

$$\Rightarrow\ g''(f(x)) = -\frac{f''(x)}{(f'(x))^3}$$

6. **(A)**

$$F'(x) = \left[f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right) + g\left(\frac{x}{2}\right)g'\left(\frac{x}{2}\right)\right]$$

here $g(x) = f'(x)$ & $g'(x) = f''(x) = -f(x)$

so $$F'(x) = f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) - f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) = 0$$

$\Rightarrow$ F(x) is constant function

so $F(10) = 5$

7. $x^y . y^x = 1 \quad \Rightarrow\ y\,\ell nx + x\,\ell ny = 0$

Diff. w.r.t. x, we get

$$\left(\frac{dy}{dx}\right).\ell nx + \frac{y}{x} + \ell ny + \frac{x}{y}\frac{dy}{dx} = 0$$

$$\Rightarrow\ \frac{dy}{dx} = -\frac{\left(\frac{y}{x} + \ell ny\right)}{\frac{x}{y} + \ell nx} = -\frac{y}{x}\frac{(x\ \ell n\ y + y)}{(y\ell n\ x + x)}$$

8. **(D)**

$$f'(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$f''(x) = \begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$f''(0) = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} = 0$$

9. $y = x(\ell nx - \ell n(a+bx))$ ......**(i)**

$\Rightarrow \frac{dy}{dx} = \ell nx - \ell n(a+bx) + \frac{a}{a+bx}$ ......**(ii)**

$\Rightarrow \frac{d^2y}{dx^2} = \frac{a}{x(a+bx)} - \frac{ab}{(a+bx)^2}$

$\Rightarrow x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2}$ ......**(iii)**

By **(i)** and **(ii)**, $x\frac{dy}{dx} - y = \frac{ax}{a+bx}$

$\Rightarrow x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$

10. (B)

S_1: $y = \sin 2x$

$\frac{dy}{dx} = 2\cos 2x$

$\frac{d^2y}{dx^2} = -4\sin 2x$

$\frac{d^3y}{dx^3} = -8\cos 2x$

$\frac{d^4y}{dx^4} = 16\sin 2x$

$\frac{d^6y}{dx^6} = -64\sin 2x$

$\therefore \frac{d^6y}{dx^6}$ at $x = \frac{\pi}{2}$ is $= 0$ $\therefore$ Statement is false.

S_2: $x = e^{y+e^{y+....\infty}} = e^{y+x}$

$\therefore 1 = e^{y+x}\left(1+\frac{dy}{dx}\right) = x\left(1+\frac{dy}{dx}\right)$

$\therefore \frac{dy}{dx} = \frac{1}{x} - 1$

$\therefore \left.\frac{dy}{dx}\right|_{at\ x=1} = 0$ $\therefore$ Statement is true.

S_3: $y = 2t^2$ $x = 4t$

$\therefore y = 2\left(\frac{x}{4}\right)^2 = \frac{x^2}{8}$

$\therefore \frac{dy}{dx} = \frac{x}{4}$ $\therefore \frac{d^2y}{dx^2} = \frac{1}{4}$

$\therefore$ Statement is false.

S_4: $x = t^2 + 3t - 8, y = 2t^2 - 2t - 5$

$\therefore \frac{dx}{dt} = 2t + 3$; $\frac{dy}{dt} = 4t - 2$

$\therefore \frac{dy}{dx} = \frac{4t-2}{2t+3}$

$\therefore \frac{d^2y}{dx^2} = \frac{(2t+3)\,4 - (4t-2)\,2}{(2t+3)^2} \cdot \frac{dt}{dx}$

$= \frac{16}{(2t+3)^2} \times \frac{1}{(2t+3)}$

$\therefore \frac{d^2y}{dx^2} = \frac{16}{(2t+3)^3}$

When $x = 2$ and $y = -1$ then $t = 2$

$\therefore \frac{d^2y}{dx^2}$ at $(2, -1)$ is $= \frac{16}{7^3}$ $\therefore$ Statement is true.

11. (A, B)

$y = \sec(\tan^{-1}x)$

$\therefore \frac{dy}{dx} = \frac{\sec(\tan^{-1}x)\,.\,\tan(\tan^{-1}x)}{1+x^2}$

$\therefore \left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{\sqrt{2}} = \sin^{-1}\left(\sin\frac{1}{\sqrt{2}}\right)$

12. (D)

$y = \tan^{-1}\left(\frac{2^{x+1} - 2^x}{1 + 2^x \cdot 2^{x+1}}\right) = \tan^{-1} 2^{(x+1)} - \tan^{-1} 2^x$

$\Rightarrow y' = \frac{2^{x+1}\ln 2}{1+(2^{x+1})^2} - \frac{2^x \ln 2}{1+(2^x)^2}$

$\Rightarrow y'(0) = -\frac{1}{10}\ln 2$

13. (B, C)

$y = 10^{10^x}$

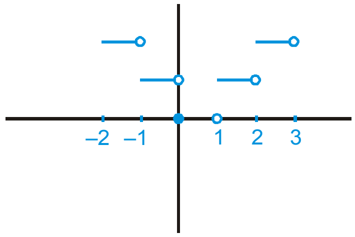
$\therefore \frac{dy}{dx} = 10^{10^x}\,\ell n\,10\,.\,10^x\,\ell n\,10 = y\,10^x\,(\ell n\,10)^2$

$\therefore \frac{1}{y}\frac{dy}{dx} = 10^x(\ell n\,10)^2$

$\therefore \lambda = (\ell n\,10)^2 = e^{\ell n(\ell n 10)^2}$

14. (B)

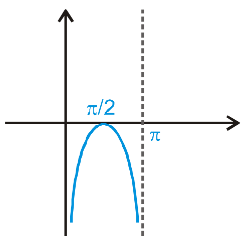
S_1 : the graph of function $f(|f(x)|) = [|[x]|]$



it is clear from the graph $f\left(\left|f\left(\frac{1}{2}\right)\right|\right) = 0$

S_2 : Function is not defined at $x = n\pi$

S_3 : Graph of $f(x) = \log|\sin x|$



clearly $f'(x) < 0 \ \forall \ x \in \left(\frac{\pi}{2}, \pi\right)$

S_4 : $f'\left(\frac{\pi}{2}\right) = \lim_{h\to 0} \frac{e^{\cos h} - e}{|h|}$

$\Rightarrow$ does not exist

15. (A, D)

$f(x) = -\frac{x^3}{3} + x^2 \sin 1.5\,a - x \sin a.\ \sin 2a - 5\sin^{-1}(a^2 - 8a + 17)$

f(x) is defined when $-1 \le a^2 - 8a + 17 \le 1$

$-1 \le (a-4)^2 + 1 \le 1$

$\Rightarrow$ a = 4

$\therefore$ $f(x) = -\frac{x^3}{3} + x^2 \sin 6 - x \sin 4 \sin 8 - \frac{5\pi}{2}$

$\therefore$ $f'(x) = -x^2 + 2x \sin 6 - \sin 4 \sin 8$

$f'(\sin 8) = -\sin^2 8 + 2 \sin 8 \sin 6 - \sin 4 \sin 8 = \sin 8\,[2 \sin 6 - (\sin 8 + \sin 4)]$

$= \sin 8\,[2 \sin 6 - 2 \sin 6 \cos 2] = 2 \sin 6 \sin 8\,(1 - \cos 2)$

$\sin 6 < 0,\ \sin 8 > 0,\ 1 - \cos 2 > 0$

$\therefore$ $f'(\sin 8) < 0$

16. Statement I : f(x) is constant function

Statement II : It is true

17. (A)

$$\frac{d}{dx}\ell n|x| = \frac{d}{dx}\ell n(-x) = \frac{1}{(-x)}(-1) = \frac{1}{x}$$

19. (B)

$$\frac{d}{dx}\{\tan^{-1}(\sec x + \tan x)\} = \frac{d}{dx}\left\{\tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)\right\}$$

$$= \frac{d}{dx}\left\{\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)\right\} = \frac{d}{dx}\left(\frac{\pi}{4}+\frac{x}{2}\right) = \frac{1}{2}$$

$$\frac{d}{dx}\{\cot^{-1}(\operatorname{cosec} x + \cot x)\} = \frac{d}{dx}\left\{\cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)\right\}$$

$$= \frac{d}{dx}\left\{\cot^{-1}\left(\cot\frac{x}{2}\right)\right\} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}$$

20. $y = x^2$

$$\frac{dy}{dx} = 2x$$

$$\frac{d^2y}{dx^2} = 2$$

again $\quad 2x\frac{dx}{dy} = 1$

$$2\left(\frac{dx}{dy}\right)^2 + 2x\frac{d^2x}{dy^2} = 0 \quad \Rightarrow \quad x\frac{d^2x}{dy^2} = -\left(\frac{dx}{dy}\right)^2$$

$$\Rightarrow \frac{d^2x}{dy^2} = -\frac{1}{4x^2} \quad \Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right) \ne 1$$

Statement-II :

$$\because \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{\left(\frac{dx}{dy}\right)^2}\cdot\frac{d^2x}{dy^2}\cdot\frac{dy}{dx} = -\left(\frac{dy}{dx}\right)^3\cdot\frac{d^2x}{dy^2}$$

21. (A) → (q), (B) → (p), (C) → (t), (D) → (p)

(A) $\lim_{x\to\frac{1}{\sqrt{2}}} \frac{x - \cos(\sin^{-1}x)}{1 - \tan(\sin^{-1}x)}$

$$= \lim_{x\to\frac{1}{\sqrt{2}}} \frac{x-\sqrt{1-x^2}}{1-\frac{x}{\sqrt{1-x^2}}} = \lim_{x\to\frac{1}{\sqrt{2}}} \left(-\sqrt{1-x^2}\right) = -\frac{1}{\sqrt{2}}$$

(B) $x = \frac{1}{2}$ is not in the domain

$\therefore$ $f'\left(\frac{1}{2}\right)$ does not exists

(C) $y = f(x) = \ell n \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$

$$\therefore \frac{dy}{dx} = \frac{\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)} \cdot \frac{1}{2}$$

$$= \frac{1}{2} \frac{1}{\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)} = \frac{1}{\sin\left(\frac{\pi}{2}+x\right)} = \sec x$$

(D) $\lim_{x\to 0} \frac{1}{x}\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \lim_{x\to 0}\frac{|\tan x|}{x}$ = does not exists

22. (A) $\to$ **(t), (B)** $\to$ **(q), (C)** $\to$ **(r), (D)** $\to$ **(s)**

(A) $y = \cos^{-1}(\cos x)$

$$y' = \frac{-1}{\sqrt{1-\cos^2 x}} \cdot (-\sin x) = \frac{\sin x}{|\sin x|}$$

$\therefore$ y' at $x = 5$ is -1

(B) $y = f(x) = \ell n |\tan x|$

$$\therefore f'(x) = (1/\tan x)(\sec^2 x) \cdot \left(\frac{|\tan x|}{\tan x}\right)$$

$$f'\left(-\frac{\pi}{4}\right) = 2$$

(C) $$\frac{d}{dx}\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2} \cdot \frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$= \frac{(1-x)^2}{2(1+x^2)} \cdot \frac{2}{(1-x)^2} = \frac{1}{1+x^2}$$

at $x = -1$

$$\frac{d}{dx}\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{1}{2}$$

(D) $$\frac{d}{dx}\frac{\ell n|x|}{x} = \frac{x\cdot\frac{1}{x}-\ell n|x|}{x^2} = \frac{1-\ell n|x|}{x^2}$$

$$\Rightarrow \frac{d}{dx}\frac{\ell n|x|}{x} = 1 \quad \text{at } x = -1$$

24.

1. (C)

$f(x)\,g(x) = x^3 g(x)$

$3x^2 . g'(x) = 3x^2 g(x) + x^3 g'(x)$

$3g'(x) = 3g(x) + xg'(x)$

$(3-x)\,g'(x) = 3g(x)$

$$\int \frac{g'(x)}{g(x)}dx = \int \frac{3}{3-x}dx$$

$\ell n\, g(x) = -3\ \ell n\,|3-x| + \ell n c$

$$\therefore g(x) = \frac{c}{|3-x|^3}$$

$$g(0) = \frac{c}{27} = \frac{1}{3} \qquad \therefore c = 9$$

$$\therefore g(x) = \frac{9}{|3-x|^3}$$

2. (A)

$f(x-3).\,g(x) = (x-3)^3.\,g(x) = 9$

$\therefore$ derivative of $f(x-3).\,g(x)$ is 0

3. (A)

$$\lim_{x\to 0}\frac{f(x).g(x)}{x(1+g(x))} = \lim_{x\to 0}\frac{x^3\frac{9}{|3-x|^3}}{x\left(1+\frac{9}{|3-x|^3}\right)} = 0$$

25.

1. (D)

$$f'(x) = -\frac{2x}{(1+x^2)^2}$$

$$f''(x) = \frac{-2+6x^2}{(1+x^2)^3} \quad f''(x) = 0 \text{ if } x = \pm\frac{1}{\sqrt{3}}$$

$\therefore$ $f'(x)$ is greatest at $x = -\frac{1}{\sqrt{3}}$

2. (A)

Equation of tangent at x = α is

$$y-\frac{1}{1+\alpha^2}=\frac{-2\alpha}{(1+\alpha^2)^2}(x-\alpha)$$

$$\therefore \quad b=\frac{1}{1+\alpha^2}+\frac{2\alpha^2}{(1+\alpha^2)^2}=\frac{1+3\alpha^2}{(1+\alpha^2)^2}$$

$$\therefore \quad \frac{db}{d\alpha}=\frac{(1+\alpha^2)^2\,.\,6\alpha-2(1+3\alpha^2)(1+\alpha^2)\,2\alpha}{(1+\alpha^2)^4}$$

$$=\frac{2\alpha(1-3\alpha^2)}{(1+\alpha^2)^3}$$

$$\frac{db}{d\alpha}=0 \quad \text{if} \quad \alpha=0,\pm\frac{1}{\sqrt{3}}$$

$$\text{at } \alpha=\pm\frac{1}{\sqrt{3}}, \qquad b=\frac{9}{8}$$

3. (A)

$$a=\frac{1+3\alpha^2}{2\alpha}$$

$$\therefore \quad \frac{1}{a}=\frac{2\alpha}{1+3\alpha^2} \text{ its greatest value is } \frac{1}{\sqrt{3}}$$

26. (5)

$$y=\tan^{-1}\frac{4x}{1+5x^2}+\tan^{-1}\frac{2+3x}{3-2x}$$

$$=\tan^{-1}\frac{5x-x}{1+5x.x}+\tan^{-1}\frac{\frac{2}{3}+x}{1-\frac{2}{3}.x}$$

$$=\tan^{-1}5x-\tan^{-1}x+\tan^{-1}\frac{2}{3}+\tan^{-1}x$$

$$=\tan^{-1}5x+\tan^{-1}\frac{2}{3}$$

$$\therefore \quad \frac{dy}{dx}=\frac{5}{1+25x^2}$$

27. $\because \quad y=\left[\ell n\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)\right]^2+k\,\ell n\,(x+\sqrt{x^2-a^2})$

Differentiating both sides w. r. t. x, we get

$$\frac{dy}{dx}=2\left[\ell n\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)\right]\cdot\frac{a}{(x+\sqrt{x^2-a^2})}$$

$$\frac{(x+\sqrt{x^2-a^2})}{a\sqrt{x^2-a^2}}+\frac{k}{(x+\sqrt{x^2-a^2})}\cdot\frac{(x+\sqrt{x^2-a^2})}{\sqrt{(x^2-a^2)}}$$

$$\frac{dy}{dx}=\frac{2}{\sqrt{x^2-a^2}}\,\ell n\left[\frac{x+\sqrt{x^2-a^2}}{a}\right]+\frac{k}{\sqrt{x^2-a^2}}$$

$$\Rightarrow \quad \sqrt{(x^2-a^2)}\frac{dy}{dx}=2\,\ell n\,(x+\sqrt{x^2-a^2})-2\,\ell n\,a+k$$

Differentiating both sides w.r.t. x, we get

$$\sqrt{(x^2-a^2)}\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{x}{\sqrt{x^2-a^2}}$$

$$=\frac{2}{(x+\sqrt{x^2-a^2})}\cdot\frac{(x+\sqrt{x^2-a^2})}{\sqrt{x^2-a^2}}$$

or $\sqrt{x^2-a^2}\dfrac{d^2y}{dx^2}+\dfrac{x}{\sqrt{x^2-a^2}}\dfrac{dy}{dx}=\dfrac{2}{\sqrt{x^2-a^2}}$

or $(x^2-a^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}=2$

hence value of $(x^2-a^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}$ is 2

28. Let $y_1=\tan^{-1}\left(\dfrac{\ell n(e/x^2)}{\ell n(ex^2)}\right)$

and $y_2=\tan^{-1}\left(\dfrac{3+2\ell nx}{1-6\ell nx}\right)$

Let $a=2\,\ell n\,x$,

then $y_1=\tan^{-1}\left(\dfrac{\ell ne-\ell nx^2}{\ell ne+\ell nx^2}\right)=\tan^{-1}\left(\dfrac{1-a}{1+a}\right)=\tan^{-}$

$1\left(\tan\left(\dfrac{\pi}{4}-\alpha\right)\right)=\dfrac{\pi}{4}-\alpha$, where $a=\tan\alpha$

Similarly $y_2=\tan^{-1}\left(\dfrac{3+a}{1-3a}\right)=\beta+\alpha$, where $\tan\beta=3$

$$\Rightarrow \quad y=\left(\frac{\pi}{4}-\alpha\right)+(\beta+\alpha)=\frac{\pi}{4}+\tan^{-1}3=\text{constant}$$

$$\Rightarrow \quad \frac{dy}{dx}=0 \quad \Rightarrow \quad \frac{d^2y}{dx^2}=0$$

29. $y = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left(\frac{\sqrt{a-b}}{\sqrt{a+b}} \tan\frac{x}{2}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{a^2-b^2}} \cdot \frac{\sec^2\left(\frac{x}{2}\right)\cdot\left(\frac{a-b}{a+b}\right)^{1/2}}{1+\left(\frac{a-b}{a+b}\right)\tan^2\frac{x}{2}} \cdot \frac{1}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(a+b)\cos^2\frac{x}{2} + (a-b)\sin^2\frac{x}{2}} = \frac{1}{a + b\cos x}$$

at $a = 1, b = 1, x = 0$

$\frac{dy}{dx} = 1/2$

30. (2)

$x = e^t \sin t$ and $y = e^t \cos t$

$\Rightarrow x^2 + y^2 = e^{2t} \quad \Rightarrow e^t = \sqrt{x^2 + y^2}$ ....**(i)**

and $\tan t = \frac{x}{y} \quad \Rightarrow t = \tan^{-1}\left(\frac{x}{y}\right)$ put in (i)

$\therefore \quad e^{\tan^{-1}\left(\frac{x}{y}\right)} = \sqrt{x^2 + y^2}$ ....**(ii)**

taking ℓn of both sides.

$\tan^{-1}\left(\frac{x}{y}\right) = \frac{1}{2}\ell n(x^2 + y^2)$

differentiate both sides w.r.t. 'x'

$$\Rightarrow \left(\frac{1}{1+\frac{x^2}{y^2}}\right)\left(\frac{y.1 - x.y'}{y^2}\right) = \frac{1}{2}\cdot\frac{(2x+2yy')}{(x^2+y^2)}$$

$\Rightarrow y' = \frac{y-x}{x+y}$ ....**(iii)**

again differentiate equation **(iii)** w.r.t. 'x'

$$\Rightarrow y'' = \frac{(x+y)(y'-1)-(y-x)(1+y')}{(x+y)^2}$$

$\Rightarrow y''(x+y)^2 = y'(2x) - 2y$

$\Rightarrow y''(x+y)^2 = 2(xy' - y)$. Hence proved.