METHOD OF DIFFERENTIATION

DERIVATIVE BY FIRST PRINCIPLE

Let $y = f(x)$; $y + \Delta y = f(x + \Delta x)$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \text{(average rate of change of function)}$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \text{....(i)}$$

(i) denotes the instantaneous rate of change of function.

Finding the value of the limit given by (i) in respect of variety of functions is called finding the derivative by first principle / by delta method / by ab-initio / by fundamental definition of calculus.

Note that if $y = f(x)$ then the symbols

$\frac{dy}{dx} = Dy = f'(x) = y_1$ or y' have the same meaning.

However a dot, denotes the time derivative.

e.g. $\dot{S} = \frac{dS}{dt}$; $\dot{\theta} = \frac{d\theta}{dt}$ etc.

Ex. Differentiate each of following functions by first principle :

(i) $f(x) = \tan x$ **(ii)** $f(x) = e^{\sin x}$

Sol. (i) $f(x) = \lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h} = \lim_{h \to 0} \frac{\tan(x+h-x)[1 + \tan x \tan(x+h)]}{h}$

$$= \lim_{h \to 0} \frac{\tanh}{h} . (1 + \tan^2 x) = \sec^2 x. \quad \textbf{Ans.}$$

(ii) $f(x) = \lim_{h \to 0} \frac{e^{\sin(x+h)} - e^{\sin x}}{h} = \lim_{h \to 0} e^{\sin x} \frac{\left[e^{\sin(x+h) - \sin x} - 1\right]}{\sin(x+h) - \sin x} \left(\frac{\sin(x+h) - \sin x}{h}\right)$

$$= e^{\sin x} \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = e^{\sin x} \cos x \quad \textbf{Ans.}$$

DERIVATIVE OF STANDARD FUNCTIONS

f(x)	f'(x)	f'(x)	f'(x)
(i) x^n	nx^{n-1}	(ii) e^x	e^x
(iii) a^x	$a^x \ell na,\ a > 0$	(iv) ℓnx	$1/x$
(v) $\log_a x$	$(1/x)\log_a e,\ a > 0,\ a \neq 1$	(vi) sinx	cosx
(vii) cosx	– sinx	(viii) tanx	$\sec^2 x$
(ix) secx	secx tanx	(x) cosecx	– cosecx . cotx
(xi) cotx	$-\text{cosec}^2 x$	(xii) constant	0
(xiii) $\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$	(xiv) $\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$
(xv) $\tan^{-1} x$	$\frac{1}{1+x^2},\ x \in R$	(xviii) $\cot^{-1} x$	$\frac{-1}{1+x^2},\ x \in R$
(xvii) $\text{cosec}^{-1} x$	$\frac{-1}{\lvert x \rvert \sqrt{x^2-1}}, \lvert x \rvert > 1$	(xvi) $\sec^{-1} x$	$\frac{1}{\lvert x \rvert \sqrt{x^2-1}}, \lvert x \rvert > 1$

FUNDAMENTAL THEOREMS

Sum of two differentiable functions is always differentiable.

Sum of two non-differentiable functions may be differentiable.

There are certain basic theorems in differentiation:

1. $\frac{d}{dx}(f \pm g) = f'(x) \pm g'(x)$

2. $\frac{d}{dx}(k\, f(x)) = k \frac{d}{dx} f(x)$

3. $\frac{d}{dx}(f(x)\,.\,g(x)) = f(x)\,g'(x) + g(x)\,f'(x)$

4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\,f'(x) - f(x)\,g'(x)}{g^2(x)}$

5. $\frac{d}{dx}(f(g(x))) = f'(g(x))\,g'(x)$

This rule is also called the chain rule of differentiation and can be written as

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

Note that an important inference obtained from the chain rule is that

$$\frac{dy}{dy} = 1 = \frac{dy}{dx} \cdot \frac{dx}{dy} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{dx/dy}$$

another way of expressing the same concept is by considering y = f(x) and x = g(y) as inverse functions of each other.

$$\frac{dy}{dx} = f'(x) \qquad \text{and} \qquad \frac{dx}{dy} = g'(y)$$

$$\Rightarrow \quad g'(y) = \frac{1}{f'(x)}$$

Ex. If $y = e^x \tan x + x\log_e x$, find $\frac{dy}{dx}$.

Sol. $y = ex.\tan x + x \cdot \log_e x$

On differentiating we get,

$$\frac{dy}{dx} = e^x \cdot \tan x + e^x \cdot \sec^2 x + 1 \cdot \log x + x \cdot \frac{1}{x}$$

Hence, $\frac{dy}{dx} = e^x(\tan x + \sec^2 x) + (\log x + 1)$

Ex. Find the derivative of the following functions with respect to x.

(i) $f(x) = \sqrt{\sin(2x+3)}$ **(ii)** $f(x) = \frac{x}{1+x^2}$ **(iii)** $f(x) = x \,.\, \sin x$

Sol. **(i)** $f(x) = \sqrt{\sin(2x+3)}$

$\Rightarrow \quad f'(x) = \frac{d}{dx}\left(\sqrt{\sin(2x+3)}\right) = \frac{1}{2\sqrt{\sin(2x+3)}} \cdot \frac{d}{dx}(\sin(2x+3))$ (chain rule)

$$= \frac{\cos(2x+3)}{\sqrt{\sin(2x+3)}}$$

(ii) $f(x) = \frac{x}{1+x^2} \Rightarrow f'(x) = \frac{(1+x^2) - x(2x)}{(1+x^2)^2}$ (Quotient rule)

$$= \frac{1-x^2}{(1+x^2)^2}$$

(iii) $f(x) = x \sin x$

$\Rightarrow \quad f'(x) = x.\cos x + \sin x$ (Product rule)

Ex. If $x = \exp\left(\tan^{-1}\left(\frac{y-x^2}{x^2}\right)\right)$, then $\frac{dy}{dx}$ equals -

(A) $x[1 + \tan(\log x) + \sec^2 x]$ **(B)** $2x[1 + \tan(\log x)] + \sec^2 x$

(C) $2x[1 + \tan(\log x)] + \sec x$ **(D)** $2x + x[1 + \tan(\log x)]^2$

Sol. Taking log on both sides, we get

$\log x = \tan^{-1}\left(\frac{y-x^2}{x^2}\right) \Rightarrow \tan(\log x) = (y - x^2)/x^2$

$\Rightarrow \quad y = x^2 + x^2\tan(\log x)$

On differentiating, we get

$\therefore \quad \frac{dy}{dx} = 2x + 2x\tan(\log x) + x\sec^2(\log x) \Rightarrow 2x[1 + \tan(\log x)] + x\sec^2(\log x)$

$$= 2x + x[1 + \tan(\log x)]^2$$

Ex. If $f(x) = 2x\sec^{-1}x - \text{cosec}^{-1}(x)$, then find $f'(-2)$.

Sol. $f'(x) = 2\sec^{-1}(x) + \dfrac{2x}{|x|\sqrt{x^2-1}} + \dfrac{1}{|x|\sqrt{x^2-1}}$

Hence, $f'(-2) = 2.\sec^{-1}(-2) - \dfrac{2}{\sqrt{3}} + \dfrac{1}{2\sqrt{3}}$

$f'(-2) = \dfrac{4\pi}{3} - \dfrac{\sqrt{3}}{2}$

(A) LOGARITHMIC DIFFERENTIATION

To find the derivative of :

(i) a function which is the product or quotient of a number of functions

(ii) a function of the form $[f(x)]^{g(x)}$ where f & g are both derivable, it will be found convinient to take the logarithm of the function first & then differentiate

express $y = (f(x))^{g(x)} = e^{g(x).\ln(f(x))}$ and then differentiate.

Ex. If $y = (\sin x)^{\ell n\, x}$, find $\dfrac{dy}{dx}$

Sol. $\ell n\, y = \ell n\, x \,.\, \ell n\,(\sin x)$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x}\,\ell n\,(\sin x) + \ell n\, x.\dfrac{\cos x}{\sin x}$

$\Rightarrow \quad \dfrac{dy}{dx} = (\sin x)^{\ell n\, x}\left[\dfrac{\ell n(\sin x)}{x} + \cot x\;\ell n\, x\right]$

Ex. If $y = \dfrac{x^{1/2}(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/5}}$ find $\dfrac{dy}{dx}$

Sol. $\ell n\, y = \dfrac{1}{2}\,\ell n\, x + \dfrac{2}{3}\,\ell n\,(1-2x) - \dfrac{3}{4}\,\ell n\,(2-3x) - \dfrac{4}{5}\,\ell n\,(3-4x)$

On differentiating we get,

$\Rightarrow \quad \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{2x} - \dfrac{4}{3(1-2x)} + \dfrac{9}{4(2-3x)} + \dfrac{16}{5(3-4x)}$

$\dfrac{dy}{dx} = y\left(\dfrac{1}{2x} - \dfrac{4}{3(1-2x)} + \dfrac{9}{4(2-3x)} + \dfrac{16}{5(3-4x)}\right)$

PARAMETRIC DIFFERENTIATION

In some situation curves are represented by the equations e.g. $x = \sin t$ & $y = \cos t$

If $x = f(t)$ and $y = g(t)$ then

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{g'(t)}{f'(t)}$$

Ex. If $y = a \cos t$ and $x = a(t - \sin t)$ find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{2}$

Sol. $\frac{dy}{dx} = \frac{-a\sin t}{a(1-\cos t)}$ $\Rightarrow$ $\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = -1$

Ex. If $x = a\cos^3 t$ and $y = a\sin^3 t$, then find the value of $\frac{dy}{dx}$.

Sol. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3a\sin^2 t\cos t}{3a\cos^2 t\sin t} = -\tan t$

DERIVAIVE OF IMPLICIT FUNCTION : $\phi(x, y) = 0$

(i) In order to find dy/dx, in the case of implicit functions, we differentiate each term w.r.t. x regarding y as a functions of x & then collect terms in dy/dx together on one side to finally find dy/dx.

(ii) In answers of dy/dx in the case of implicit functions, both x & y are present.

Corresponding to every curve represented by an implicit equation, there exist one or more explicit functions representing that equation. It can be shown that dy / dx at any point on the curve remains the same whether the process of differentiation is done explictly or implicitly.

Ex. If $x^3 + y^3 = 3xy$, then find $\frac{dy}{dx}$.

Sol. Differentiating both sides w.r.t.x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

Note that above result holds only for points where $y^2 - x \neq 0$

Ex. If $y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \cfrac{\sin x}{1 + \cos x.....}}}$, prove that $\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1 + 2y + \cos x - \sin x}$.

Sol. Given function is $y = \dfrac{\sin x}{1+\dfrac{\cos x}{1+y}} = \dfrac{(1+y)\sin x}{1+y+\cos x}$

or $y + y^2 + y\cos x = (1+y)\sin x$

Differentiate both sides with respect to x,

$$\frac{dy}{dx} + 2y\frac{dy}{dx} + \frac{dy}{dx}\cos x - y\sin x = (1+y)\cos x + \frac{dy}{dx}\sin x$$

$$\frac{dy}{dx}(1 + 2y + \cos x - \sin x) = (1+y)\cos x + y\sin x$$

or $$\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1 + 2y + \cos x - \sin x}$$

DIFFERENTIATION USING SUBSTITUTION

In certain situations as mentioned below, substitution simplifies differentiation. For each of the following expression, appropriate substitution is as follows

(i) $\sqrt{x^2+a^2}$: $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ or $x = a\cot\theta$, where $0 < \theta < \pi$

(ii) $\sqrt{a^2-x^2}$: $x = a\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ or $x = a\cos\theta$, where $0 \le \theta \le \pi$

(iii) $\sqrt{x^2-a^2}$: $x = a\sec\theta$, where $\theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$ or $x = a\,\text{cosec}\,\theta$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(iv) $\sqrt{\dfrac{x+a}{a-x}}$: $x = a\cos\theta$, where $0 < \theta \le \pi$

Ex. Differentiate $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ with respect to x.

Sol. Let $x = \tan\theta$, where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$

$$y = \tan^{-1}\left(\frac{|\sec\theta|-1}{\tan\theta}\right) \qquad \left\{|\sec\theta| = \sec\theta\ \forall\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$$

$\Rightarrow$ $y = \tan^{-1}\left(\dfrac{1-\cos\theta}{\sin\theta}\right)$ $\Rightarrow$ $y = \tan^{-1}\left(\tan\dfrac{\theta}{2}\right)$

$\Rightarrow$ $y = \dfrac{\theta}{2}$ $\left\{\tan^{-1}(\tan x) = x \text{ for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$

$\Rightarrow$ $y = \dfrac{1}{2}\tan^{-1} x$ $\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{1}{2(1+x^2)}$

DERIVATIVE OF f (x) w.r.t. g (x)

If y = f (x) and z = g (x) then derivative of f (x) w.r.t. g (x) is given by

$$\frac{dy}{dz} = \frac{dy}{dx}\cdot\frac{dx}{dz} = \frac{f'(x)}{g'(x)}$$

$\therefore$ Differential coefficient of f (x) w.r.t. g(x) $= \dfrac{\text{derivative of f(x) w.r.t. x}}{\text{derivative of g(x) w.r.t. x}} = \dfrac{f'(x)}{g'(x)}$

Ex. Find derivative of $y = \ell n\, x$ with respect to z.

Sol. $\dfrac{dy}{dz} = \dfrac{dy/dx}{dz/dx} = \dfrac{1}{xe^x}$

Ex. Differentiate $\log_e(\tan x)$ with respect to $\sin^{-1}(e^x)$.

Sol. $\dfrac{d(\log_e \tan x)}{d(\sin^{-1}(e^x))} = \dfrac{\frac{d}{dx}(\log_e \tan x)}{\frac{d}{dx}\sin^{-1}(e^x)} = \dfrac{\cot x.\sec^2 x}{e^x.1/\sqrt{1-e^{2x}}} = \dfrac{e^{-x}\sqrt{1-e^{2x}}}{\sin x \cos x}$

DERIVATIVE OF INVERSE FUNCTION

Theorem : If the inverse functions f & g are defined by y = f(x) & x = g(y) & if $f'(x)$ exists & $f'(x) \neq 0$ then $g'(y) = \dfrac{1}{f'(x)}$. This result can also be written as, if $\dfrac{dy}{dx}$ exists & $\dfrac{dy}{dx} \neq 0$,

then $\dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}$ or $\dfrac{dy}{dx}.\dfrac{dx}{dy} = 1$ or $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$ $[\dfrac{dx}{dy} \neq 0]$

If g is inverse of f, then

(A) $g\{f(x)\} = x$

$g'\{f(x)\}f'(x) = 1$

(B) $f\{g(x)\} = x$

$f'\{g(x)\}g'(x) = 1$

Ex. If g is inverse of f and $f'(x) = \dfrac{1}{1+x^n}$, then g'(x) equals :-

(A) $1 + x^n$ **(B)** $1 + [f(x)]^n$ **(C)** $1 + [g(x)]^n$ **(D)** none of these

Sol. Since g is the inverse of f. Therefore

$f(g(x)) = x$ for all x

$\Rightarrow \dfrac{d}{dx} f(g(x)) = 1$ for all x

$\Rightarrow f'(g(x))\, g'(x) = 1 \Rightarrow g'(x) = \dfrac{1}{f'(g(x))} = 1 + (g(x))^n$

HIGHER ORDER DERIVATIVES

Let a function y = f(x) be defined on an open interval (a, b). It's derivative, if it exists in (a, b) is a certain function $f'(x)$ [or (dy/dx) or y'] & is called the first derivative of y w. r. t. x.

If it happens that the first derivative has a derivative in (a, b) then this derivative is called the second derivative of y w. r. t. x & is denoted by $f''(x)$ or (d^2y/dx^2) or y''. While the first derivative denotes slope of the graph, the second derivative denotes it's concavity.

Similarly, the 3rd order derivative of y w. r. t. x, if it exists, is defined by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$, it is also denoted by $f'''(x)$ or y'''.

It must be carefully noted that in case of parametric functions

although $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ but $\frac{d^2y}{dx^2} \neq \frac{d^2y/dt^2}{dx^2/dt^2}$ rather $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right)$

which on applying chain rule can be resolved as

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left\{\frac{dy/dt}{dx/dt}\right\}.\frac{dt}{dx} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\left(\frac{dx}{dt}.\frac{d^2y}{dt^2} - \frac{dy}{dt}.\frac{d^2x}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^2}.\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{\left[\frac{dx}{dt}.\frac{d^2y}{dt^2} - \frac{dy}{dt}.\frac{d^2x}{dt^2}\right]}{\left(\frac{dx}{dt}\right)^3}$$

Ex. If $y = x^3 \ell n\, x$, then find y'' and y'''

Sol. $y' = 3x^2 \ell n\, x + x^3 \frac{1}{x} = 3x^2 \ell n\, x + x^2$

$y'' = 6x\, \ell n\, x + 3x^2 . \frac{1}{x} + 2x = 6x\, \ell n\, x + 5x$

$y''' = 6\, \ell n\, x + 11$

Ex. If $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$ for all $x \in R$. Then find f(x) independent of $f'(1)$, $f''(2)$ and $f'''(3)$.

Sol. Here, $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$

put $f'(1) = a,\ f''(2) = b,\ f'''(3) = c$ ....**(i)**

$\therefore$ $f(x) = x^3 + ax^2 + bx + c$

$\Rightarrow$ $f'(x) = 3x^2 + 2ax + b$ or $f'(1) = 3 + 2a + b$ ....**(ii)**

$\Rightarrow$ $f''(x) = 6x + 2a$ or $f''(2) = 12 + 2a$ ....**(iii)**

$\Rightarrow$ $f'''(x) = 6$ or $f'''(3) = 6$ ....**(iv)**

from **(i)** and **(iv)**, c = 6

from **(i)**, **(ii)** and **(iii)** we have, a = –5, b = 2

$\therefore$ $f(x) = x^3 - 5x^2 + 2x + 6$

Ex. If $x = t + 1$ and $y = t^2 + t^3$, then find $\frac{d^2y}{dx^2}$.

Sol. $\frac{dy}{dt} = 2t + 3t^2$; $\frac{dx}{dt} = 1$

$\Rightarrow \frac{dy}{dx} = 2t + 3t^2$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dt}(2t + 3t^2) \cdot \frac{dt}{dx}$

$\frac{d^2y}{dx^2} = 2 + 6t.$

DERIVATIVE OF A DETERMINANT

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h, l, m, n, u, v, w are differentiable functions of x, then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Ex. If $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$, find f'(x).

Sol. Here, $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

On differentiating, we get,

$$\Rightarrow f'(x) = \begin{vmatrix} \frac{d}{dx}(x) & \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ \frac{d}{dx}(1) & \frac{d}{dx}(2x) & \frac{d}{dx}(3x^2) \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ \frac{d}{dx}(0) & \frac{d}{dx}(2) & \frac{d}{dx}(6x) \end{vmatrix}$$

or $$f'(x) = \begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 0 & 2 & 6x \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix}$$

As we know if any two rows or columns are equal, then value of determinant is zero.

$$= 0 + 0 + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix} \qquad \therefore \quad f'(x) = 6(2x^2 - x^2)$$

Therefore, $f'(x) = 6x^2$

L'HOSPITAL'S RULE

(A) This rule is applicable for the indeterminate forms of the type $\frac{0}{0}, \frac{\infty}{\infty}$. If the function f(x) and g(x) are differentiable in certain neighbourhood of the point 'a', except, may be, at the point 'a' itself and $g'(x) \neq 0$, and if

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \text{or} \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty,$$

then $$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided the limit $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists (L' Hôpital's rule). The point 'a' may be either finite or improper $(+\infty$ or $-\infty)$.

(B) Indeterminate forms of the type $0.\infty$ or $\infty-\infty$ are reduced to forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic transformations.

(C) Indeterminate forms of the type $1^{\infty}, \infty^0$ or 0^0 are reduced to forms of the type $0\times\infty$ by taking logarithms or by the transformation $[f(x)]^{\phi(x)} = e^{\phi(x).\ell nf(x)}$.

Ex. Evaluate $\lim_{x\to 0}|x|^{\sin x}$

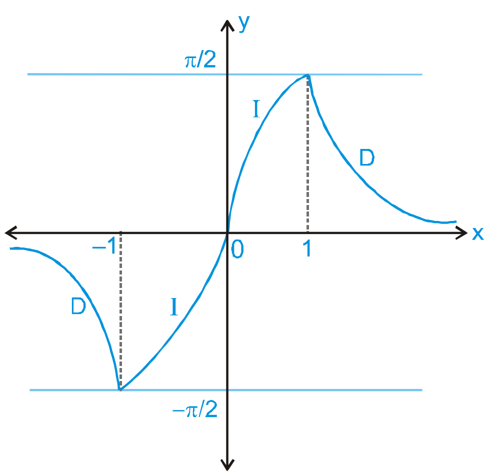
Sol. $$\lim_{x\to 0}|x|^{\sin x} = \lim_{x\to 0} e^{\sin x \log_e|x|} = e^{\lim_{x\to 0}\frac{\log_e|x|}{\operatorname{cosec} x}}$$

$$= e^{\lim_{x\to 0}\frac{1/x}{-\operatorname{cosec} x \cot x}} \qquad \text{(applying L'Hôpital's rule)}$$

$$= e^{\lim_{x\to 0} -\frac{\sin^2 x}{x\cos x}} = e^{\lim_{x\to 0} -\left(\frac{\sin x}{x}\right)^2 \cdot \left(\frac{x}{\cos x}\right)} = e^{-(1)^2\cdot(0)} = e^0 = 1$$

ANALYSIS AND GRAPHS OF SOME INVERSE TRIGONOMETRIC FUNCTIONS

(A) $$y = f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & |x| \le 1 \\ \pi - 2\tan^{-1}x & x > 1 \\ -(\pi + 2\tan^{-1}x) & x < -1 \end{cases}$$



Important points

(i) Domain is $x \in R$ & range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) f is continuous for all x but not differentiable at $x = 1, -1$

(iii) $\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & \text{for} \quad |x| < 1 \\ \text{non existent} & \text{for} \quad |x| = 1 \\ \frac{-2}{1+x^2} & \text{for} \quad |x| > 1 \end{cases}$

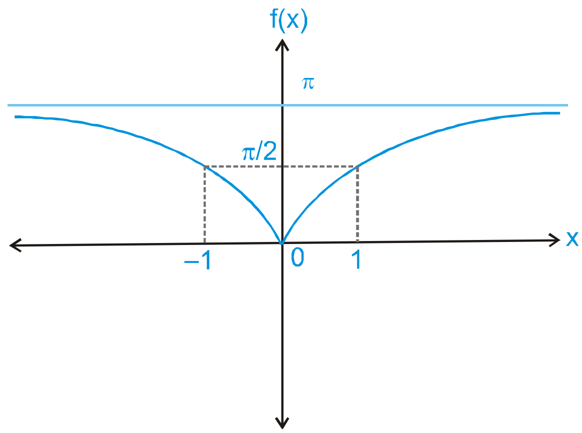
(iv) Increasing in $(-1, 1)$ & Decreasing in $(-\infty, -1) \cup (1, \infty)$

(B) Consider $y = f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & \text{if} \quad x \geq 0 \\ -2\tan^{-1}x & \text{if} \quad x < 0 \end{cases}$

Important points

(i) Domain is $x \in R$ & range is $[0, \pi)$

(ii) Continuous for all x but not differentiable at $x = 0$



(iii) $\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & \text{for} \quad x > 0 \\ \text{non existent} & \text{for} \quad x = 0 \\ -\frac{2}{1+x^2} & \text{for} \quad x < 0 \end{cases}$

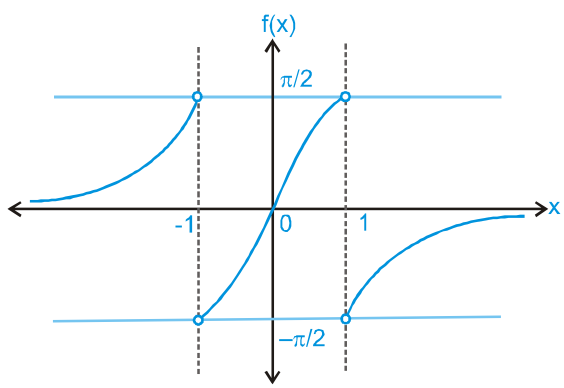
(iv) Increasing in $(0, \infty)$ & Decreasing in $(-\infty, 0)$

(C) $y = f(x) = \tan^{-1}\frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & |x| < 1 \\ \pi + 2\tan^{-1}x & x < -1 \\ -(\pi - 2\tan^{-1}x) & x > 1 \end{cases}$

Important points

(i) Domain is $R - \{1, -1\}$ & range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(ii) It is neither continuous nor differentiable at $x = 1, -1$



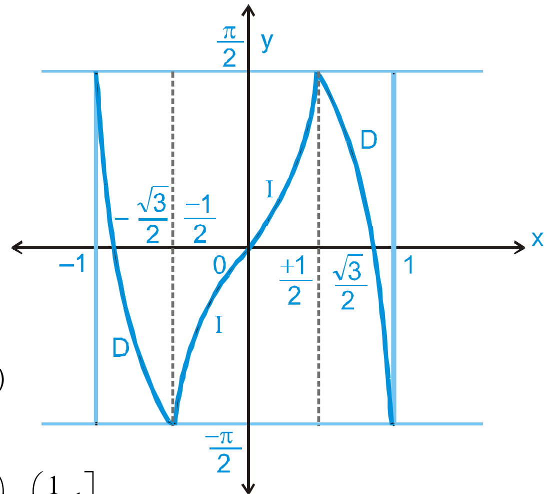
(iii) $\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & |x| \neq 1 \\ \text{non existent} & |x| = 1 \end{cases}$

(iv) Increasing $\forall$ x in its domain

(v) It is bounded for all x

(D) $y = f(x) = \sin^{-1}(3x - 4x^3) = \begin{cases} -(\pi + 3\sin^{-1} x) & \text{if } -1 \le x < -\frac{1}{2} \\ 3\sin^{-1} x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - 3\sin^{-1} x & \text{if } \frac{1}{2} < x \le 1 \end{cases}$

Important points



(i) Domain is $x \in [-1, 1]$ & range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) Continuous everywhere in its domain

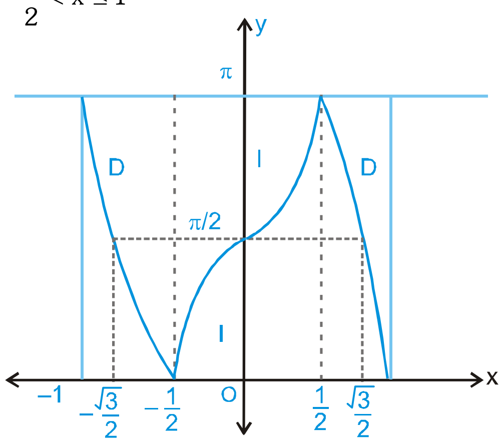
(iii) Not derivable at $x = -\frac{1}{2}, \frac{1}{2}$

(iv) $\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}} & \text{if } x \in (-\frac{1}{2}, \frac{1}{2}) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if } x \in (-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1) \end{cases}$

(v) Increasing in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and Decreasing in $\left[-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$

(E) $y = f(x) = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1} x - 2\pi & \text{if } -1 \le x < -\frac{1}{2} \\ 2\pi - 3\cos^{-1} x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1} x & \text{if } \frac{1}{2} < x \le 1 \end{cases}$

Important points



(i) Domain is $x \in [-1, 1]$ & range is $[0, \pi]$

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(v) Increasing in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ & Decreasing in $\left[-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$

GENERAL NOTE

Concavity is decided by the sign of 2nd derivative as :

$\frac{d^2y}{dx^2} > 0 \Rightarrow$ Concave upwards ; $\frac{d^2y}{dx^2} < 0 \Rightarrow$ Concave downwards

Ex. $\frac{d}{dx}\left\{\sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)\right\} =$

Sol. Let $y = \sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)$. Put $x = \cos 2\theta$ $\theta \in \left(0, \frac{\pi}{2}\right]$

$\therefore$ $y = \sin^2 \cot^{-1}\left(\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\right) = \sin^2 \cot^{-1}(\cot\theta)$

$\therefore$ $y = \sin^2\theta = \frac{1-\cos 2\theta}{2} = \frac{1-x}{2} = \frac{1}{2} - \frac{x}{2}$

$\therefore$ $\frac{dy}{dx} = -\frac{1}{2}$.

TIPS & FORMULAS

1. Derivative of f(x) from the First Principle

Obtaining the derivative using the definition $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = f'(x) = \frac{dy}{dx}$ is called calculating derivative using first principle or ab initio or delta method.

2. Fundamental Theorems

If f and g are derivable functions of x, then,

(A) $\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$

(B) $\frac{d}{dx}(cf) = c\frac{df}{dx}$, where c is any constant

(C) $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$ known as **"Product Rule"**

(D) $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\left(\frac{df}{dx}\right) - f\left(\frac{dg}{dx}\right)}{g^2}$ where $g \neq 0$ known as **"Quotient Rule"**

(E) If $y = f(u)$ & $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ known as **"Chain Rule"**

Note : In general if $y = f(u)$ then $\frac{dy}{dx} = f'(u).\frac{du}{dx}$.

3. Derivative of Standard Functions

f(x)	f'(x)	f(x)	f'(x)
(i) x^n	nx^{n-1}	(ii) e^x	e^x
(iii) a^x	$a^x \ell na$, $a > 0$	(iv) ℓnx	$1/x$
(v) $\log_a x$	$(1/x) \log_a e$, $a > 0$, $a \neq 1$	(vi) sinx	cosx
(vii) cosx	– sinx	(viii) tanx	$\sec^2 x$
(ix) secx	secx tanx	(x) cosecx	– cosecx . cotx
(xi) cotx	$-\text{cosec}^2 x$	(xii) constant	0
(xiii) $\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$	(xiv) $\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$
(xv) $\tan^{-1} x$	$\frac{1}{1+x^2}$, $x \in R$	(xviii) $\cot^{-1} x$	$\frac{-1}{1+x^2}$, $x \in R$
(xvii) $\text{cosec}^{-1} x$	$\frac{-1}{\lvert x \rvert \sqrt{x^2-1}}, \lvert x \rvert > 1$	(xvi) $\sec^{-1} x$	$\frac{1}{\lvert x \rvert \sqrt{x^2-1}}, \lvert x \rvert > 1$

4. Logarithmic Differentiation

To find the derivative of :

(A) A function which is the product or quotient of a number of functions or

(B) A function of the form $[f(x)]^{g(x)}$ where f & g are both derivable.

It is convenient to take the logarithm of the function first & then differentiate.

5. Differentiation of Implicit Functions

(A) Let function is $\phi(x, y) = 0$ then to find dy /dx, in the case of implicit functions, we differentiate each term w.r.t. x regarding y as a functions of x & then collect terms in dy/dx together on one side to finally find dy/dx.

OR $\dfrac{dy}{dx} = \dfrac{-\partial\phi/\partial x}{\partial\phi/\partial y}$ where $\dfrac{\partial\phi}{\partial x}$ & $\dfrac{\partial\phi}{\partial y}$ are partial differential coefficient of f(x, y) w.r.t to x & y respectively.

(B) In answer of dy/dx in the case of implicit functions, generally, both x & y are present.

6. Parametric Differentiation

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$.

7. Derivative of a Function w.r.t. another Function

Let $y = f(x)$; $z = g(x)$ then $\dfrac{dy}{dz} = \dfrac{dy/dx}{dz/dx} = \dfrac{f'(x)}{g'(x)}$

8. Derivative of a Function and its Inverse Function

If inverse of $y = f(x)$

$x = f^{-1}(y)$ is denoted by $x = g(y)$ then $g(f(x)) = x$

$g'(f(x))f'(x)=1$

9. Higher Order Derivatives

Let a function $y = f(x)$ be defined on an open interval (a, b). It's derivative, if it exists on (a,b) is a certain function $f'(x)$ [or (dy/dx) or y'] is called the first derivative of y w.r.t. x. If it happens that the first derivative has a derivative on (a,b) then derivative is called second derivative of y w.r.t. x & is denoted by $f''(x)$ [or d^2y/dx^2 or y''] . Similarly, the 3rd order derivative of y w.r.t x, if it exists, is defined by $\dfrac{d^3y}{dx^3} = \dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)$. It is also denoted by $f'''(x)$ or y''' and so on.

10. Differentiation of Determinants

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h. l, m, n, u, v, w are differentiable functions of x then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

11. L'HOSPITAL'S RULE

(A) Applicable while calculating limits of indeterminate forms of the type $\frac{0}{0}, \frac{\infty}{\infty}$. If the function f(x) and g(x) are differentiable in certain neighbourhood of the point 'a', except, may be, at the point 'a' itself and $g'(x) \neq 0$, and if

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \text{or} \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty,$$

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

provided the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists (L' Hospital's rule). The point 'a' may be either finite or improper ($+\infty$ or $-\infty$).

(B) Indeterminate forms of the type $0.\infty$ or $\infty - \infty$ are reduced to forms of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic transformations.

(C) Indeterminate forms of the type 1^{∞}, ∞^0 or 0^0 are reduced to forms of the type $0 \times \infty$ by taking logarithms or by the transformation $[f(x)]^{\phi(x)} = e^{\phi(x).\ell nf(x)}$.