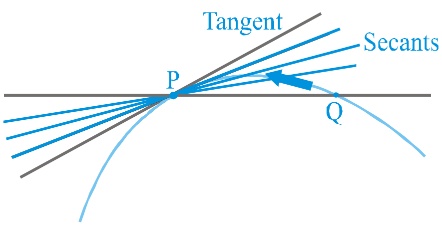
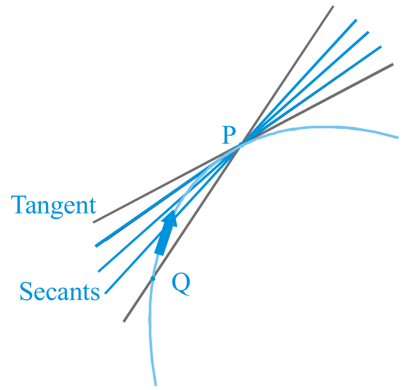
APPLICATION OF DERIVATIVE

TANGENT & NORMAL

Let y = f(x) be function with graph as shown in figure. Consider secant PQ. If Q tends to P along the curve passing through the points Q_1, Q_2, ...

I.e. $Q \rightarrow P$, secant PQ will become tangent at P. A line through P perpendicular to tangent is called normal at P.

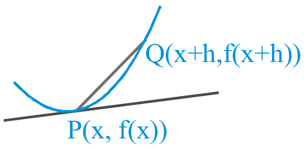




GEOMETRICAL MEANING OF $\frac{dy}{dx}$

As $Q \rightarrow P$, $h \rightarrow 0$ and slope of chord PQ tends to slope of tangent at P (see figure).

Slope of chord PQ = $\frac{f(x+h)-f(x)}{h}$



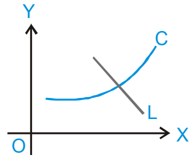
$\lim\limits_{Q\rightarrow P}$ slope of chord PQ = $\lim\limits_{h\rightarrow 0}\frac{f(x+h)-f(x)}{h}$

$\Rightarrow$ slope of tangent at P = $f'(x) = \frac{dy}{dx}$

MYTHS ABOUT TANGENT

(A) **Myth :** A line meeting the curve only at one point is a tangent to the curve.

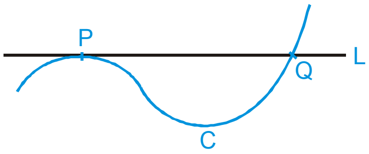
Explanation : A line meeting the curve in one point is not necessarily tangent to it.



Here L is not tangent to C

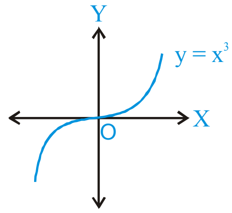
(B) **Myth :** A line meeting the curve at more than one point is not a tangent to the curve.

Explanation : A line may meet the curve at several points and may still be tangent to it at some point



Here L is tangent to C at P, and cutting it again at Q.

(C) **Myth :** Tangent at a point to the curve can not cross it at the same point.
Explanation : A line may be tangent to the curve and also cross it.



Here X-axis is tangent to $y = x^3$ at origin.

EQUATION OF TANGENT AND NORMAL

(A) The value of the derivative at $P(x_1, y_1)$ gives the slope of the tangent to the curve at P. Symbolically

$$f'(x_1) = \left.\frac{dy}{dx}\right]_{(x_1, y_1)} = \text{Slope of tangent at } P(x_1, y_1) = m(\text{say}).$$

(B) Equation of tangent at (x_1, y_1) is ; $y - y_1 = \left.\frac{dy}{dx}\right]_{(x_1, y_1)} (x - x_1)$

(C) Equation of normal at (x_1, y_1) is; $y - y_1 = -\frac{1}{\left.\frac{dy}{dx}\right]_{(x_1, y_1)}} (x - x_1)$.

(i) The point P (x_1, y_1) will satisfy the equation of the curve & the equation of tangent & normal line.

(ii) If the tangent at any point P on the curve is parallel to the axis of x then dy/dx = 0 at the point P.

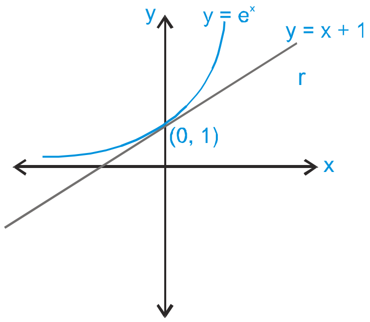
(iii) If the tangent at any point on the curve is parallel to the axis of y, then dy/dx not defined or dx/dy = 0.

(iv) If the tangent at any point on the curve is equally inclined to both the axes then dy/dx = ±1.

(v) If a curve passing through the origin be given by a rational integral algebraic equation, then the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation. e.g. If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. $x + y = 0$ and $x - y = 0$

Ex. Find equation of tangent to $y = e^x$ at $x = 0$. Hence draw graph

Sol. At $x = 0$ $\Rightarrow$ $y = e^0 = 1$



Hence point of tangent is (0, 1)

$$\frac{dy}{dx} = e^x \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=0} = 1$$

Hence equation of tangent is

$$1 (x - 0) = (y - 1)$$

$\Rightarrow$ $y = x + 1$

Ex. Find the equation of the tangent to the curve $y=(x^3-1)(x-2)$ at the points where the curve cuts the x-axis.

Sol. The equation of the curve is $y=(x^3-1)(x-2)$ .... **(i)**

It cuts x-axis at y = 0. So, putting y = 0 in (i), we get $(x^3-1)(x-2)=0$

$\Rightarrow (x-1)(x-2)(x^2+x+1)=0 \Rightarrow x-1=0,\ x-2=0 \quad [\because x^2+x+1\neq 0]$

$\Rightarrow x=1,\ 2.$

Thus, the points of intersection of curve **(i)** with x-axis are (1, 0) and (2, 0). Now,

$y=(x^3-1)(x-2)$

$\Rightarrow \frac{dy}{dx}=3x^2(x-2)+(x^3-1)$

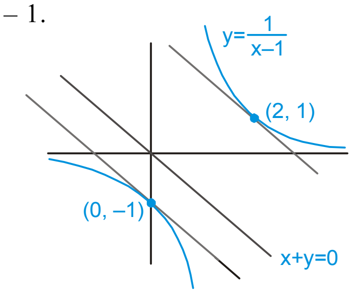
$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)}=-3$ and $\left(\frac{dy}{dx}\right)_{(2,0)}=7$

The equations of the tangents at (1, 0) and (2, 0) are respectively

$y-0=-3(x-1)$ and $y-0=7(x-2) \Rightarrow y+3x-3=0$ and $7x-y-14=0$

Ex. Find the equation of all straight lines which are tangent to curve $y=\frac{1}{x-1}$ and which are parallel to the line x + y = 0.

Sol. Suppose the tangent is at (x_1, y_1) and it has slope – 1.



$\Rightarrow \left.\frac{dy}{dx}\right|_{(x_1,y_1)}=-1.$

$\Rightarrow -\frac{1}{(x_1-1)^2}=-1.$

$\Rightarrow x_1=0$ or 2

$\Rightarrow y_1=-1$ or 1

Hence tangent at (0, – 1) and (2, 1) are the required lines (see figure) with equations

$-1(x-0)=(y+1)$ and $-1(x-2)=(y-1)$

$\Rightarrow x+y+1=0$ and $y+x=3$

Ex. The equation of the normal to the curve $y=x+\sin x\cos x$ at $x=\frac{\pi}{2}$ is -

Sol. $\because x=\frac{\pi}{2} \Rightarrow y=\frac{\pi}{2}+0=\frac{\pi}{2}$, so the given point $=\left(\frac{\pi}{2},\frac{\pi}{2}\right)$

Now from the given equation $\frac{dy}{dx}=1+\cos^2 x-\sin^2 x \Rightarrow \left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{2},\frac{\pi}{2}\right)}=1+0-1=0$

$\Rightarrow$ The curve has vertical normal at $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$.

The equation to this normal is $x=\frac{\pi}{2} \Rightarrow x-\frac{\pi}{2}=0 \Rightarrow 2x=\pi$

Ex. Find equation of normal to the curve $y = |x^2 - |x||$ at $x = -2$.

Sol. In the neighborhood of $x = -2$, $y = x^2 + x$.

Hence the point of contact is $(-2, 2)$

$$\frac{dy}{dx} = 2x + 1 \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=-2} = -3.$$

So the slope of normal at $(-2, 2)$ is $\frac{1}{3}$.

Hence equation of normal is

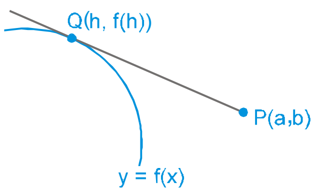
$$\frac{1}{3}(x + 2) = y - 2.$$

$$\Rightarrow \quad 3y = x + 8.$$

TANGENT FROM AN EXTERNAL POINT

Given a point P(a, b) which does not lie on the curve $y = f(x)$, then the equation of possible tangents to the curve $y = f(x)$, passing through (a, b) can be found by solving for the point of contact Q.

$$f'(h) = \frac{f(h) - b}{h - a}$$



Figure

And equation of tangent is $y - b = \frac{f(h) - b}{h - a}(x - a)$

Ex. Find value of c such that line joining points (0, 3) and (5, – 2) becomes tangent to curve $y = \frac{c}{x+1}$.

Sol. Equation of line joining A & B is $x + y = 3$

Solving this line and curve we get

$$3 - x = \frac{c}{x+1} \quad \Rightarrow \quad x^2 - 2x + (c - 3) = 0 \qquad \text{......(i)}$$

For tangency, roots of this equation must be equal.

Hence discriminant of quadratic equation = 0

$$\Rightarrow \quad 4 = 4(c - 3) \quad \Rightarrow \quad c = 4$$

Putting c = 4, equation **(i)** becomes

$$x^2 - 2x + 1 = 0 \quad \Rightarrow \quad x = 1$$

Hence point of contact becomes (1, 2).

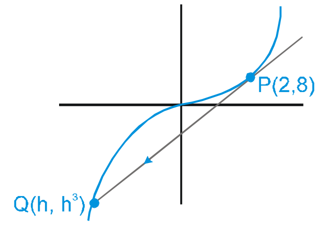
Ex. Tangent at P(2, 8) on the curve $y = x^3$ meets the curve again at Q. Find coordinates of Q.

Sol. Equation of tangent at (2, 8) is

$y = 12x - 16$

Solving this with $y = x^3$

$x^3 - 12x + 16 = 0$



This cubic will give all points of intersection of line and curve $y = x^3$ i.e., point P and Q. (see figure)

But, since line is tangent at P so x = 2 will be a repeated root of equation $x^3 - 12x + 16 = 0$ and another root will be x = h. Using theory of equations :

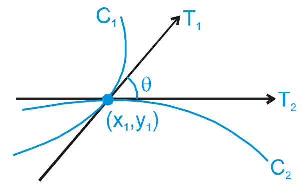
sum of roots $\Rightarrow$ $2 + 2 + h = 0$ $\Rightarrow$ $h = -4$

Hence coordinates of Q are (– 4, – 64)

ANGLE BETWEEN THE CURVES

Angle between two intersecting curves is defined as the acute angle between their tangents (or normals) at the point of intersection of two curves (as shown in figure).

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$



where m_1 & m_2 are the slopes of tangents at the intersection point (x_1, y_1).

Orthogonal Curve

(i) If the angle between two curves at each point of intersection is 90° then they are called orthogonal curves.

(ii) Two curves are said to be orthogonal if angle between them at each point of intersection is right angle.
i.e. $m_1 m_2 = -1$.
For example, the curves $x^2 + y^2 = r^2$ & $y = mx$ are orthogonal curves.

> **(i)** The angle is defined between two curves if the curves are intersecting. This can be ensured by finding their point of intersection or graphically.
>
> **(ii)** If the curves intersect at more than one point then angle between curves is found out with respect to the point of intersection.

Ex. The angle of intersection between the curve $x^2 = 32y$ and $y^2 = 4x$ at point (16, 8) is -

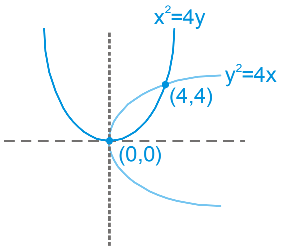
Sol. $x^2 = 32y \Rightarrow \frac{dy}{dx} = \frac{x}{16} \Rightarrow y^2 = 4x \Rightarrow \frac{dy}{dx} = \frac{2}{y}$

$\therefore$ at $(16, 8), \left(\frac{dy}{dx}\right)_1 = 1, \left(\frac{dy}{dx}\right)_2 = \frac{1}{4}$

So required angle $= \tan^{-1}\left(\frac{1 - \frac{1}{4}}{1 + 1\left(\frac{1}{4}\right)}\right) = \tan^{-1}\left(\frac{3}{5}\right)$

Ex. Find angle between $y^2 = 4x$ and $x^2 = 4y$. Are these two curves orthogonal?

Sol. $y^2 = 4x$ and $x^2 = 4y$ intersect at point (0, 0) and (4, 4) (see figure).



$C_1 : y^2 = 4x$ $\qquad$ $C_2 : x^2 = 4y$

$$\frac{dy}{dx} = \frac{2}{y} \qquad \frac{dy}{dx} = \frac{x}{2}$$

$$\left.\frac{dy}{dx}\right|_{(0,0)} \to \infty \qquad \left.\frac{dy}{dx}\right|_{(0,0)} = 0$$

Hence $\tan\theta = 90°$ at point (0, 0)

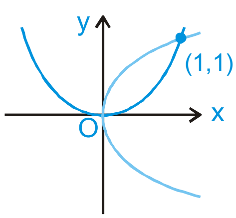
$$\left.\frac{dy}{dx}\right|_{(4,4)} = \frac{1}{2} \qquad \left.\frac{dy}{dx}\right|_{(4,4)} = 2$$

$$\tan\theta = \left|\frac{2-\frac{1}{2}}{1+2.\frac{1}{2}}\right| = \frac{3}{4}$$

Two curves are not orthogonal because angle between them at (4, 4) is not 90°.

Ex. Check the orthogonality of the curves $y^2 = x$ & $x^2 = y$.

Sol. Solving the curves simultaneously we get points of intersection as (1, 1) and (0, 0).



At (1,1) for first curve $\quad 2y\left(\frac{dy}{dx}\right)_1 = 1 \Rightarrow \cdot\, m_1 = \frac{1}{2}$

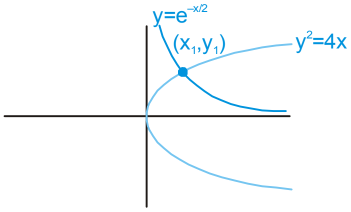
& for second curve $\quad 2x = \left(\frac{dy}{dx}\right)_2 \Rightarrow m_2 = 2$

$m_1 m_2 \neq -1$ at (1,1).

But at (0, 0) clearly x-axis & y-axis are their respective tangents hence they are orthogonal at (0,0) but not at (1, 1). Hence these curves are not said to be orthogonal.

Ex. Find the angle between curves $y^2 = 4x$ and $y = e^{-x/2}$

Sol. Let the curves intersect at point (x_1, y_1) (see figure).



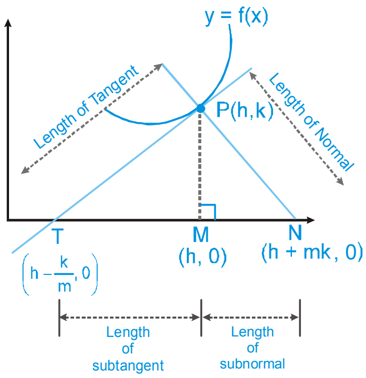
for $y^2 = 4x$, $\quad \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{2}{y_1}$

and for $y = e^{-x/2}$ $\quad \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = -\frac{1}{2}\, e^{-x_1/2} = -\frac{y_1}{2}$

$\Rightarrow \quad m_1 m_2 = -1 \qquad$ Hence $\theta = 90°$

LENGTHS OF TANGENT, NORMAL, SUBTANGENT AND SUBNORMAL

Let P (h, k) be any point on curve y = f(x). Let tangent drawn at point P meets x-axis at T & normal at point P meets x-axis at N. Then the length PT is called the length of tangent and PN is called length of normal. (as shown in figure)



Projection of segment PT on x-axis, TM, is called the subtangent and similarly projection of line segment PN on x axis, MN is called subnormal.

Let $m = \left.\frac{dy}{dx}\right]_{(h,k)}$ = slope of tangent.

Hence equation of tangent is $m(x - h) = (y - k)$.

Putting y = 0, we get x - intercept of tangent is $x = h - \frac{k}{m}$

Similarly, the x-intercept of normal is $x = h + km$

Now, length PT, PN, TM, MN can be easily evaluated using distance formula

(i) $PT = |k|\sqrt{1+\frac{1}{m^2}}$ = Length of Tangent **(ii)** $PN = |k|\sqrt{1+m^2}$ = Length of Normal

(iii) $TM = \left|\frac{k}{m}\right|$ = Length of subtangent **(iv)** $MN = |km|$ = Length of subnormal.

Ex. Find the length of tangent for the curve $y = x^3 + 3x^2 + 4x - 1$ at point x = 0.

Sol. Here, $m = \left.\frac{dy}{dx}\right|_{x=0}$

$\frac{dy}{dx} = 3x^2 + 6x + 4 \Rightarrow m = 4$

and, $k = y(0) \Rightarrow k = -1$

$\ell = |k|\sqrt{1 + \frac{1}{m^2}} \Rightarrow \ell = |(-1)|\sqrt{1+\frac{1}{16}} = \frac{\sqrt{17}}{4}$

Ex. The length of the tangent to the curve $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$ is

Sol. $\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right) = \frac{a\cos t}{a\left(-\sin t + \frac{1}{\sin t}\right)} = \tan t$

$\therefore$ length of the tangent $= y\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\left(\frac{dy}{dx}\right)} = a\sin t\,\frac{\sqrt{1+\tan^2 t}}{\tan t} = a\sin t\left(\frac{\sec t}{\tan t}\right) = a$

Ex. Prove that for the curve $y = be^{x/a}$, the length of subtangent at any point is always constant.

Sol. $y = be^{x/a}$

Let the point be (x_1, y_1)

$\Rightarrow \quad m = \left.\frac{dy}{dx}\right|_{x=x_1} = \frac{b.e^{x_1/a}}{a} = \frac{y_1}{a}$

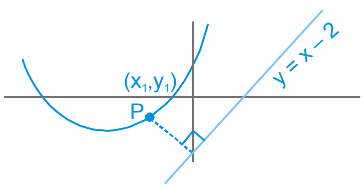
Now, length of subtangent $= \left|\frac{y_1}{m}\right| = \left|\frac{y_1}{y_1/a}\right| = |a|$; which is always constant.

SHORTEST DISTANCE BETWEEN TWO CURVES

Shortest distance between two non-intersecting differentiable curves is always along their common normal. (Wherever defined)

Ex. Find the shortest distance between the line $y = x - 2$ and the parabola $y = x^2 + 3x + 2$.

Sol. Let $P(x_1, y_1)$ be a point closest to the line $y = x - 2$



then $\left.\frac{dy}{dx}\right|_{(x_1, y_1)} =$ slope of line

$\Rightarrow \quad 2x_1 + 3 = 1 \quad \Rightarrow \quad x_1 = -1 \quad \Rightarrow \quad y_1 = 0$

Hence point $(-1, 0)$ is the closest and its perpendicular distance from the line $y = x - 2$ will give the shortest distance

$\Rightarrow \quad p = \frac{3}{\sqrt{2}}$.

ERROR AND APPROXIMATION

In order to calculate the approximate value of a function, differentials may be used where the differential of a function is equal to its derivative multiplied by the differential of the independent variable. Thus if, y = tanx then dy = $\sec^2 x$ dx.

In general dy = f'(x)dx or df(x) = f'(x)dx

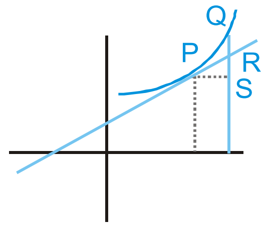
Let y = f(x) be a function. If these is an error δx in x then corresponding error in y is δy = f(x + δx) – f(x).

We have $\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} = \frac{dy}{dx} = f'(x)$

We define the differential of y, at point x, corresponding to the increment δx as f'(x) δx and denote it by dy.

i.e. dy = f'(x) δx.

Let P(x, f(x)), Q((x + δx), f(x + δx)) (as shown in figure)



δy = QS,

δx = PS,

dy = RS

In many practical situations, it is easier to evaluate dy but not δy.

Ex. Find the approximate value of square root of 25.2.

Sol. Let $f(x) = \sqrt{x}$

Now, $f(x + \Delta x) - f(x) = f'(x) . \Delta x = \frac{\Delta x}{2\sqrt{x}}$

we may write, 25.2 = 25 + 0.2

Taking x = 25 and Δx = 0.2,

we have $f(25.2) - f(25) = \frac{0.2}{2\sqrt{25}}$

or $f(25.2) - \sqrt{25} = \frac{0.2}{10} = 0.02 \Rightarrow f(25.2) = 5.02$

or $\sqrt{(25.2)} = 5.02$

RATE MEASUREMENT

Whenever one quantity y varies with another quantity x, satisfying some rule y = *f*(x), then $\frac{dy}{dx}$ (or *f* '(x)) represents the rate of change of y with respect to x and $\left.\frac{dy}{dx}\right]_{x=a}$ (or *f* '(a)) represents the rate of change of y with respect to x at x = a.

Ex. How fast the area of a circle increases when its radius is 5cm;
(i) with respect to radius **(ii)** with respect to diameter

Sol. **(i)** $A = \pi r^2$, $\frac{dA}{dr} = 2\pi r$

$\therefore \quad \left.\frac{dA}{dr}\right]_{r=5} = 10\pi \text{ cm}^2/\text{cm}.$

(ii) $A = \frac{\pi}{4} D^2$, $\frac{dA}{dD} = \frac{\pi}{2} D$

$\therefore \quad \left.\frac{dA}{dD}\right]_{D=10} = \frac{\pi}{2} . 10 = 5\pi \text{ cm}^2/\text{cm}.$

Ex. The volume of a cube is increasing at a rate of $9\text{cm}^3/\text{s}$. How fast is the surface area increasing when the length of an edge is 10cm ?

Sol. Let x be the length of side, V be the volume and S be the surface area of the cube. Then $V = x^3$ and $S = 6x^2$, where x is a function of time t.

$$\frac{dV}{dt} = 9\text{cm}^3/\text{s} = \frac{d}{dt}(x^3) = 3x^2 \frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3}{x^2}$$

$$\frac{dS}{dt} = \frac{d}{dt}(6x^2) = 12x\left(\frac{3}{x^2}\right) = \frac{36}{x}$$

$$\left.\frac{dS}{dt}\right]_{x=10\text{cm}} = 3.6 \text{ cm}^2/\text{s}.$$

MONOTONICITY

Let f be a real valued function having domain D($D \subset R$) and S be a subset of D. f is said to be monotonically increasing (non decreasing) (increasing) in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$. f is said to be monotonically decreasing (non increasing) (decreasing) in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

f is said to be strictly increasing in S if for $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. Similarly, f is said to be strictly decreasing in S if for $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

(i) f is strictly increasing $\Rightarrow$ f is monotonically increasing (non decreasing). But converse need not be true.

(ii) f is strictly decreasing $\Rightarrow$ f is monotonically decreasing (non increasing). Again, converse need not be true.

(iii) If f(x) = constant in S, then f is increasing as well as decreasing in S

(iv) A function f is said to be an increasing function if it is increasing in the domain. Similarly, if f is decreasing in the domain, we say that f is monotonically decreasing

(v) f is said to be a monotonic function if either it is monotonically increasing or monotonically decreasing

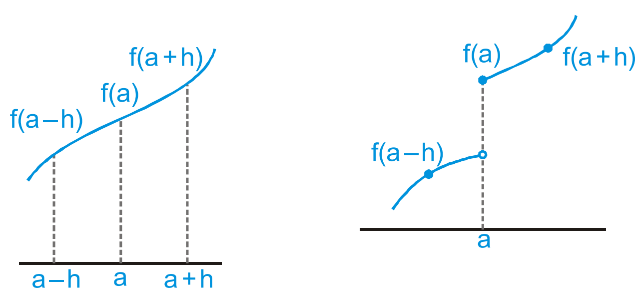
(vi) If f is increasing in a subset of S and decreasing in another subset of S, then f is non monotonic in S.

(vii) A function is said to be monotonic if it's either increasing or decreasing.

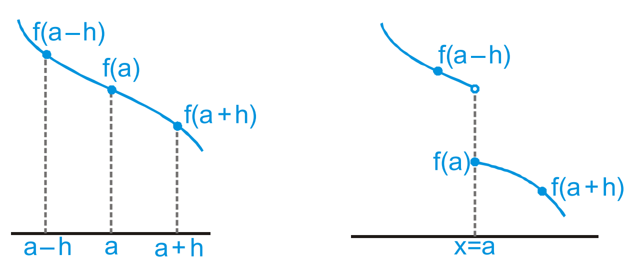
(viii) If a function is invertible it has to be either increasing or decreasing.

MONOTONICITY AT A POINT

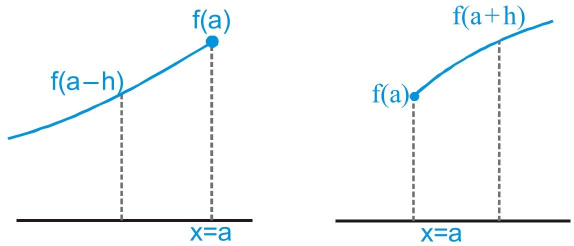
(A) A function f(x) is called an increasing function at point x = a, if in a sufficiently small neighbourhood of $x = a$; $f(a-h) < f(a) < f(a+h)$



(B) A function f(x) is called a decreasing function at point x = a, if in a sufficiently small neighbourhood of $x = a$; $f(a-h) > f(a) > f(a+h)$



❖ If x = a is a boundary point, then use the appropriate one sides inequality to test Monotonicity of f(x).



$f(a) > f(a-h)$ increasing at x = a

$f(a+h) > f(a)$ decreasing at x = a

(C) Testing of monotonicity of differentiable function at a point.

(i) If $f'(a) > 0$, then f(x) is increasing at x = a.

(ii) If $f'(a) < 0$, then f(x) is decreasing at x = a.

(iii) If $f'(a) = 0$, then examine the sign of $f'(a^+)$ and $f'(a^-)$.

(1) If $f'(a^+) > 0$ and $f'(a^-) > 0$, then increasing

(2) If $f'(a^+) < 0$ and $f'(a^-) < 0$, then decreasing

(3) otherwise neither increasing nor decreasing.

Ex. Let $f(x) = x^3 - 3x + 2$. Examine the monotonicity of function at points x = 0, 1, 2.

Sol. $f(x) = x^3 - 3x + 2$

$f'(x) = 3(x^2 - 1)$

(i) $f'(0) = -3$ $\Rightarrow$ decreasing at x = 0

(ii) $f'(1) = 0$

also, f′(x) is positive on left neighbourhood and f′(x) is negative in right neighbourhood.

$\Rightarrow$ neither increasing nor decreasing at x = 1.

(iii) $f'(2) = 9$ $\Rightarrow$ increasing at x = 2

MONOTONICITY OVER AN INTERVAL

(A) A function f(x) is said to be monotonically increasing (MI) in (a, b) if $f'(x) \geq 0$ where equality holds only for discrete values of x i.e. f'(x) does not identically become zero for $x \in (a, b)$ or any sub interval.

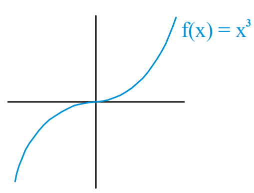
(B) f(x) is said to be monotonically decreasing (MD) in (a, b) if $f'(x) \leq 0$ where equality holds only for discrete values of x i.e. f'(x) does not identically become zero for $x \in (a, b)$ or any sub interval.

❖ By discrete points, we mean that points where f'(x) = 0 does not form an interval.

Ex. Let $f(x) = x^3$. Find the intervals of monotonicity.

Sol. $f'(x) = 3x^2$

$f'(x) > 0$ everywhere except at x = 0. Hence f(x) will be strictly increasing function for $x \in R$ {see figure}



Ex. Prove that the function $f(x)=\log\left(x^3+\sqrt{x^6+1}\right)$ is entirely increasing.

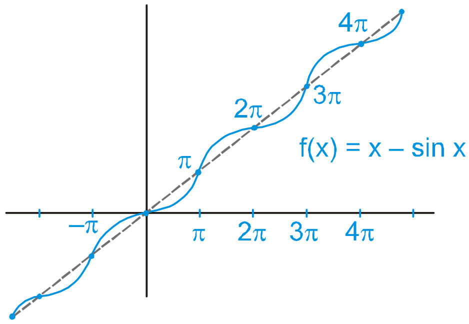
Sol. Now, $f(x)=\log\left(x^3+\sqrt{x^6+1}\right) \Rightarrow f'(x)=\frac{1}{x^3+\sqrt{x^6+1}}\left(3x^2+\frac{6x^5}{2\sqrt{x^6+1}}\right)=\frac{3x^2}{\sqrt{x^6+1}}>0$

$\Rightarrow$ f(x) is increasing.

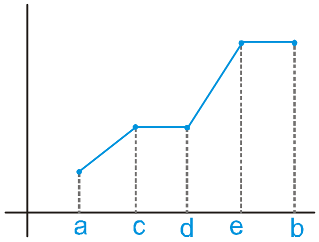
Ex. Let f(x) = x – sinx. Find the intervals of monotonicity.

Sol. f′(x) = 1 – cosx

Now, f′(x) > 0 every where, except at x = 0, ± 2π, ± 4π etc. But all these points are discrete (countable) and do not form an interval. Hence we can conclude that f(x) is strictly increasing in R. In fact we can also see it graphically.



Ex. Let us consider another function whose graph is shown below for x ∈ (a, b).



Sol. Here also f′(x) ≥ 0 for all x ∈ (a, b). But, note that in this case, f′(x) = 0 holds for all x ∈ (c, d) and (e, b). Thus the given function is increasing (monotonically increasing) in (a, b), but not strictly increasing.

Ex. Prove the following

(i) $y = e^x + \sin x$ is increasing in $x \in R^+$

(ii) y = 2x – sin x – tan x is decreasing in x ∈ (0, π/2)

Sol. **(i)** $f(x) = e^x + \sin x,\ x \in R^+ \Rightarrow f'(x) = e^x + \cos x$

Clearly $f'(x) > 0\ \forall\ x \in R^+$ (as $e^x > 1, x \in R^+$ and $-1 \le \cos x \le 1, x \in R^+$)

Hence f(x) is increasing.

(ii) f(x) = 2x – sin x – tan x x ∈ (0, π/2)

$\Rightarrow$ $f'(x) = 2 - \cos x - \sec^2 x$

$\Rightarrow$ $f'(x) = \cos^2 x - \cos x - (\cos^2 x + \sec^2 x - 2)$

$= \cos^2 x - \cos x - (\cos x - \sec x)^2$

$\therefore$ $f'(x) < 0, x \in (0, \pi/2)$ $\because$ $\cos^2 x < \cos x, x \in (0, \pi/2)$

Hence f(x) is decreasing in (0, π/2)

Ex. Find the intervals in which $f(x) = x^3 - 3x + 2$ is increasing.

Sol. $f(x) = x^3 - 3x + 2$

$f'(x) = 3(x^2 - 1)$

$f'(x) = 3(x - 1)(x + 1)$

for M.I. $f'(x) \geq 0 \Rightarrow 3(x - 1)(x + 1) \geq 0$

$\Rightarrow x \in (-\infty, -1] \cup [1, \infty)$, thus f is increasing in $(-\infty, -1]$ and also in $[1, \infty)$

PROVING INEQUALITIES USING MONOTONICITY

Comparison of two functions f(x) and g(x) can be done by analysing the monotonic behaviour of $h(x) = f(x) - g(x)$

> In proving inequalities, we must always check when does the equality takes place because the point of equality is very important in this method. Normally point of equality occur at end point of the interval or will be easily predicted by hit and trial.

Ex. For $x \in \left(0, \frac{\pi}{2}\right)$ prove that $\sin x < x < \tan x$

Sol. Let $f(x) = x - \sin x \Rightarrow f'(x) = 1 - \cos x$

$f'(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow$ f(x) is M.I. $\Rightarrow f(x) > f(0)$

$\Rightarrow x - \sin x > 0 \Rightarrow x > \sin x$

Similarly consider another function $g(x) = x - \tan x \Rightarrow g'(x) = 1 - \sec^2 x$

$g'(x) < 0$ for $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow$ g(x) is M.D.

Hence $g(x) < g(0)$

$x - \tan x < 0 \Rightarrow x < \tan x$

$\sin x < x < \tan x$ Hence proved

Ex. For $x \in \left(0, \frac{\pi}{2}\right)$, prove that $\sin x > x - \frac{x^3}{6}$

Sol. Let $f(x) = \sin x - x + \frac{x^3}{6}$

$f'(x) = \cos x - 1 + \frac{x^2}{2}$

we cannot decide at this point whether $f'(x)$ is positive or negative, hence let us check for monotonic nature of $f'(x)$

$f''(x) = x - \sin x$

Since $f''(x) > 0 \Rightarrow$ $f'(x)$ is M.I. for $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow$ $f'(x) > f'(0)$

$\Rightarrow$ $f'(x) > 0$ $\Rightarrow$ $f(x)$ is M.I.

$\Rightarrow$ $f(x) > f(0)$

$\Rightarrow$ $\sin x - x + \frac{x^3}{6} > 0$

$\Rightarrow$ $\sin x > x - \frac{x^3}{6}$ Hence proved

Ex. Compare which of the two is greater $(100)^{1/100}$ or $(101)^{1/101}$.

Sol. Assume $f(x) = x^{1/x}$ and let us examine monotonic nature of f(x)

$$f'(x) = x^{1/x} \cdot \left(\frac{1 - \ell nx}{x^2}\right)$$

$f'(x) > 0$ $\Rightarrow$ $x \in (0,e)$

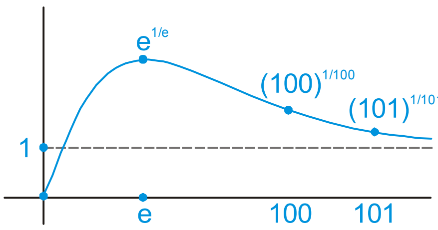
and $f'(x) < 0$ $\Rightarrow$ $x \in (e,\infty)$

Hence f(x) is M.D. for $x \geq e$

and since $100 < 101$

$\Rightarrow$ $f(100) > f(101)$

$\Rightarrow$ $(100)^{1/100} > (101)^{1/101}$



GREATEST AND LEAST VALUE OF A FUNCTION

(A) If a continuous function $y = f(x)$ is increasing in the closed interval [a, b], then $f(a)$ is the least value and $f(b)$ is the greatest value of $f(x)$ in [a, b] (figure-1)

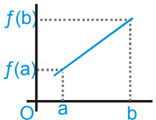


figure-1

(B) If a continuous function $y = f(x)$ is decreasing in [a, b], then $f(b)$ is the least and $f(a)$ is the greatest value of $f(x)$ in [a, b]. (figure-2)

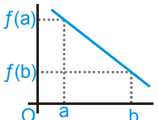
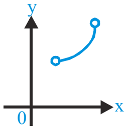
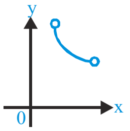


figure-2

(C) If a continuous function $y = f(x)$ is increasing/decreasing in the (a,b), then no greatest and least value exist.





Ex. Show that $f(x) = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}} - \ell nx$ is decreasing in $x \in \left[\dfrac{1}{\sqrt{3}}, \sqrt{3}\right]$. Also find its range.

Sol. $f(x) = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}} - \ell nx = \tan^{-1}x - \ell nx \Rightarrow f'(x) = \dfrac{1}{1+x^2} - \dfrac{1}{x} = \dfrac{-(1+x^2-x)}{x(1+x^2)}$

$\therefore$ $f'(x) < 0 \;\forall\; x \in \left[\dfrac{1}{\sqrt{3}}, \sqrt{3}\right]$

$\Rightarrow$ f(x) is decreasing $f(x)|_{max} = f\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6} + \dfrac{1}{2}\ell n3$ & $f(x)|_{min} = f(\sqrt{3}) = \dfrac{\pi}{3} - \dfrac{1}{2}\ell n3$

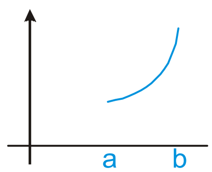
$\therefore$ Range of $f(x) = \left[\dfrac{\pi}{3} - \dfrac{1}{2}\ell n3, \dfrac{\pi}{6} + \dfrac{1}{2}\ell n3\right]$

CONCAVITY, CONVEXITY, POINT OF INFLECTION

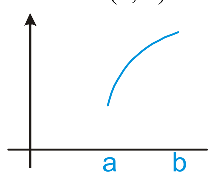
A function f(x) is concave in (a, b) if tangent drawn at every point $(x_0, (f(x_0))$, for $x_0 \in (a, b)$ lie below the curve.
f(x) is convex in (a,b) if tangent drawn at each point $(x_0, f(x_0))$, $x_0 \in (a, b)$ lie above the curve.
A point (c, f(c)) of the graph y = f(x) is said to be a point of inflection of the graph, if f(x) is concave in $(c - \delta, c)$ and convex in $(c, c + \delta)$ (or vice verse), for some $\delta \in R^+$.

Results

1. If $f''(x) > 0 \;\forall\; x \in (a, b)$, then the curve y = f(x) is concave in (a, b)



2. If $f''(x) < 0 \;\forall\; x \in (a, b)$ then the curve y = f(x) is convex in (a, b)



3. If f is continuous at x = c and $f''(x)$ has opposite signs on either sides of c, then the point (c, f(c)) is a point of inflection of the curve

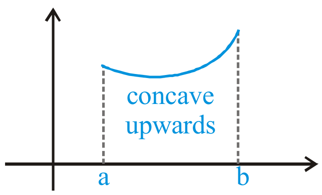
4. If $f''(c) = 0$ and $f'''(c) \neq 0$, then the point (c, f(c)) is a point of inflection

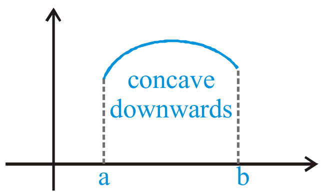
SIGNIFICANCE OF THE SIGN OF IIND ORDER DERIVATIVE :

The sign of the 2nd order derivative determines the concavity of the curve.

If $f''(x) > 0 \ \forall \ x \in (a, b)$ then graph of f(x) is concave upward in (a, b).

Similarly if $f''(x) < 0 \ \forall \ x \in (a, b)$ then graph of f(x) is concave downward in (a, b).



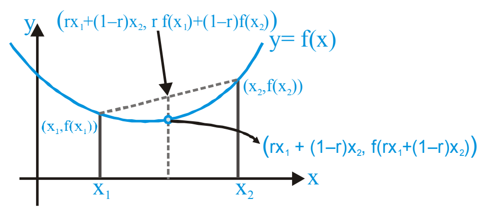


PROVING INEQUALITIES USING GRAPHS & CONCAVITY

Generally these inequalities involve comparison between values of two function at some particular points.

(i) If function f(x) is concave upward, then

$f\left(rx_1 + (1-r)x_2\right) < rf(x_1) + (1-r)f(x_2) \ \forall$ distinct x_1 & $x_2 \in$ domain of f(x) and $r > 0$.

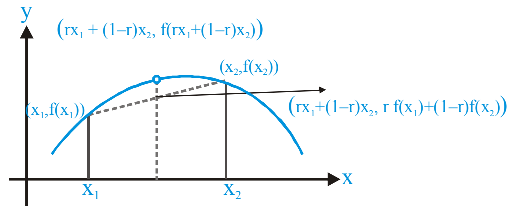


(fig. 1)

(ii) If function f(x) is concave downward, then

$f\left(rx_1 + (1-r)x_2\right) > rf(x_1) + (1-r)f(x_2) \ \forall$ distinct x_1 & $x_2 \in$ domain of f(x) and $r > 0$.

Note : Equality hold when x_1 and x_2 coincide.



(fig. 2)

Use of Monotonicity in identifying the number of roots of the equation in a given interval. Suppose a and b are two real numbers such that,

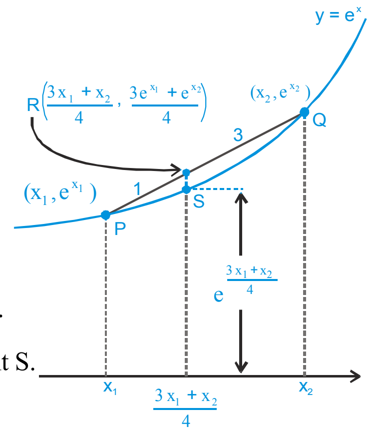
(iii) Let f(x) is differentiable & either MI or MD for $0 \le x \le b$.

&

(iv) f(a) and f(b) have opposite signs.

Then there is one & only one root of the equation $f(x) = 0$ in (a, b).

Ex. Prove that for any two numbers x_1 & x_2, $\dfrac{3e^{x_1}+e^{x_2}}{4} > e^{\frac{3x_1+x_2}{4}}$.



Sol. Assume $f(x) = e^x$ and let x_1 & x_2 be two points on the curve y = Ex.

Let R be another point which divides line segment PQ in ratio 1 : 3.

y coordinate of point R is $\dfrac{3e^{x_1}+e^{x_2}}{4}$ and y coordinate of point S is $e^{\frac{3x_1+x_2}{4}}$.

Since $f(x) = e^x$ is always concave up, hence point R will always be above point S.

$$\Rightarrow \quad \frac{3e^{x_1}+e^{x_2}}{4} > e^{\frac{3x_1+x_2}{4}}$$

ROLLE'S THEOREM

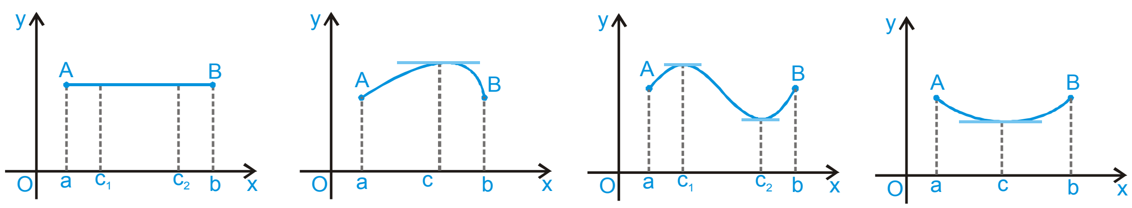
Let f be a function that satisfies the following three hypotheses :

(A) f is continuous on the closed interval [a, b].

(B) f is differentiable on the open interval (a, b)

(C) f(a) = f(b)

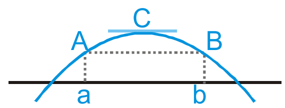
Then there exist at least one number c in (a, b) such that f'(c) = 0.



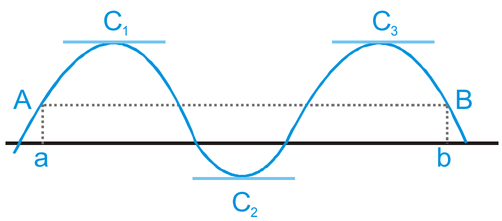
(a) (b) (c) (d)

Geometrical Explanation of Rolle's Theorem

Let the curve y = f(x), which is continuous on [a, b] and derivable on (a, b), be drawn (as shown in figure).



A(a, f(a)), B(b, f(b)), f(a) = f(b), C(c, f(c)), f'(c) = 0.



C_1 $(c_1, f(c_1))$, $f'(c_1) = 0$ $\qquad$ C_2 $(c_2, f(c_2))$, $f'(c_2) = 0$

C_3 $(c_3, f(c_3))$, $f'(c_3) = 0$

The theorem simply states that between two points with equal ordinates on the graph of f(x), there exists at least one point where the tangent is parallel to x-axis.

Algebraic Interpretation of Rolle's Theorem

Between two zeros a and b of f(x) (i.e. between two roots a and b of f(x) = 0) there exists at least one zero of f′(x)

Ex. Verify Rolle's theorem for $f(x) = (x-a)^n (x-b)^m$, where m, n are positive real numbers, for $x \in [a, b]$.

Sol. Being a polynomial function f(x) is continuous as well as differentiable. Also f(a) = f(b)

$\Rightarrow$ $f'(x) = 0$ for some $x \in (a, b)$

$n(x-a)^{n-1}(x-b)^m + m(x-a)^n(x-b)^{m-1} = 0$

$\Rightarrow$ $(x-a)^{n-1}(x-b)^{m-1}\ [(m+n)x - (nb+ma)] = 0$

$\Rightarrow$ $x = \dfrac{nb+ma}{m+n}$, which lies in the interval (a, b), as $m, n \in R^+$.

Ex. Verify Rolle's theorem for the function $f(x) = x^3 - 3x^2 + 2x$ in the interval [0, 2].

Sol. Here we observe that

(A) f(x) is polynomial and since polynomial are always continuous, as well as differentiable. Hence f(x) is continuous in the [0,2] and differentiable in the (0, 2).

&

(B) $f(0) = 0,\ f(2) = 2^3 - 3.(2)^2 + 2(2) = 0$

$\therefore$ $f(0) = f(2)$

Thus, all the condition of Rolle's theorem are satisfied.

So, there must exists some $c \in (0, 2)$ such that $f'(c) = 0$

$\Rightarrow$ $f'(c) = 3c^2 - 6c + 2 = 0 \Rightarrow c = 1 \pm \dfrac{1}{\sqrt{3}}$

where both $c = 1 \pm \dfrac{1}{\sqrt{3}} \in (0, 2)$ thus Rolle's theorem is verified.

Ex. If $2a + 3b + 6c = 0$ then prove that the equation $ax^2 + bx + c = 0$ has at least one real root between 0 and 1.

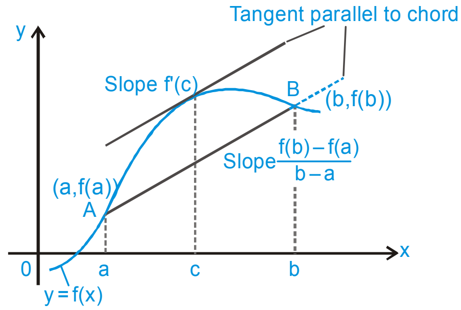
Sol. Let $f(x) = \dfrac{ax^3}{3} + \dfrac{bx^2}{2} + cx$

$f(0) = 0$ and $f(1) = \dfrac{a}{3} + \dfrac{b}{2} + c = 2a + 3b + 6c = 0$

If $f(0) = f(1)$ then $f'(x) = 0$ for some value of $x \in (0, 1)$

$\Rightarrow$ $ax^2 + bx + c = 0$ for at least one $x \in (0, 1)$

LAGRANGE'S MEAN VALUE THEOREM (LMVT)



If a function f defined on [a, b] is

(i) continuous on [a, b] and

(ii) derivable on (a, b)

then there exists at least one real numbers between a and b ($a < c < b$) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$

Proof Let us consider a function $g(x) = f(x) + \lambda x$, $x \in [a, b]$

where λ is a constant to b determined such that g(a) = g(b).

$\therefore \quad \lambda = -\frac{f(b)-f(a)}{b-a}$

Now the function g(x), being the sum of two continuous and derivable functions it self

(i) continuous on [a, b]

(ii) derivable on (a, b) and

(iii) g(a) = g(b).

Therefore, by Rolle's theorem there exists a real number $c \in (a, b)$ such that $g'(c) = 0$

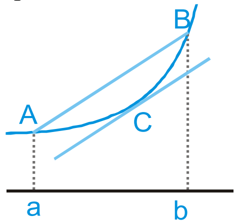
But $g'(x) = f'(x) + \lambda$

$\therefore \quad 0 = g'(c) = f'(c) + \lambda$

$$f'(c) = -\lambda = \frac{f(b)-f(a)}{b-a}$$

Geometrical Interpretation of LMVT

The theorem simply states that between two points A and B of the graph of f(x) there exists at least one point where tangent is parallel to chord AB.



C(c, f(c)), $f'(c)$ = slope of AB.

Alternative Statement : If in the statement of LMVT, b is replaced by a + h, then number c between a and b may be written as $a + \theta h$, where $0 < \theta < 1$. Thus

$$\frac{f(a+h)-f(a)}{h} = f'(a+\theta h) \qquad \text{or} \qquad f(a+h) = f(a) + hf'(a+\theta h),\ 0 < \theta < 1$$

Physical Interpretations of LMVT

If we think of the number (f(b) – f(a))/(b – a) as the average change in f over [a, b] and f'(c) as an instantaneous change, then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

Ex. Verify LMVT for $f(x) = -x^2 + 4x - 5$ and $x \in [-1, 1]$

Sol. $f(1) = -2$; $f(-1) = -10$

$$\Rightarrow \quad f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$$

$$\Rightarrow \quad -2c + 4 = 4 \quad \Rightarrow \quad c = 0$$

Ex. Find c of the Lagrange's mean value theorem for the function $f(x) = 3x^2 + 5x + 7$ in the interval [1, 3].

Sol. Given $f(x) = 3x^2 + 5x + 7$ ...... **(i)**

$\Rightarrow$ $f(1) = 3 + 5 + 7 = 15$ and $f(3) = 27 + 15 + 7 = 49$

Again $f'(x) = 6x + 5$

Here $a = 1, b = 3$

Now from Lagrange's mean value theorem

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \Rightarrow \quad 6c + 5 = \frac{f(3) - f(1)}{3 - 1} = \frac{49 - 15}{2} = 17 \quad \text{or} \quad c = 2.$$

Ex. Using Lagrange's mean value theorem, prove that if $b > a > 0$,

then $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$

Sol. Let $f(x) = \tan^{-1} x$; $x \in [a, b]$ applying LMVT

$$f'(c) = \frac{\tan^{-1} b - \tan^{-1} a}{b - a} \text{ for } a < c < b \quad \text{and} \quad f'(x) = \frac{1}{1+x^2},$$

Now $f'(x)$ is a monotonically decreasing function

Hence if $a < c < b$ $\Rightarrow$ $f'(b) < f'(c) < f'(a)$

$$\Rightarrow \quad \frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b - a} < \frac{1}{1+a^2} \quad \text{Hence proved}$$

Ex. If f(x) is continuous and differentiable over [–2, 5] and $-4 \le f'(x) \le 3$ for all x in (–2, 5), then the greatest possible value of $f(5) - f(-2)$ is -

Sol. Apply LMVT

$$f'(x) = \frac{f(5) - f(-2)}{5 - (-2)} \text{ for some x in } (-2, 5)$$

Now, $-4 \le \frac{f(5) - f(-2)}{7} \le 3$

$-28 \le f(5) - f(-2) \le 21$

$\therefore$ Greatest possible value of $f(5) - f(-2)$ is 21.

MAXIMA AND MINIMA

INTRODUCTION

Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal (best) way of doing something. Here are examples of such problems that we will solve in this chapter

- What is the shape of a vessel that can with-stand maximum pressure ?
- What is the maximum acceleration of a space shuttle ? (This is an important question to the astronauts who have to withstand the effects of acceleration)
- What is the radius of a contracted windpipe that expels air most rapidly during a cough ?

These problems can be reduced to finding the maximum or minimum values of a function. Let's first explain exactly what we mean by maxima and minima.

(A) Maxima (Local/Relative maxima)

A function f(x) is said to have a maximum at $x = a$ if there exist a neighbourhood $(a - h, a + h) - \{a\}$ such that $f(a) > f(x)\ \forall\ x \in (a - h, a + h) - \{a\}$

(B) Minima (Local/Relative minima)

A function f(x) is said to have a minimum at $x = a$ if there exist a neighbourhood $(a - h, a + h) - \{a\}$ such that

$f(a) < f(x)\ \forall\ x \in (a - h, a + h) - \{a\}$

(C) Absolute maximum (Global maximum)

A function f has an absolute maximum (or global maximum) at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f. The number f(c) is called the maximum value of f on D.

(D) Absolute minimum (Global minimum)

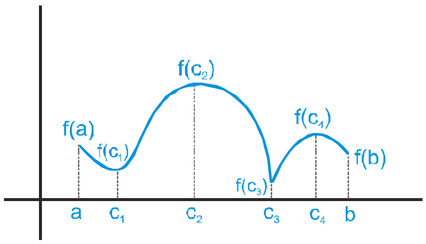
A function f has an absolute minimum at c if $f(c) \leq f(x)$ for all x in D and the number f(c) is called the minimum value of f on D. The maximum and minimum values of f are called the extreme values of f.

(E) Extrema

A maxima or a minima is called an extrema.

Explanation :

Consider graph of $y = f(x), x \in [a, b]$



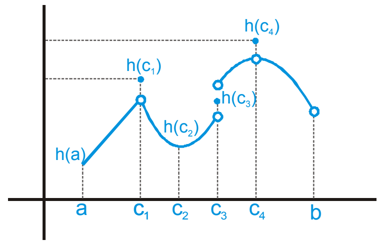
$x = a, x = c_2, x = c_4$ are points of local maxima, with maximum values $f(a), f(c_2), f(c_4)$ respectively.

$x = c_1, x = c_3, x = b$ are points of local minima, with minimum values $f(c_1), f(c_3), f(b)$ respectively

$x = c_2$ is a point of global maximum

$x = c_3$ is a point of global minimum

Consider the graph of $y = h(x), x \in [a, b)$



$x = c_1$, $x = c_4$ are points of local maxima, with maximum values $h(c_1)$, $h(c_4)$ respectively.

$x = a$, $x = c_2$ are points of local minima, with minimum values $h(a)$, $h(c_2)$ respectively.

$x = c_3$ is neither a point of maxima nor a minima.

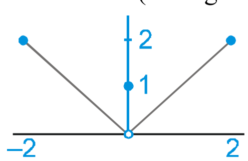
Global maximum is $h(c_4)$

Global minimum is $h(a)$

(i) The maximum & minimum values of a function are also known as **local/relative maxima or local/relative minima** as these are the greatest & least values of the function relative to some neighbourhood of the point in question.
(ii) The term 'extremum' (or extremal) or 'turning value' is used both for maximum or a minimum value.
(iii) A maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.
(iv) A function can have several maximum & minimum values & a minimum value may even be greater than a maximum value.
(v) Local maximum & local minimum values of a continuous function occur alternately & between two consecutive local maximum values there is a local minimum value & vice versa.

Ex. Let $f(x) = \begin{cases} |x| & 0 < |x| \le 2 \\ 1 & x = 0 \end{cases}$. Examine the behaviour of f(x) at x = 0.

Sol. f(x) has local maxima at x = 0 (see figure).



Ex. Let $f(x) = \begin{cases} -x^3 + \dfrac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} & 0 \le x < 1 \\ 2x - 3 & 1 \le x \le 3 \end{cases}$

Find all possible values of b such that f(x) has the smallest value at x = 1.

Sol. Such problems can easily be solved by graphical approach (as in figure).

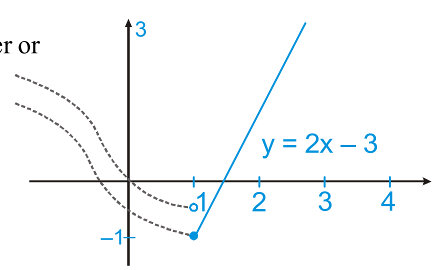
Hence the limiting value of f(x) from left of x = 1 should be either greater or equal to the value of function at

x = 1.

$$\lim_{x \to 1^-} f(x) \ge f(1)$$

$$\Rightarrow \quad -1 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} \ge -1$$

$$\Rightarrow \quad \frac{(b^2+1)(b-1)}{(b+1)(b+2)} \ge 0 \qquad \Rightarrow \qquad b \in (-2, -1) \cup [1, +\infty)$$



DERIVATIVE TEST FOR ASSERTATINING MAXIMA/MINIMA

Mere definition of maxima, minima becomes tedious in solving problems. We use derivative as a tool to overcome this difficulty.

First Derivative Test

(A) Let f(x) be continuous and differentiable function.

Step - I. Find $f'(x)$

Step - II. Solve $f'(x) = 0$, let $x = c$ be a solution. (i.e. Find stationary points)

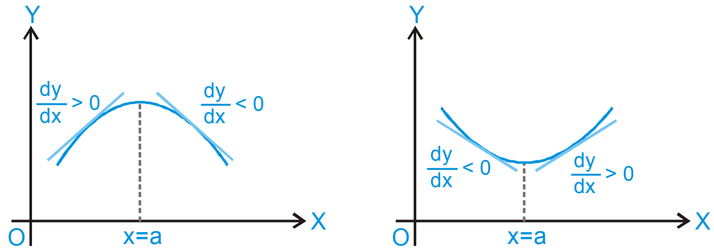
Step - III. Observe change of sign

(i) If $f'(x)$ changes sign from negative to positive as x crosses c from left to right then $x = c$ is a point of local minima

(ii) If $f'(x)$ changes sign from positive to negative as x crosses c from left to right then $x = c$ is a point of local maxima.

(iii) If $f'(x)$ does not changes sign as x crosses c then $x = c$ is neither a point of maxima nor minima.

(iv) If f'(x) does not change sign i.e. has the same sign in a certain complete neighbourhood of a, then f(x) is either strictly increasing or decreasing throughout this neighbourhood implying that f(a) is not an extreme value of f.



Stationary points

The points on graph of function f(x) where $f'(x) = 0$ are called stationary points.

Rate of change of f(x) is zero at a stationary point.

Critical points

The points where $f'(x) = 0$ or f(x) is not differentiable are called critical points.
Stationary points $\subseteq$ Critical points.

(B) Function defined on closed interval

Let f(x), $x \in [a, b]$ be a continuous function

Step - I : Find critical points. Let it be $c_1, c_2, \ldots\ldots, c_n$

Step - II : Find $f(a), f(c_1) \ldots\ldots\ldots, f(c_n), f(b)$

Let $M = \max \cdot \{f(a), f(c_1), \ldots\ldots\ldots, f(c_n), f(b)\}$

$m = \min \cdot \{f(a), f(c_1), \ldots\ldots f(c_n), f(b)\}$

Step - III : M is global maximum.

m is global minimum.

(C) Function defined on open interval.

Let f(x), $x \in (a, b)$ be continuous function.

Step - I Find critical points . Let it be $c_1, c_2, \ldots\ldots c_n$

Step - II Find $f(c_1), f(c_2), \ldots\ldots\ldots, f(c_n)$

Let $M = \max \cdot \{f(c_1), \ldots\ldots. f(c_n)\}$

$m = \min \cdot \{f(c_1), \ldots\ldots\ldots\ldots, f(c_n)\}$

Step - III $\lim\limits_{x \to a^+} f(x) = \ell_1$ (say), $\lim\limits_{x \to b^-} f(x) = \ell_2$ (say).

Let $\ell = \min. \{\ell_1, \ell_2\}$, $L = \max. \{\ell_1, \ell_2\}$

Step - IV

(i) If $m \le \ell$ then m is global minimum

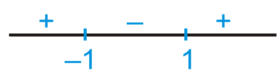
(ii) If $m > \ell$ then f(x) has no global minimum

(iii) If $M \ge L$ then M is global maximum

(iv) If $M < L$, then f(x) has no global maximum

Ex. Let $f(x) = x + \dfrac{1}{x}$; $x \neq 0$. Discuss the local maximum and local minimum values of f(x).

Sol. Here, $f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{x^2 - 1}{x^2} = \dfrac{(x-1)(x+1)}{x^2}$



Using number line rule, f(x) will have local maximum at x = –1 and local minimum at x = 1

∴ local maximum value of f(x) = –2 at x = –1

and local minimum value of f(x) = 2 at x = 1

Ex. Find stationary points of the function $f(x) = 4x^3 - 6x^2 - 24x + 9$.

Sol. $f'(x) = 12x^2 - 12x - 24$

$f'(x) = 0 \quad \Rightarrow \quad x = -1, 2$

$f(-1) = 23$, $f(2) = -31$

(– 1, 23), (2, – 31) are stationary points

Ex. If $f(x) = x^3 + ax^2 + bx + c$ has extreme values at x = – 1 and x = 3. Find a, b, c.

Sol. Extreme values basically mean maximum or minimum values, since f(x) is differentiable function so

$f'(-1) = 0 = f'(3)$

$f'(x) = 3x^2 + 2ax + b$

$f'(3) = 27 + 6a + b = 0$

$f'(-1) = 3 - 2a + b = 0$

$\Rightarrow \quad a = -3, b = -9, c \in R$

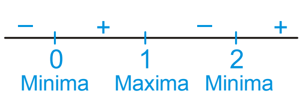
Ex. Find the points of maxima or minima of $f(x) = x^2 (x-2)^2$.

Sol. $f(x) = x^2 (x-2)^2$

$f'(x) = 4x(x-1)(x-2)$

$f'(x) = 0 \quad \Rightarrow \quad x = 0, 1, 2$

examining the sign change of f′(x)



Hence x = 1 is point of maxima, x = 0, 2 are points of minima.

Ex. Let $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$. If f(x) has positive point of maxima, then find possible values of 'a'.

Sol. $f'(x) = 3[x^2 + 2(a-7)x + (a^2-9)]$

Let α, β be roots of $f'(x) = 0$ and let α be the smaller root. Examining sign change of $f'(x)$.

Maxima occurs at smaller root α which has to be positive. This basically implies that both roots of $f'(x) = 0$ must be positive and distinct.

(i) $D > 0 \Rightarrow a < \frac{29}{7}$

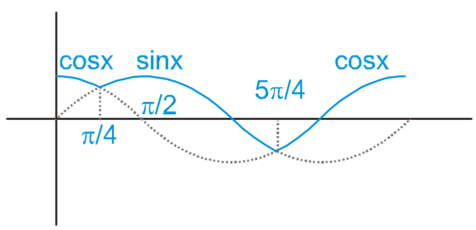
(ii) $-\frac{b}{2a} > 0 \Rightarrow a < 7$

(iii) $f'(0) > 0 \Rightarrow a \in (-\infty, -3) \cup (3, \infty)$

from **(i)**, **(ii)** and **(iii)** $\Rightarrow a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)$

Ex. Find critical points of $f(x) = \max(\sin x, \cos x)\ \forall,\ x \in (0, 2\pi)$.

Sol.



From the figure it is clear that f(x) has three critical points $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}$.

Ex. Find the greatest and least values of $f(x) = x^3 - 12x \quad x \in [-1, 3]$

Sol. The possible points of maxima/minima are critical points and the boundary points.

for $x \in [-1, 3]$ and $f(x) = x^3 - 12x$

$x = 2$ is the only critical point.

Examining the value of f(x) at points $x = -1, 2, 3$. We can find greatest and least values.

x	f(x)
–1	11
2	–16
3	–9

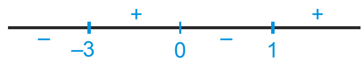
$\therefore$ Minimum $f(x) = -16$ & Maximum $f(x) = 11$.

Ex Find extrema of $f(x) = 3x^4 + 8x^3 - 18x^2 + 60$. Draw graph of $g(x) = \dfrac{40}{f(x)}$ and comment on its local and global extrema.

Sol. $f'(x) = 0$

$\Rightarrow \quad 12x(x^2 + 2x - 3) = 0 \quad \Rightarrow \quad 12x(x-1)(x+3) = 0 \quad \Rightarrow \quad x = -3, 0, 1$

$f'(x) = 12(x+3)x(x-1)$



local minima occurs at $x = -3, 1$

local maxima occurs at $x = 0$

$f(-3) = -75$, $f(1) = 53$ are local minima

$f(0) = 60$ is local maxima

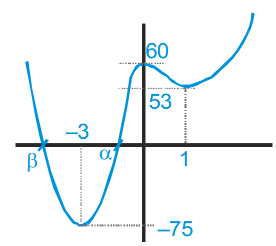
$\lim\limits_{x\to\infty} f(x) = \infty$, $\quad \lim\limits_{x\to-\infty} f(x) = \infty$

Hence global maxima does not exists : Global minima is -75

$g'(x) = \dfrac{-40}{(f(x))^2} f'(x)$

$\Rightarrow$ g(x) has same critical points as that of f(x).

A rough sketch of $y = f(x)$ is



Let zeros of f(x) be α, β

$g(\alpha), g(\beta)$ are undefined,

$\lim\limits_{x\to\beta^-} g(x) = \infty$, $\lim\limits_{x\to\beta^+} g(x) = -\infty$, $\lim\limits_{x\to\alpha^-} g(x) = -\infty$, $\lim\limits_{x\to\alpha^+} g(x) = \infty$

$x = \alpha$, $x = \beta$ are asymptotes of $y = g(x)$.

$\lim\limits_{x\to\infty} g(x) = 0$, $\quad \lim\limits_{x\to-\infty} g(x) = 0$

$\Rightarrow$ $y = 0$ is also an asymptote.

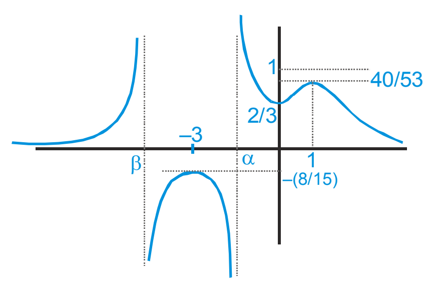
$\because$ $x = -3, x = 1$ are local minima of

$y = f(x) \quad \Rightarrow \quad x = -3, x = 1$ are local maxima of $y = g(x)$

similarly, $x = 0$ is local minima of $y = g(x)$

Global extrema of g(x) does not exists.

A rough sketch of $y = g(x)$ is



Second derivative test

If f(x) is continuous and differentiable at $x = a$ where $f'(a) = 0$ and $f''(a)$ also exists then for ascertaining maxima/minima at $x = a$, 2nd derivative test can be used -

(i) If $f''(a) > 0 \quad \Rightarrow \quad x = a$ is a point of local minima

(ii) If $f''(a) < 0 \quad \Rightarrow \quad x = a$ is a point of local maxima

(iii) If $f''(a) = 0 \quad \Rightarrow$ second derivative test fails. To identify maxima/minima at this point either first derivative test or higher derivative test can be used.

Ex. Find the points of local maxima or minima for $f(x) = \sin 2x - x$, $x \in (0, \pi)$.

Sol. $f(x) = \sin 2x - x$

$f'(x) = 2\cos 2x - 1$

$f'(x) = 0 \Rightarrow \cos 2x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

$f''(x) = -4 \sin 2x$

$f''\left(\frac{\pi}{6}\right) < 0 \Rightarrow$ Maxima at $x = \frac{\pi}{6}$

$f''\left(\frac{5\pi}{6}\right) > 0 \Rightarrow$ Minima at $x = \frac{5\pi}{6}$

Ex. If $f(x) = 2x^3 - 3x^2 - 36x + 6$ has local maximum and minimum at $x = a$ and $x = b$ respectively, then ordered pair (a, b) is

Sol. $f(x) = 2x^3 - 3x^2 - 36x + 6$

$f'(x) = 6x^2 - 6x - 36$ & $f''(x) = 12x - 6$

Now $f'(x) = 0 \Rightarrow 6(x^2 - x - 6) = 0$

$\Rightarrow (x - 3)(x + 2) = 0 \Rightarrow x = -2, 3$

$f''(-2) = -30$

$\therefore$ $x = -2$ is a point of local maximum

$f''(3) = 30$

$\therefore$ $x = 3$ is a point of local minimum

Hence, (–2, 3) is the required ordered pair.

Ex. Find the point of local maxima of $f(x) = \sin x\,(1+\cos x)$ in $x \in (0, \pi/2)$.

Sol. Let $f(x) = \sin x\,(1 + \cos x) = \sin x + \frac{1}{2}\sin 2x$

$\Rightarrow f'(x) = \cos x + \cos 2x$

$f''(x) = -\sin x - 2\sin 2x$

Now $f'(x) = 0 \Rightarrow \cos x + \cos 2x = 0$

$\Rightarrow \cos 2x = \cos(\pi - x) \Rightarrow x = \pi/3$

Also $f''(\pi/3) = -\sqrt{3}/2 - \sqrt{3} < 0$ $\therefore$ f(x) has a maxima at $x = \pi/3$

n^{th} Derivative test

Let f(x) have derivatives up to nth order

If $f'(c) = f''(c) = \ldots\ldots\ldots = f^{n-1}(c) = 0$ and

$f^n(c) \neq 0$ then we have following possibilities

(i) n is even, $f^{(n)}(c) < 0 \Rightarrow x = c$ is point of maxima

(ii) n is even, $f^{(n)}(c) > 0 \Rightarrow x = c$ is point of minima.

(iii) n is odd, $f^{(n)}(c) < 0 \Rightarrow$ f(x) is decreasing about x = c

(iv) n is odd, $f^{(n)} > 0 \Rightarrow$ f(x) is increasing about x = c.

Ex. Identify a point of maxima/minima in $f(x) = (x + 1)^4$.

Sol. $f(x) = (x + 1)^4$

$f'(x) = 4(x+1)^3$

$f''(x) = 12(x+1)^2$

$f'''(x) = 24(x+1)$

$f''''(x) = 24$

Now $f'(x) = 0 \Rightarrow x = -1$

$f''(-1) = 0, f'''(-1) = 0, f''''(-1) = 24 > 0$

$\therefore$ at x = 1 f(x) has point of minima.

Ex. Find points of local maxima or minima of $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Sol. $f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f'(x) = 5x^2 (x - 1)(x - 3)$

$f'(x) = 0 \Rightarrow x = 0, 1, 3$

$f''(x) = 10x(2x^2 - 6x + 3)$

Now, $f''(1) < 0 \Rightarrow$ Maxima at x = 1

$f''(3) > 0 \Rightarrow$ Minima at x = 3

and, $f''(0) = 0 \Rightarrow$ II^{nd} derivative test fails

so, $f'''(x) = 30(2x^2 - 4x + 1)$

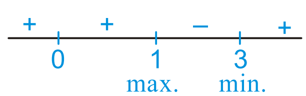
$f'''(0) = 30$

$\Rightarrow$ Neither maxima nor minima at x = 0.

❖ It was very convenient to check maxima/minima at first step by examining the sign change of f′(x)

no sign change of f′(x) at x = 0

$f'(x) = 5x^2 (x - 1)(x - 3)$



APPLICATION OF MAXIMA AND MINIMA

Summary of Working Rule for solving real life optimization problem

First : When possible, draw a figure to illustrate the problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

Second : Write an equation for the quantity that is to be maximized or minimized. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. This may require some algebraic manipulations.

Third : If y = f(x) is a quantity to be maximum or minimum, find those values of x for which $dy/dx = f'(x) = 0$.

Fourth : Using derivative test, test each value of x for which $f'(x) = 0$ to determine whether it provides a maximum or minimum or neither.

Fifth : If the derivative fails to exist at some point, examine this point as possible maximum or minimum.

Sixth : If the function y = f(x) is defined only for $x \in [a,b]$ then examine x = a & x = b for possible extreme values.

Useful Formulae

1. Volume of a cuboid = ℓbh.
2. Surface area of cuboid = $2(\ell b + bh + h\ell)$.
3. Volume of cube = a^3
4. Surface area of cube = $6a^2$
5. Volume of a cone = $\frac{1}{3}\pi r^2 h$.
6. Curved surface area of cone = $\pi r\ell$ (ℓ = slant height)
7. Curved surface area of a cylinder = $2\pi rh$.
8. Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.
9. Volume of a sphere = $\frac{4}{3}\pi r^3$.
10. Surface area of a sphere = $4\pi r^2$.
11. Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.
12. Volume of a prism = (area of the base) × (height).
13. Lateral surface area of a prism = (perimeter of the base) × (height).
14. Total surface area of a prism = (lateral surface area) + 2 (area of the base)
    (Note that lateral surfaces of a prism are all rectangle).
15. Volume of a pyramid = $\frac{1}{3}$ (area of the base) × (height).
16. Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) × (slant height).
    (Note that slant surfaces of a pyramid are triangles).

(i) If the sum of two real numbers x and y is constant then their product is maximum if they are equal.

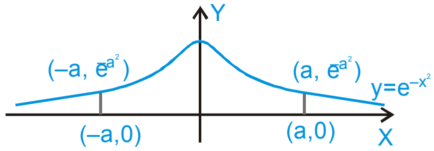
i.e. $xy = \frac{1}{4}[(x+y)^2 - (x-y)^2]$

(ii) If the product of two positive numbers is constant then their sum is least if they are equal.

i.e. $(x+y)^2 = (x-y)^2 + 4xy$

Ex. Determine the largest area of the rectangle whose base is on the x-axis and two of its vertices lie on the curve $y = e^{-x^2}$.

Sol. Area of the rectangle will be $A = 2a.\ e^{-a^2}$



For max. area, $\frac{dA}{da} = \frac{d}{da}(2ae^{-a^2}) = e^{-a^2}[2 - 4a^2]$

$\frac{dA}{da} = 0 \quad \Rightarrow \quad a = \frac{1}{\sqrt{2}}$

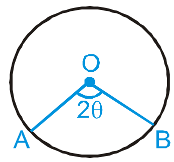
sign of $\frac{dA}{da}$ changes from positive to negative

$\Rightarrow \quad x = \frac{1}{\sqrt{2}}$ is a point of maxima $\Rightarrow A_{max} = \frac{2}{\sqrt{2}}.e^{-\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{\sqrt{2}}{e^{1/2}}$ sq units .

Ex. A Conical vessel is to be prepared out of a circular sheet of gold of unit radius. How much sectorial area is to be removed from the sheet so that vessel has maximum volume.

Sol. Lateral height of cone = Radius of circle = 1

Lateral area of cone = Area of circle with sector removed



i.e. $\pi r(1) = \frac{\pi(1)^2}{2\pi}(2\pi - 2\theta)$

i.e. $r = \frac{\pi - \theta}{\pi}$ (here r is radius of cone)

Height 'h' of cone = $\sqrt{1^2 - r^2}$

Volume of cone $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{\pi-\theta}{\pi}\right)^2 \times \sqrt{1^2 - \left(\frac{\pi-\theta}{\pi}\right)^2}$

upon maximizing V, we get $\frac{\pi-\theta}{\pi} = \sqrt{\frac{2}{3}} \Rightarrow \theta = \pi\left(1 - \sqrt{\frac{2}{3}}\right)$

Area of sector removed = $\frac{1}{2}(1)^2(2\theta) = \pi\left(1 - \sqrt{\frac{2}{3}}\right)$

Ex. If the equation $x^3 + px + q = 0$ has three real roots, then show that $4p^3 + 27q^2 < 0$.

Sol. $f(x) = x^3 + px + q$, $f'(x) = 3x^2 + p$

$\therefore$ f(x) must have one maximum > 0 and one minimum < 0. $f'(x) = 0$

$\Rightarrow$ $x = \pm\sqrt{\frac{-p}{3}}$, $p < 0$

f is maximum at $x = -\sqrt{\frac{-p}{3}}$ and minimum at $x = \sqrt{\frac{-p}{3}}$

$f\left(-\sqrt{\frac{-p}{3}}\right) f\left(\sqrt{\frac{-p}{3}}\right) < 0$

$\left(q - \frac{2p}{3}\sqrt{\frac{-p}{3}}\right)\left(q + \frac{2p}{3}\sqrt{\frac{-p}{3}}\right) < 0$

$q^2 + \frac{4p^3}{27} < 0$, $4p^3 + 27q^2 < 0$.

Ex. Find two positive numbers x and y such that x + y = 60 and xy^3 is maximum.

Sol. $x + y = 60$

$\Rightarrow$ $x = 60 - y$ $\Rightarrow$ $xy^3 = (60 - y)y^3$

Let $f(y) = (60 - y)\,y^3$; $y \in (0, 60)$

for maximizing f(y) let us find critical points

$f'(y) = 3y^2(60 - y) - y^3 = 0$

$f'(y) = y^2(180 - 4y) = 0$

$\Rightarrow$ $y = 45$

$f'(45^+) < 0$ and $f'(45^-) > 0$. Hence local maxima at y = 45.

So x = 15 and y = 45.

Ex. Let A(1, 2) and B(– 2, – 4) be two fixed points. A variable point P is chosen on the straight line y = x such that perimeter of ΔPAB is minimum. Find coordinates of P.

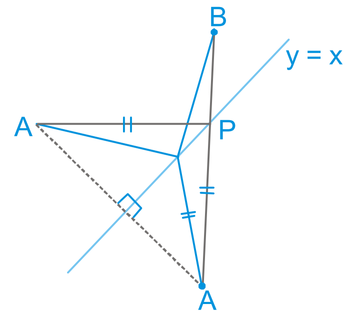
Sol. Since distance AB is fixed so for minimizing the perimeter of ΔPAB, we basically have to minimize (PA + PB)

Let A′ be the mirror image of A in the line y = x (see figure).

F(P) = PA + PB

F(P) = PA′ + PB

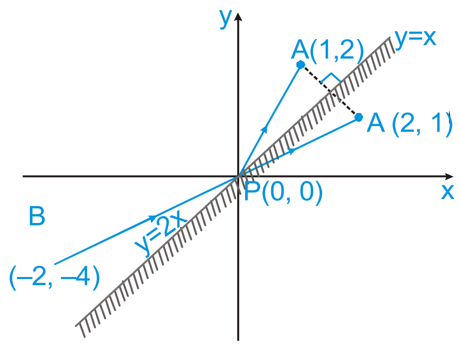
But for ΔPA′B



PA′ + PB ≥ A′B and equality hold when P, A′ and B becomes collinear. Thus for minimum path length point P is that special point for which PA and PB become incident and reflected rays with respect to the mirror y = x.

Equation of line joining A′ and B is y = 2x intersection of this line with y = x is the point P.

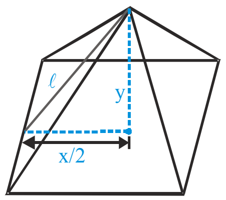
Hence P ≡ (0, 0).



Ex. Among all regular square pyramids of volume $36\sqrt{2}$ cm^3. Find dimensions of the pyramid having least lateral surface area.

Sol. Let the length of a side of base be x cm and y cm be the perpendicular height of the pyramid

$$V = \frac{1}{3} \text{ area of base} \times \text{height}$$



$$\Rightarrow \quad V = \frac{1}{3} x^2 y = 36\sqrt{2}$$

$$\Rightarrow \quad y = \frac{108\sqrt{2}}{x^2}$$

and $$S = \frac{1}{2} \text{ perimeter of base} \times \text{ slant height}$$

$$= \frac{1}{2}(Ax) \cdot \ell$$

but $$\ell = \sqrt{\frac{x^2}{4} + y^2} \qquad \Rightarrow \qquad S = 2x\sqrt{\frac{x^2}{4} + y^2} = \sqrt{x^4 + 4x^2y^2}$$

$\Rightarrow \quad S = \sqrt{x^4 + 4x^2\left(\frac{108\sqrt{2}}{x^2}\right)^2} \quad \Rightarrow \quad S(x) = \sqrt{x^4 + \frac{8.(108)^2}{x^2}}$

Let $\quad f(x) = x^4 + \frac{8.(108)^2}{x^2}$

for minimizing f(x)

$f'(x) = 4x^3 - \frac{16(108)^2}{x^3} = 0 \quad \Rightarrow \quad f'(x) = 4\frac{(x^6 - 6^6)}{x^3} = 0$

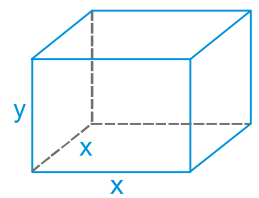
$\Rightarrow$ x = 6 which a point of minima

Hence x = 6 cm and y = $3\sqrt{2}$.

Ex. A sheet of area 40 m^2 is used to make an open tank with square base. Find the dimensions of the base such that volume of this tank is maximum.

Sol. Let length of base be x meter and height be y meter (as shown in figure).

$V = x^2y$



again x and y are related to surface area of this tank which is equal to 40 m^2.

$\Rightarrow \quad x^2 + 4xy = 40$

$y = \frac{40 - x^2}{4x} \qquad x \in (0, \sqrt{40})$

$\Rightarrow \quad V(x) = x^2\left(\frac{40 - x^2}{4x}\right)$

$V(x) = \frac{(40x - x^3)}{4}$

maximizing volume,

$V'(x) = \frac{(40 - 3x^2)}{4} = 0 \quad \Rightarrow \quad x = \sqrt{\frac{40}{3}}$ m

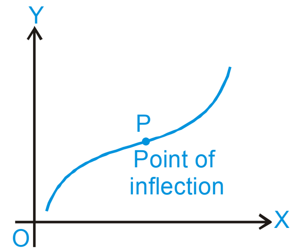
and $\quad V''(x) = -\frac{3x}{2} \quad \Rightarrow \quad V''\left(\sqrt{\frac{40}{3}}\right) < 0.$

Confirming that volume is maximum at $x = \sqrt{\frac{40}{3}}$ m.

POINT OF INFLECTION

A point where the graph of a function has a tangent line and where the concavity changes is called a point of inflection.

If function $y = f(x)$ is double differentiable then the point at which $\frac{d^2y}{dx^2} = 0$ & changes its sign is the point of inflection.



If at any point $\frac{d^2y}{dx^2}$ does not exist but sign of $\frac{d^2y}{dx^2}$ changes about this point then it is also called point of inflection. e.g. for $y = x^{1/3}$, $x = 0$ is point of inflection.

Ex. The point of inflexion for the curve $y = x^{\frac{5}{3}}$ is -

Sol. Here $\frac{d^2y}{dx^2} = \frac{10}{9x^{1/3}}$

From the given points we find that (0, 0) is the point of the curve where

$\frac{d^2y}{dx^2}$ does not exist but sign of $\frac{d^2y}{dx^2}$ changes about this point.

$\therefore$ (0, 0) is the required point

Ex. Divide 64 into two parts such that sum of the cubes of two parts is minimum.

Sol. Let x and y be two positive number such that

$x + y = 64$ .... **(i)**

Let $u = x^3 + y^3$ .... **(ii)**

Eliminate x from **(ii)** with the help of **(i)**, then $u = (64 - y)^3 + y^3$

$\therefore$ $\frac{du}{dy} = -3(64 - y)^2 + 3y^2$ .... **(iii)**

and $\frac{d^2u}{dy^2} = 6(64 - y) + 6y = 384 > 0$ .... **(iv)**

For maximum or minimum of u, $\frac{du}{dy} = 0$

Then $3(64)(2y - 64) = 0$

$\therefore$ $y = 32$

From **(i)**, $x = 32$

It is clear from **(iv)**, u is minimum.

Hence $x = 32$, $y = 32$.

TIPS & FORMULAS

TANGENT & NORMAL

1. Tangent to The Curve at a Point

The tangent to the curve at 'P' is the line through P whose slope is limit of the secant's slope as $Q \to P$ from either side.

2. Normal to The Curve at a Point

A line which is perpendicular to the tangent at the point of contact is called normal to the curve at that point.

3. Things to Remember

(A) The value of the derivative at $P(x_1, y_1)$ gives the slope of the tangent to the curve at P. Symbolically

$$f'(x_1) = \left.\frac{dy}{dx}\right]_{(x_1,y_1)} = \text{Slope of tangent at } P(x_1,y_1) = m(\text{say}).$$

(B) Equation of tangent at (x_1, y_1) is ;

$$y - y_1 = \left.\frac{dy}{dx}\right]_{(x_1,y_1)} (x - x_1)$$

(C) Equation of normal at (x_1, y_1) is ; $y - y_1 = -\dfrac{1}{\left.\frac{dy}{dx}\right]_{(x_1,y_1)}}(x - x_1)$.

Note

(i) The point P(x1, y1) will satisfy the equation of the curve & the equation of tangent & normal line.

(ii) If the tangent at any point P on the curve is parallel to the axis of x then dy/dx = 0 at the point P.

(iii) If the tangent at any point on the curve is parallel to the axis of y, then dy/dx is not defined or dx/dy = 0 at that point.

(iv) If the tangent at any point on the curve is equally inclined to both the axes then $dy/dx = \pm 1$.

(v) If a curve passing through the origin be given by a rational integral algebraic equation, then the equation of the tangent (or tangents) at the origin is obtained by equation to zero the terms of the lowest degree in the equation.
e.g. If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$
i.e., $x + y = 0$ and $x - y = 0$

4. Angle of Intersection between two Curves

Angle of intersection between two curves is defined as the angle between the two tangents drawn to the two curves at their point of intersection. If the angle between two curves is 90° then they are called orthogonal curves.

5. Length of Tangent, Subtangent, Normal & Subnormal

(A) Length of the tangent (PT) = $\dfrac{y_1\sqrt{1+[f'(x_1)]^2}}{f'(x_1)}$

(B) Length of Subtangent (MT) = $\dfrac{y_1}{f'(x_1)}$

(C) Length of Normal (PN) = $y_1\sqrt{1+[f'(x_1)]^2}$

(D) Length of Subnormal (MN) = $y_1 f'(x_1)$

6. Differentials

The differential of a function is equal to its derivative multiplied by the differential of the independent variable. Thus, if, $y = \tan x$ then $dy = \sec^2 x\, dx$. In general $dy = f'(x)\, dx$ or $df(x) = f'(x)\, dx$

Note

(i) $d(c) = 0$ where 'c' is a constant

(ii) $d(u + v) = du + dv$

(iii) $d(uv) = udv + vdu$

(iv) $d(u - v) = du - dv$

(v) $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$

(vi) For the independent variable 'x', increment Δx and differential dx are equal but this is not the case with the dependent variable 'y' i.e., $\Delta y \neq dy$.

$\therefore$ Approximate value of y when increment Δx is given to independent variable x in $y = f(x)$ is $y + \Delta y = f(x + \Delta x)$

$= f(x) + \frac{dy}{dx} . \Delta x,$

(vii) The relation $dy = f'(x)\, dx$ can be written as $\frac{dy}{dx} = f'(x)$; thus the quotient of the differentials of 'y' and 'x' is equal to the derivative of 'y' w.r.t. 'x'.

MONOTONICITY

1. Monotonicity at a Point

(A) A function f(x) is called an increasing function at point x = a, if in a sufficiently small neighbourhood of x = a ; $f(a - h) < f(a) < f(a + h)$

(B) A function f(x) is called a decreasing function at point x = a if in a sufficiently small neighbourhood of x = a ; $f(a - h) > f(a) > f(a + h)$

Note

If x = a is a boundary point then use the appropriate one sides inequality to test Monotonicity of f(x).

(C) Derivative Test for Increasing and Decreasing Functions at a Point :

(i) If $f'(a) > 0$ then f(x) is increasing at x = a.

(ii) If $f'(a) < 0$ then f(x) is decreasing at x = a.

(iii) If $f'(a) = 0$ then examine the sign of $f'(a^+)$ and $f'(a^-)$.

(1) If $f'(a^+) > 0$ and $f'(a^-) > 0$ then increasing

(2) If $f'(a^+) < 0$ and $f'(a^-) < 0$ then decreasing

(3) Otherwise neither increasing nor decreasing

Note

Above rule is applicable only for functions that are differentiable at x = a.

2. Monotonicity over an Interval

(A) A function f(x) is said to be monotonically increasing (MI) in (a, b) if $f'(x) \geq 0$ where equality holds only for discrete values of x i.e., $f'(x)$ does not identically become zero for $x \in (a, b)$ or any sub interval.

(B) f(x) is said to be monotonically decreasing (MD) in (a, b) if $f'(x) \leq 0$ where equality holds only for discrete values of x i.e., $f'(x)$ does not identically become zero for $x \in (a, b)$ or any sub interval.

❖ By discrete points we mean that points where $f'(x) = 0$ does not form an interval.

Note

A function is said to be monotonic if it's either increasing or decreasing.

3. Special Points

(A) **Critical Points :** The points of domain for which $f'(x)$ is equal to zero or doesn't exist are called critical points.

(B) **Stationary Points :** The stationary points are the points of domain where $f'(x) = 0$.

Every stationary point is a critical point.

4. Rolle's Theorem

Let f be a function that satisfies the following three hypotheses :

(A) f is continuous in the closed interval [a, b]

(B) f is differentiable in the open interval (a, b)

(C) f(a) = f(b)

Then there is a number c in (a, b) such that $f'(c) = 0$

Conclusion : If f is a differentiable function then between any two consecutive roots of f(x) = 0, there is atleast one root of the equation $f'(x) = 0$

5. Lagrange's Mean value Theorem (LVMT)

Let f be a function that satisfies the following hypotheses.

(i) f is continuous in a closed interval [a, b].

(ii) f is differentiable in the open interval (a, b). Then there is a number c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

(A) Geometrical Interpretation

Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least tangent parallel to chord AB.

(B) Physical Interpretations

If we think of the number (f(b) – f(a)) / (b – a) as the average change in f over [a, b] and $f'(c)$ as an instantaneous change, then the Mean Value Theorem says that at some interior point the instantaneous change must equal to average change over the entire interval.

6. Special Note

Use of Monotonicity in identifying the number of roots of the equation in a given interval. Suppose a and b are two real numbers such that,

(A) $f(x)$ & its first derivative $f'(x)$ are continuous for $a \le x \le b$.

(B) $f(a)$ and $f(b)$ have opposite signs.

(C) $f'(x)$ is different from zero for all values of x between a & b.

Then there is one & only one root of the equation $f(x) = 0$ in (a, b).

MAXIMA - MINIMA

1. Introduction

(A) Maxima (Local Maxima)

A function $f(x)$ is said to have a maximum at $x = a$ if there exist a neighbourhood $(a - h, a + h) - (a)$ such that $f(a) > f(x)\ \forall\ x \in (a - h, a + h) - \{a\}$

(B) Minima (Local Minima)

A function $f(x)$ is said to have a minimum at $x = a$ if there exist a neighbourhood $(a - h, a + h) - \{a\}$ such that $f(a) < f(x)\ \forall\ x \in (a - h, a + h) - \{a\}$

(C) Absolute Maximum (Global Maximuma)

A function f has an absolute maximum (or global) at c if $f(c) \ge f(x)$ for all x in D, where D is the domain of f. The number of $f(c)$ is called the maximum value of f on D.

(D) Absolute Minimum (Global Minimum)

A function f has an absolute minimum at c if $f(c) \le f(x)$ for all x in D and the number $f(c)$ is called the minimum value of f on D. The maximum and minimum values of f are called the extreme values of f.

Note

(i) The maximum & minimum values of a function are also known as local/relative maximua or local/relative minima as these are the greatest & least values of the function relative to some neighbourhood of the point in questions.

(ii) The term ‘extremum’ or ‘turning value’ is used both for maximu or a minimum value.

(iii) A maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(iv) A function can have several maximum & minimum values & a minimum value may be greater than a maximum value.

(v) Local maximum & local minimum values of a continuous function occur alternately & between two consecutive local maximum values there is a local minimum value & vice versa.

(vi) Monotonic function do not have extrems points.

2. Derivative Test for Ascertaining Maxima/Minima

(A) First Derivative Test

Find the point (say $x = a$) where $f'(x) = 0$ and

- **(i)** If $f'(x)$ changes sign from positive to negative while graph of the function passes through $x = a$ then $x = a$ is said to be a point of local maxima.
- **(ii)** If $f'(x)$ changes sign from negative to positive while graph of the function passes through $x = a$ then $x = a$ is said to be a point of local minima.

Note

If $f'(x)$ does not change sign i.e. has the same sign in a certain complete neighbourhood of a, then $f(x)$ is either strictly increasing or decreasing this neighbourhood implying that $f(a)$ is not an extreme value of f.

(B) Second Derivative Test

If f(x) is continuous and differentiable at x = a where $f'(a) = 0$ and $f''(a)$ also exists then for ascertaining maxima/minima at x = a, 2nd derivative test can be used.

(i) If $f''(a) > 0$ $\Rightarrow$ x = a is a point of local minima

(ii) If $f''(a) < 0$ $\Rightarrow$ x = a is a point of local maxima

(iii) If $f''(x) = 0$ $\Rightarrow$ second derivative test fails. To identify maxima/minima at this point either first derivative test or higher derivative can be used.

3. Useful Formulae or Mensuration of Remember

(A) Volume of a cuboid = ℓbh.

(B) Surface area of a cuboid = $2(\ell b + bh + h\ell)$.

(C) Volume of a prism = area of the base x height.

(D) Lateral surface area of prism = perimeter of the base x height.

(E) Total surface of a prism = lateral surface area + 2 area of the base (Note that lateral surfaces of a prism are all rectangles).

(F) Volume of a pyramid = $\frac{1}{3}$ area of the base x height.

(G) Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) x slant height.
(Note that slant surfaces of a pyramid are triangles)

(H) Volume of a cone = $\frac{1}{3}\pi r^2 h$.

(I) Curved surface area of a cylinder = $2\pi rh$.

(J) Total surface of a cylinder = $2\pi rh + 2\pi r^2$.

(K) Volume of a sphere = $\frac{4}{3}\pi r^3$.

(L) Surface area of a sphere = $4\pi r^2$.

(M) Area of a circular sector = $\frac{1}{2} r^2\theta$, when θ is in radians.

(N) Perimeter of circular sector = $2r + r\theta$.

4. Significance of the Sign of 2nd order Derivative and Point of Inflection

The sign of the 2nd order derivative determines the concavity of the curve.

If $f''(x) > 0 \ \forall \ x \in (a, b)$ then graph of f(x) is concave upward in (a, b).

Similarly if $f''(x) < 0 \ \forall \ x \in (a, b)$ then graph of f(x) is concave downward in (a, b).

Point of Inflection

A point where the graph of a function has a tangent line and where the concavity changes as called a point of inflection.

For finding point of inflection of any function, compute the solutions of $\frac{d^2y}{dx^2} = 0$ or does not exist. Let the solution is x = a, if sign of $\frac{d^2y}{dx^2}$ changes about this point then it is called point of inflection.

Note

If at any point $\frac{d^2y}{dx^2}$ does not exist but sign of $\frac{d^2y}{dx^2}$ changes about this point then it is also called point of inflection.

5. Some Standard Results

(A) Rectangle of largest area inscribed in a circle is a square.

(B) The function $y = \sin^m x \cos^n x$ attains the max value at $x = \tan^{-1}\sqrt{\frac{m}{n}}$

(C) If $0 < a < b$ then $|x - a| + |x - b| \geq b - a$ and equality hold when $x \in [a, b]$

If $0 < a < b < c$ then $|x - a| + |x - b| + |x - c| \geq c - a$ and equality hold when $x \in [a, c]$

If $0 < a < b < c < d$ then $|x - a| + |x - b| + |x - c| + |x - d| \geq d - a$ and equality hold when $x \in [a, d]$

6. Shortest Distance between two Curves

Shortest distance between two non-intersecting curves always along the common normal. (Wherever defined)