FUNCTION

DEFINITION

A relation R from a set A to a set B is called a function if each element of A has unique image in B.
It is denoted by the symbol.

$f: A \rightarrow B$ or $A \xrightarrow{f} B$

which reads f ' is a function from A to B 'or' f maps A to B,
If an element $a \in A$ is associated with an element $b \in B$, then b is called 'the f image of a' or 'image of a under f 'or' the value of the function f at a'. Also a is called the pre-image of b or argument of b under the function f. We write it as

$b = f(A)$ or $f: a \rightarrow b$ or $f: (a, b)$

Thus a function 'f ' from a set A to a set B is a subset of $A \times B$ in which each 'a' belonging to A appears in one and only one ordered pair belonging to f.

Every function from $A \rightarrow B$ satisfies the following conditions .
(I) $f \subset A \times B$ **(ii)** $\forall\ a \in A \Rightarrow (a, f(A)) \in f$ and **(iii)** $(a, b) \in f$ & $(a, c) \in f \Rightarrow b = c$

REPRESENTATION OF FUNCTION

(A) **Ordered pair :** Every function from $A \rightarrow B$ satisfies the following conditions :
(i) $f \subset A \times B$ **(ii)** $\forall\ a \in A$ there exist $b \in B$ and **(iii)** $(a, b) \in f$ & $(a, c) \in f \Rightarrow b = c$

(B) **Formula based (uniformly/nonuniformly) :**

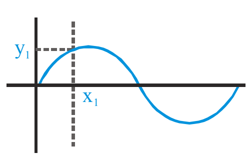
e.g.

(i) $f: R \rightarrow R, y = f(x) = 4x, f(x) = x^2$ (uniformly defined)

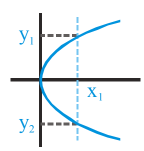
(ii) $f(x) = \begin{cases} x+1 & -1 \leq x < 4 \\ -x & 4 \leq x < 7 \end{cases}$ (non-uniformly defined)

(iii) $f(x) = \begin{cases} x^2 & x \geq 0 \\ -x-1 & x < 0 \end{cases}$ (non-uniformly defined)

(C) **Graphical representation :**



Graph (1)



Graph (2)

Graph(1) represent a function but graph(2) does not represent a function.

Domain, Co-domain & Range Of A Function

Let $f: A \rightarrow B$, then the set A is known as the domain of f & the set B is known as co-domain of f. The set of f images of all the elements of A is known as the range of f.

Thus : Domain of $f = \{a \mid a \in A, (a, f(A)) \in f\}$
Range of $f = \{f(A) \mid a \in A, f(A) \in B\}$

(i) If a vertical line cuts a given graph at more than one point then it can not be the graph of a function.
(ii) Every function is a relation but every relation is not necessarily a function.
(iii) It should be noted that range is a subset of co-domain.
(iv) If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives the range

METHODS OF DETERMINING RANGE

(i) Representing x in terms of y

If y = f(x), try to express as x = g(y), then domain of g(y) represents possible values of y, which is range of f(x).

Ex. Find the range of $f(x) = \dfrac{x^2+x+1}{x^2+x-1}$

Sol. $f(x) = \dfrac{x^2+x+1}{x^2+x-1}$ $\{x^2 + x + 1$ and $x^2 + x - 1$ have no common factor$\}$

$y = \dfrac{x^2+x+1}{x^2+x-1}$

$\Rightarrow \quad yx^2 + yx - y = x^2 + x + 1$

$\Rightarrow \quad (y-1)\,x^2 + (y-1)\,x - y - 1 = 0$

If y = 1, then the above equation reduces to –2 = 0. Which is not true.

Further if $y \neq 1$, then $(y-1)\,x^2 + (y-1)\,x - y - 1 = 0$ is a quadratic and has real roots if

$(y-1)^2 - 4\,(y-1)\,(-y-1) \geq 0$

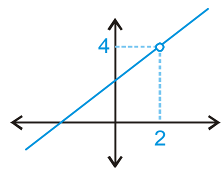
i.e. if $y \leq -3/5$ or $y \geq 1$ but $y \neq 1$

Thus the range is $(-\infty, -3/5] \cup (1, \infty)$

(ii) Graphical Method

The set of y– coordinates of the graph of a function is the range.

Ex. Find the range of $f(x) = \dfrac{x^2-4}{x-2}$

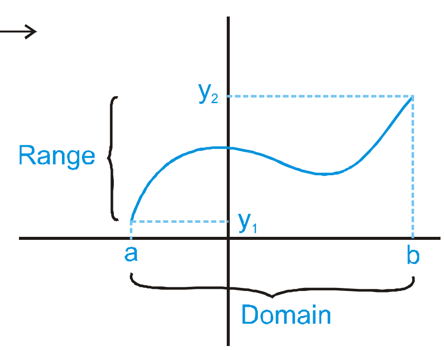


Sol. $f(x) = \dfrac{x^2-4}{x-2} = x + 2;\ x \neq 2$

$\therefore$ graph of f(x) would be

Thus the range of $f(x)$ is $R - \{4\}$

Further if f(x) happens to be continuous in its domain then range of f(x) is [min f(x), max. f(x)]. However for sectionally continuous functions, range will be union of [min f(x), max. f(x)] over all those intervals where f(x) is continuous, as shown by following example.



Ex. Find the Domain of the following function :

(i) $f(x)=\sqrt{\sin x}+\sqrt{16-x^2}$ **(ii)** $f(x)=\log_2\left(-\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)-1\right)$

Sol. **(i)** $\sin x \ge 0$ and $16-x^2\ge 0$

$\Rightarrow$ $2n\pi \le x \le (2n+1)\pi$ and $-4\le x\le 4$

$\therefore$ Domain is $[-4,-\pi]\cup[0,\pi]$

(iii) We have $f(x)=\log_2\left(-\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)-1\right)$

f(x) is defined if $-\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)-1>0$

or if $\log_{1/2}\left(1+\frac{1}{\sqrt[4]{x}}\right)<-1$ or if $\left(1+\frac{1}{\sqrt[4]{x}}\right)>(1/2)^{-1}$

or if $1+\frac{1}{\sqrt[4]{x}}>2$ or if $\frac{1}{\sqrt[4]{x}}>1$ or if $x^{1/4}<1$ or if $0<x<1$

$\therefore$ $D(F)=(0,1)$

Ex. Find the range of following functions :

(i) $f(x)=\log_2\left(\frac{\sin x-\cos x+3\sqrt{2}}{\sqrt{2}}\right)$ **(ii)** $f(x)=\log_{\sqrt{2}}\left(2-\log_2(16\sin^2 x+1)\right)$

Sol. **(i)** Let $y=\log_2\left(\frac{\sin x-\cos x+3\sqrt{2}}{\sqrt{2}}\right)$

$\Rightarrow$ $2^y=\sin\left(x-\frac{\pi}{4}\right)+3$ $\Rightarrow$ $-1\le 2^y-3\le 1$

$\Rightarrow$ $2\le 2^y\le 4$ $\Rightarrow$ y $\Rightarrow$ $[1,2]$

(ii) $f(x)=\log_{\sqrt{2}}\left(2-\log_2(16\sin^2 x+1)\right)$

$1\le 16\sin^2 x+1\le 17$

$\therefore$ $0\le\log_2(16\sin^2 x+1)\le\log_2 17$ $\therefore$ $2-\log_2 17\le 2-\log_2(16\sin^2 x+1)\le 2$

Now consider $0<2-\log_2(16\sin^2 x+1)\le 2$

$\therefore$ $-\infty<\log_{\sqrt{2}}[2-\log_2(16\sin^2 x+1)]\le\log_{\sqrt{2}}2=2$ $\therefore$ the range is $(-\infty, 2]$

NUMBER OF FUNCTION

Let A and B be two finite sets having m and n elements respectively. Then, each element of set A can be associated to any one of n elements of set A. So, total number of functions from set A to set B is equal to the number of ways of doing m jobs where each job can be done in n ways.

The total number of such ways is $n\times n\times n\ ....\times n=n^m$. (m-times)

Hence, the total number of functions from A to B is n^m i.e, . $[O(B)]^{O(A)}$.

For example, the total number of functions from a set

$A = \{a, b, c, d\}$ to a set $B = \{1, 2, 3\}$ is $3^4 = 81$.

The total number of relations from a set A having m elements to a set n having n elements is 2^{mn}. So, the number of relations from A to B which are not functions is $2^{mn} - n^m$ i.e., $2^{O(A) \times O(B)} - [O(B)]^{O(A)}$.

(i) Polynomial Function

If a function f is defined by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, ..., a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

(A) A polynomial of degree one with no constant term is called an odd linear function. i.e. $f(x) = ax$, $a \neq 0$

(B) There are two polynomial functions, satisfying the relation;

$f(x).f(1/x) = f(x) + f(1/x)$. They are :

(I) $f(x) = x^n + 1$ & **(ii)** $f(x) = 1 - x^n$, where n is a positive integer.

(ii) Algebraic Function

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form

$P_0(x) y^n + P_1(x) y^{n-1} + ... + P_{n-1}(x) y + P_n(x) = 0$

Where n is a positive integer and $P_0(x), P_1(x)$ .... are Polynomials in x.

e.g. $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

❖ That all polynomial functions are Algebraic but not the converse. A function that is not algebraic is called Transcedental Function.

(iii) Fractional Rational Function

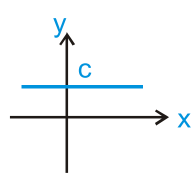
A rational function is a function of the form. $y = f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ & $h(x)$ are polynomials & $h(x) \neq 0$.

(iv) Constant function

A function $f : A \rightarrow B$ is said to be a constant function, if every element of A has the same f image in B.

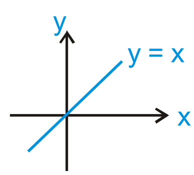
Thus $f : A \rightarrow B$;

$f(x) = c$, $\forall$ $x \in A$, $c \in B$ is a constant function.



(v) Identity function

The function $f : A \rightarrow A$ defined by, $f(x) = x$, $\forall$ $x \in A$ is called the identity function on A and is denoted by I_A. It is easy to observe that identity function is a bijection.



ALGEBRAIC FUNCTION

A function 'f' is called an algebraic function if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking radicals) starting with polynomials.

Examples $f(x) = \sqrt{x^2+1}$; $g(x) = \dfrac{x^4-16x^2}{x+\sqrt{x}} + (x-2)\sqrt[3]{x+1}$

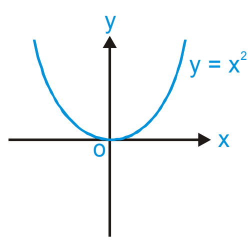
If y is an algebraic function of x, then it satisfies a polynomial equation of the form

$P_0(x)y^n + P_1(x)y^{n-1} + \ldots\ldots + P_{n-1}(x)y + P_n(x) = 0$, where 'n' is a positive integer and $P_0(x), P_1(x), \ldots\ldots$ are polynomial in x.

Note that all polynomial functions are Algebraic but the converse in not true. A functions that is not algebraic is called transcedental function.

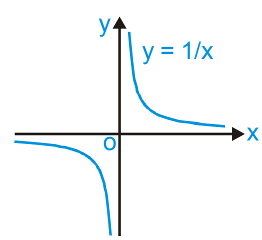
BASIC ALGEBRAIC FUNCTION

(i) $y = x^2$



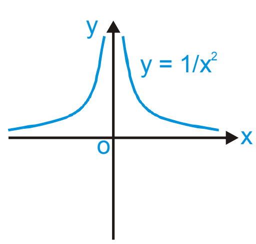
Domain : R **Range :** $R^+\{0\}$ or $[0, \infty)$

(ii) $y = \dfrac{1}{x}$



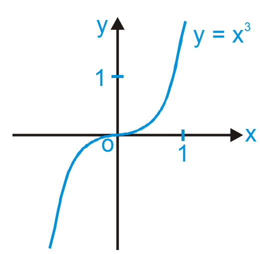
Domain : $R - \{0\}$ or R_0 **Range :** $R - \{0\}$

(iii) $y = \dfrac{1}{x^2}$



Domain : R_0 **Range :** R^+ or $(0, \infty)$

(iv) $y = x^3$



Domain : R **Range :** R

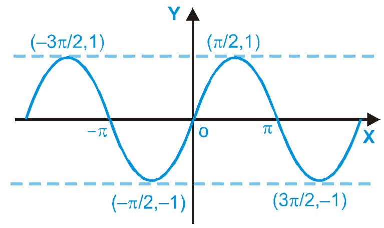
TRIGONOMETRIC FUNCTIONS

(I) Sine function

$f(x) = \sin x$

Domain : R

Range : $[-1, 1]$, period 2π

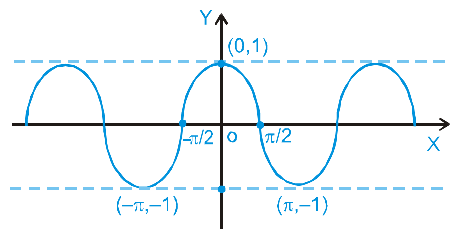


(ii) Cosine function

$f(x) = \cos x$

Domain : R

Range : $[-1, 1]$, period 2π

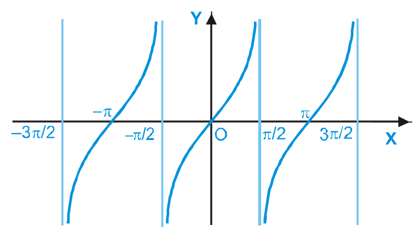


(iii) Tangent function

$f(x) = \tan x$

Domain : $R - \left\{x \mid x = \dfrac{(2n+1)\pi}{2}, n \in I\right\}$

Range : R , period π

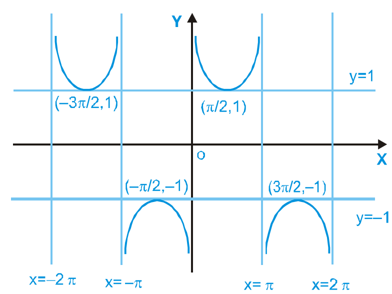


(iv) Cosecant function

$f(x) = \text{cosec } x$

Domain : $R - \{x \mid x = n\pi, n \in I\}$

Range : $R - (-1, 1)$, period 2π

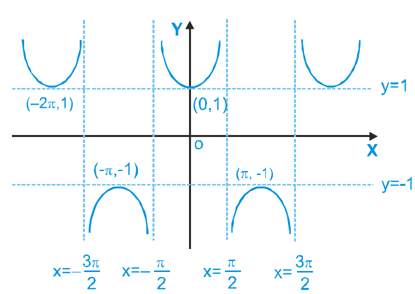


(v) Secant function

$f(x) = \sec x$

Domain : $R - \{x \mid x = (2n + 1)\pi/2 : n \in I\}$

Range : $R - (-1, 1)$, period 2π

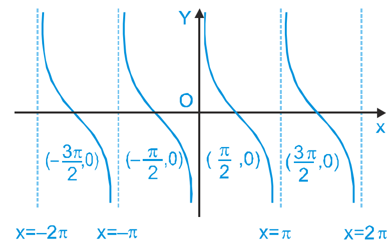


(vi) Cotangent function

$f(x) = \cot x$

Domain : $R - \{x | x = n\pi, n \in I\}$

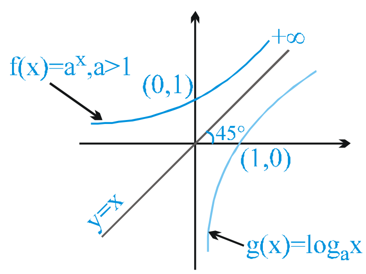
Range : R, period π

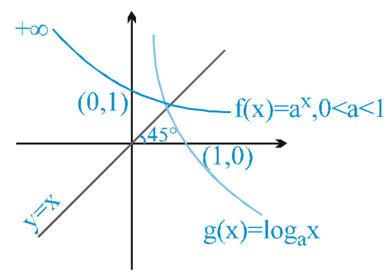


Exponential Function

A function $f(x) = a^x = e^{x \ln a}$ $(a > 0, a \neq 1, x \in R)$ is called an exponential function. The inverse of the exponential function is called the logarithmic function . i.e. $g(x) = \log_a x$.

Note that f(x) & g(x) are inverse of each other & their graphs are as shown .





Absolute Value Function

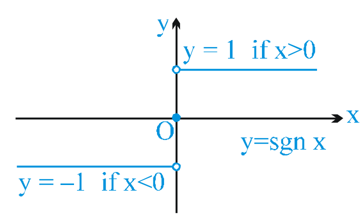
A function $y = f(x) = |x|$ is called the absolute value function or Modulus function. It is defined as :

$$y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Signum Function

A function y= f (x) = Sgn (x) is defined as follows :

$$y = f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$



It is also written as $\text{Sgn } x = |x|/x$;

$x \neq 0$; $f(0) = 0$

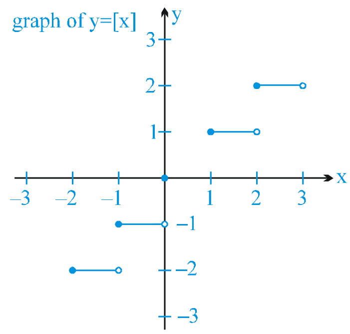
Greatest Integer Or Step Up Function

The function $y = f(x) = [x]$ is called the greatest integer function where $[x]$ denotes the greatest integer less than or equal to x . Note that for :

$-1 \leq x < 0$	;	$[x] = -1$	$0 \leq x < 1$	;	$[x] = 0$
$1 \leq x < 2$	;	$[x] = 1$	$2 \leq x < 3$	;	$[x] = 2$

and so on .

Properties of greatest integer function



(A) $[x] \le x < [x]+1$ and

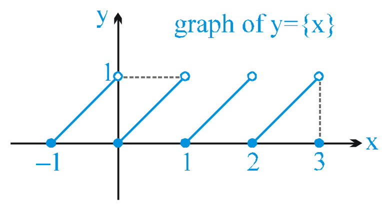
$x-1 < [x] \le x,\ 0 \le x-[x] < 1$

(B) $[x+m] = [x]+m$ if m is an integer.

(C) $[x]+[y] \le [x+y] \le [x]+[y]+1$

(D) $[x]+[-x] = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise.} \end{cases}$

Fractional Part Function



It is defined as :

$g(x) = \{x\} = x - [x]$.

e.g. the fractional part of the no. 2.1 is

$2.1 - 2 = 0.1$ and the fractional part of -3.7 is 0.3.

The period of this function is 1 and graph of this function is as shown .

Ex. Determine the values of x satisfying the equality $|x^4 - x^2 - 6| = |x^4 - 4| - |x^2 + 2|$.

Sol. The equality $|a - b| = |a| - |b|$ holds true if and only if a and b have the same sign and $|a| \ge |b|$.

In our case the equality will hold true for the value of x at which $x^4 - 4 \ge x^2 + 2$.

Hence $x^2 - 2 \ge 1;\ |x| \ge \sqrt{3}$.

Ex. If $y = 2[x] + 3$ & $y = 3[x-2] + 5$ then find $[x+y]$ where [.] denotes greatest integer function.

Sol. $y = 3[x-2] + 5 = 3[x] - 1$

So $3[x] - 1 = 2[x] + 3$

$[x] = 4 \Rightarrow 4 \le x < 5$

Then $y = 11$

So $x + y$ will lie in the interval $[15, 16)$

So $[x+y] = 15$

Ex. Solve the equation $|2x - 1| = 3[x] + 2\{x\}$ where [.] denotes greatest integer and { . } denotes fractional part function.

Sol. We are given that, $|2x - 1| = 3[x] + 2\{x\}$

Let, $2x - 1 \le 0$ i.e. $x \le \frac{1}{2}$. The given equation yields.

$1 - 2x = 3[x] + 2\{x\}$

$\Rightarrow 1 - 2[x] - 2\{x\} = 3[x] + 2\{x\} \quad \Rightarrow 1 - 5[x] = 4\{x\} \quad \Rightarrow \{x\} = \frac{1-5[x]}{4}$

$\Rightarrow 0 \le \frac{1-5[x]}{4} < 1 \quad \Rightarrow 0 \le 1 - 5[x] < 4 \quad \Rightarrow -\frac{3}{5} < [x] \le \frac{1}{5}$

Now, $[x] = 0$ as zero is the only integer lying between $-\frac{3}{5}$ and $\frac{1}{5}$

$\Rightarrow \quad \{x\} = \frac{1}{4} \quad \Rightarrow \quad x = \frac{1}{4}$ which is less than $\frac{1}{2}$, Hence $\frac{1}{4}$ is one solution.

Now, let $2x - 1 > 0$ i.e $x > \frac{1}{2}$

$\Rightarrow \quad 2x - 1 = 3[x] + 2\{x\} \quad \Rightarrow \quad 2[x] + 2\{x\} - 1 = 3[x] + 2\{x\}$

$\Rightarrow \quad [x] = -1 \quad \Rightarrow \quad -1 \le x < 0$ which is not a solution as $x > \frac{1}{2}$

$\Rightarrow \quad x = \frac{1}{4}$ is the only solution.

ALGEBRAIC OPERATIONS ON FUNCTIONS

If f and g are real valued functions of x with domain set A and B respectively, then both f and g are defined in $A \cap B$. Now we define $f + g$, $f - g$, $(f \,.\, g)$ and (f/g) as follows:

(i) $(f \pm g)(x) = f(x) \pm g(x)$

(ii) $(f.g)(x) = f(x).\,g(x)$

domain in each case is $A \cap B$

(iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain is $\{x \mid x \in A \cap B$ such that $g(x) \ne 0\}$.

(iv) $(kf)(x) = kf(x)$ where k is a scalar.

Equal or Identical Function

Two functions f & g are said to be equal if :

(i) The domain of f = the domain of g.

(ii) The range of f = the range of g and

(iii) $f(x) = g(x)$, for every x belonging to their common domain. eg.

$$f(x) = \frac{1}{x} \;\&\; g(x) = \frac{x}{x^2} \text{ are identical functions .}$$

Ex. The functions $f(x) = \log\left(\frac{x-1}{x-2}\right)(x-1) - \log(x-2)$ and $g(x) = \log$ are identical when x lies in the interval

Sol. Since $f(x) = \log(x-1) - \log(x-2)$.

Domain of $f(x)$ is $x > 2$ or $x \in (2, \infty)$ .....**(I)**

$g(x) = \log\left(\frac{x-1}{x-2}\right)$ is defined if $\frac{x-1}{x-2} > 0$

$\Rightarrow \quad x \in (-\infty, 1) \cup (2, \infty)$ .....**(ii)**

From **(I)** and **(ii)**, $x \in (2, \infty)$.

CLASSIFICATION OF FUNCTIONS

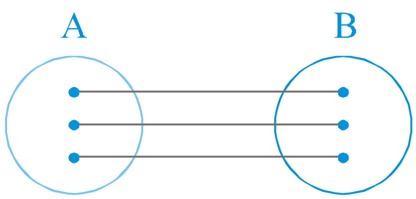
One – One Function (Injective mapping)

A function $f : A \to B$ is said to be a one–one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ & $f(x_1)$,

$f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Examples : $R \to R$ $f(x) = x^3 + 1$; $f(x) = e^{-x}$; $f(x) = \ln x$

Diagramatically an injective mapping can be shown as



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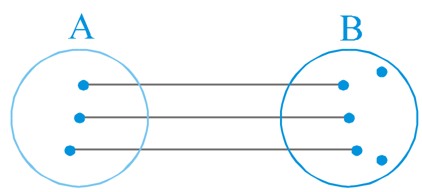


(i) A continuous function which is always increasing or decreasing in whole domain, then f(x) is one–one.

(ii) If any line parallel to x–axis cuts the graph of the function atmost at one point, then the function is one–one.

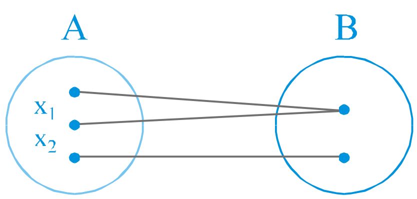
Many–one function : (not injective)

A function $f : A \to B$ is said to be a many one function if two or more elements of A have the same f image in B. Thus $f : A \to B$ is many one if for ;

$x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Ex. $R \to R$ $f(x) = [x]$; $f(x) = |x|$; $f(x) = ax^2 + bx + c$; $f(x) = \sin x$

Diagramatically a many one mapping can be shown as



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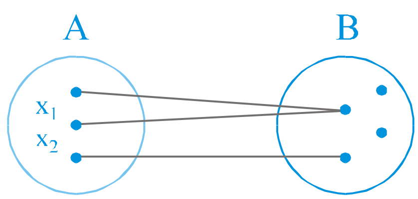


(i) Any continuous function which has atleast one local maximum or local minimum, then f(x) is many–one . In other words, if a line parallel to x–axis cuts the graph of the function atleast at two points, then f is many–one .

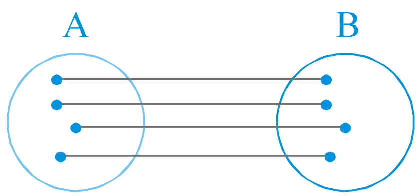
(ii) If a function is one–one, it cannot be many–one and vice versa.

One One + Many One = Total number of mappings.

Onto function (Surjective mapping)

If the function $f: A \to B$ is such that each element in B (co–domain) is the f image of atleast one element in A, then we say that f is a function of A 'onto' B. Thus $f: A \to B$ is surjective iff $\forall\ b \in B,\ \exists$ some $a \in A$ such that $f(A) = b$.

Diagramatically surjective mapping can be shown as



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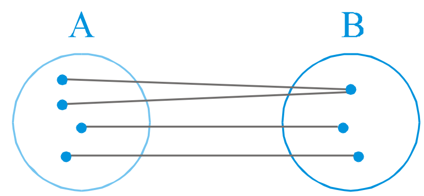
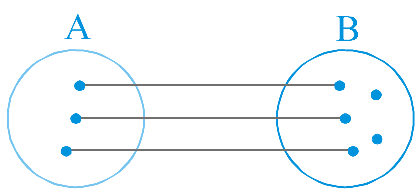


❖ If range = co–domain, then f(x) is onto .

Into function :

If $f: A \to B$ is such that there exists atleast one element in co–domain which is not the image of any element in domain, then f(x) is into .



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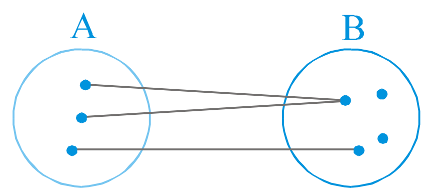
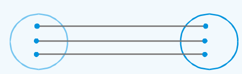
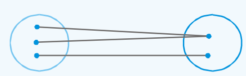


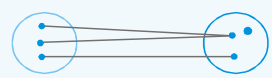
If a function is onto, it cannot be into and vice versa . A polynomial of degree even define from $R \to R$ will always be into & a polynomial of degree odd defined from $R \to R$ will always be onto.

Thus a function can be one of these four types :

(i) one–one onto (injective & surjective) $(I \cap S)$ 

(ii) one–one into (injective but not surjective) $(I \cap \overline{S})$ 

(iii) many–one onto (surjective but not injective) $(S \cap \overline{I})$ 

(iv) many–one into (neither surjective nor injective) $(\overline{I} \cap \overline{S})$ 

(v) If f is both injective & surjective, then it is called a Bijective mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(vi) If a set A contains n distinct elements then the number of different functions defined from $A \to A$ is n^n & out of it n ! are one one.

Ex. Let $f : R \to R$ be a function defined by $f(x) = x + \sqrt{x^2}$, then which type of function f is ?

Sol. We have, $f(x) = x + \sqrt{x^2} = x + |x|$

Clearly, f is not one-one as $f(-1) = f(-2) = 0$ and $-1 \neq -2$

Also, f is not onto as $f(x) \geq 0 \ \forall \ x \in R$

$\therefore$ range of $f = (0, \infty) \subset R$

Ex. Let $f(x) = \dfrac{x^2+3x+a}{x^2+x+1}$, where $f : R \to R$. Find the value of parameter 'a' so that the given function is one-one.

Sol. $f(x) = \dfrac{x^2+3x+a}{x^2+x+1}$

$$f'(x) = \frac{(x^2+x+1)(2x+3)-(x^2+3x+a)(2x+1)}{(x^2+x+1)^2} = \frac{-2x^2+2x(1-a)+(3-a)}{(x^2+x+1)^2}$$

Let, $g(x) = -2x^2 + 2x(1-a) + (3-a)$

g(x) will be negative if $4(1-a)^2 + 8(3-a) < 0$

$\Rightarrow$ $1 + a^2 - 2a + 6 - 2a < 0$

$\Rightarrow$ $(a-2)^2 + 3 < 0$ (which is not possible)

Therefore function is not monotonic.

Hence, no value of a is possible.

COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTIONS

Let $f: A \to B$ & $g: B \to C$ be two functions. Then the function $gof: A \to C$ defined by $(gof)(x) = g(f(x)) \ \forall \ x \in A$ is called the composite of the two functions f & g .

Diagramatically $\xrightarrow{x} f \xrightarrow{f(x)} g \longrightarrow g(f(x))$.

Thus the image of every $x \in A$ under the function gof is the g–image of the f–image of x .

Note that gof is defined only if $\forall \ x \in A$, f(x) is an element of the domain of g so that we can take its g image. Hence for the product gof of two functions f & g, the range of f must be a subset of the domain of g.

Properties Of Composite Functions

(i) The composite of functions is not commutative i.e. $gof \neq fog$.

(ii) The composite of functions is associative i.e. if f, g, h are three functions such that fo(goh) & (fog)oh are defined, then fo(goh) = (fog)oh .

(iii) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

HOMOGENEOUS FUNCTIONS

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables .

For example $5x^2 + 3y^2 - xy$ is homogeneous in x & y . Symbolically if , $f(tx, ty) = t^n . f(x, y)$ then $f(x, y)$ is homogeneous function of degree n .

BOUNDED FUNCTION

A function is said to be bounded if $|f(x)| \le M$, where M is a finite quantity .

IMPLICIT & EXPLICIT FUNCTION

A function defined by an equation not solved for the dependent variable is called an **Implicit Function** .

For eg. the equation $x^3 + y^3 = 1$ defines y as an implicit function.

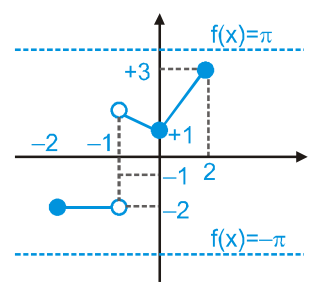
If y has been expressed in terms of x alone then it is called an **Explicit Function**.

Ex. Find the domain and range of $h(x) = g(f(x))$,

where $f(x) = \begin{cases} [x], & -2 \le x \le -1 \\ |x|+1, & -1 < x \le 2 \end{cases}$ and $g(x) = \begin{cases} [x], & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$, [.] denotes the greatest integer function.

Sol. $h(x) = g(f(x)) = \begin{cases} [f(x)], & -\pi \le f(x) < 0 \\ \sin(f(x)), & 0 \le f(x) \le \pi \end{cases}$

From graph of f(x), we get



$$h(x) = \begin{cases} [[x]], & -2 \le x \le -1 \\ \sin(|x|+1), & -1 < x \le 2 \end{cases}$$

$\Rightarrow$ Domain of h(x) is [–2, 2]

and Range of h(x) is $\{-2, 1\} \cup [\sin 3, 1]$

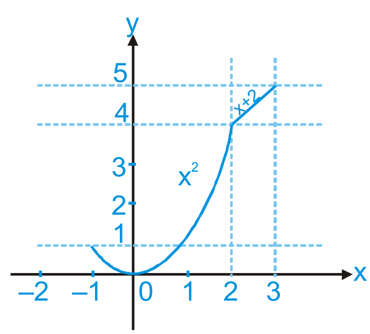
Ex. Let $f(x) = \begin{cases} x+1, & x \le 1 \\ 2x+1, & 1 < x \le 2 \end{cases}$ and $g(x) = \begin{cases} x^2, & -1 \le x < 2 \\ x+2, & 2 \le x \le 3 \end{cases}$, find (fog)

Sol. $f(g(x)) = \begin{cases} g(x)+1, & g(x) \le 1 \\ 2g(x)+1, & 1 < g(x) \le 2 \end{cases}$

Here, g(x) becomes the variable that means we should draw the graph.

It is clear that $g(x) \le 1$; $\forall\ x \in [-1, 1]$

and $1 < g(x) \le 2$; $\forall\ x \in (1, \sqrt{2}]$



$$\Rightarrow \quad f(g(x)) = \begin{cases} x^2+1, & -1 \le x \le 1 \\ 2x^2+1, & 1 < x \le \sqrt{2} \end{cases}$$

Ex. Which of the following function is not homogeneous ?

(A) $x^3 + 8x^2y + 7y^3$ **(B)** $y^2 + x^2 + 5xy$ **(C)** $\dfrac{xy}{x^2+y^2}$ **(D)** $\dfrac{2x-y+1}{2y-x+1}$

Sol. It is clear that **(D)** does not have the same degree in each term.

Ex. Which of the following function is implicit function ?

(A) $y = \dfrac{x^2 + e^x + 5}{\sqrt{1 - \cos^{-1} x}}$ **(B)** $y = x^2$ **(C)** $xy - \sin(x + y) = 0$ **(D)** $y = \dfrac{x^2 \log x}{\sin x}$

Sol. It is clear that in **(C)** y is not clearly expressed in x.

INVERSE OF A FUNCTION

Let $f: A \to B$ be a one–one & onto function, then their exists a unique function $g: B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x, \ \forall \ x \in A \ \& \ y \in B$. Then g is said to be inverse of f.

Thus $g = f^{-1}: B \to A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

Properties of Inverse Function

(I) The inverse of a bijection is unique .

(ii) If $f: A \to B$ is a bijection & $g: B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A & I_B are identity functions on the sets A & B respectively.

Note that the graphs of f & g are the mirror images of each other in the line $y = x$. As shown in the figure given below a point (x ‘,y ‘) corresponding to $y = x^2$ $(x \geq 0)$ changes to (y ‘,x ‘) corresponding to $y = +\sqrt{x}$, the changed form of $x = \sqrt{y}$.

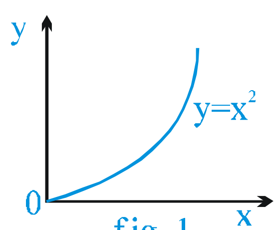


fig. 1

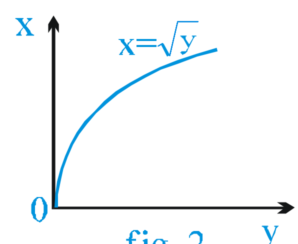


fig. 2

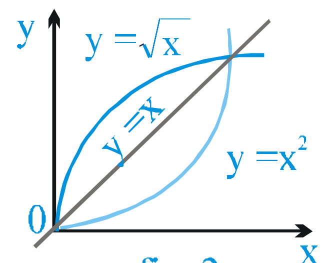


fig. 3

(iii) The inverse of a bijection is also a bijection .

(iv) If f & g are two bijections $f: A \to B$, $g: B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} \text{ o } g^{-1}$.

Ex. Find the inverse of the function f(x) = $\begin{cases} x; < 1 \\ x^2; 1 \leq x \leq 4 \\ 8\sqrt{x}; \quad x > 4 \end{cases}$

Sol. Given f(x) = $\begin{cases} x; < 1 \\ x^2; 1 \leq x \leq 4 \\ 8\sqrt{x}; \quad x > 4 \end{cases}$

Let $f(x) = y \quad \Rightarrow \quad x = f^{-1}(y)$

$$\therefore \quad x = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le \sqrt{y} \le 4 \\ \dfrac{y^2}{64}, & \dfrac{y^2}{64} > 4 \end{cases}$$

$$\Rightarrow \quad f^{-1}(y) = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le y \le 16 \\ \dfrac{y^2}{64}, & y > 16 \end{cases}$$

Hence $f^{-1}(x) = \begin{cases} x; < 1 \\ \sqrt{x}; 1 \le x \le 16 \\ \dfrac{x^2}{64}; x > 16 \end{cases}$

Ex. **(i)** Determine whether $f(x) = \dfrac{2x+3}{4}$ for $f : R \to R$, is invertible or not? If so find it.

(ii) Is the function $f(x) = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ invertible ?

Sol. **(i)** Given function is one-one and onto, therefore it is invertible.

$y = \dfrac{2x+3}{4}$

$\Rightarrow \quad x = \dfrac{4y-3}{2} \qquad \therefore \quad f^{-1}(x) = \dfrac{4x-3}{2}$

(ii) Domain of f is $[-1, 1]$

$f(0) = 0 = f(1)$

$\Rightarrow$ f is not one – one

$\Rightarrow$ f is not invertible

ODD & EVEN FUNCTIONS

If a function is such that whenever 'x' is in it's domain '–x' is also in it's domain & it satisfies

$f(-x) = f(x)$ it is an even function

$f(-x) = -f(x)$ it is an odd function

KEY POINTS

(i) A function may neither be odd nor even.

(ii) Inverse of an even function is not defined, as it is many – one function.

(iii) Every even function is symmetric about the y-axis & every odd function is symmetric about the origin.

(iv) Every function which has '–x' in it's domain whenever 'x' is in it's domain, can be expressed as the sum of an even & an odd function .

$$\text{e.g. } f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$$

(v) The only function which is defined on the entire number line & even and odd at the same time is f(x) =

f (x)	g (x)	f (x) + g (x)	f (x) - g (x)	f (x) . g (x)	f(x) / g(x)	(gof) (x)	(f o g) (x)
odd	odd	odd	odd	even	even	odd	odd
even	even	even	even	even	even	even	even
odd	even	neither odd nor even	neither odd nor even	odd	odd	even	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even	even

Ex. Show that $a^x + a^{-x}$ is an even function.

Sol. Let $f(x) = a^x + a^{-x}$

Then $f(-x) = a^{-x} + a^{-(-x)} = a^{-x} + a^x = f(x)$.

Hence f(x) is an even function

Ex. Identify the given functions as odd, even or neither :

(i) $f(x) = \dfrac{x}{e^x - 1} + \dfrac{x}{2} + 1$ **(ii)** $f(x+y) = f(x) + f(y)$ for all $x, y \in R$

Sol. **(i)** $f(x) = \dfrac{x}{e^x - 1} + \dfrac{x}{2} + 1$

Clearly domain of f(x) is $R \sim \{0\}$. We have,

$$f(-x) = \frac{-x}{e^{-x}-1} - \frac{x}{2} + 1 = \frac{-e^x.x}{1-e^x} - \frac{x}{2} + 1 = \frac{(e^x - 1 + 1)x}{(e^x - 1)} - \frac{x}{2} + 1$$

$$= x + \frac{x}{e^x - 1} - \frac{x}{2} + 1 = \frac{x}{e^x - 1} + \frac{x}{2} + 1 = f(x)$$

Hence f(x) is an even function.

(ii) $f(x+y) = f(x) + f(y)$ for all $x, y \in R$

Replacing x, y by zero, we get $f(0) = 2f(0)$ $\Rightarrow$ $f(0) = 0$

Replacing y by –x, we get $f(x) + f(-x) = f(0) = 0$ $\Rightarrow$ $f(x) = -f(-x)$

Hence f(x) is an odd function.

PERIODIC FUNCTIONS

A function f(x) is called periodic with a period T if there exists a real number $T > 0$ such that for each x in the domain of f the numbers $x - T$ and $x + T$ are also in the domain of f and $f(x) = f(x + T)$ for all x in the domain of f(x). Graph of a periodic function with period T is repeated after every interval of 'T'.

e.g. The function sin x and cos x both are periodic over 2π and tan x is periodic over π.

The least positive period is called the principal or fundamental period of f(x) or simply the period of the function.

❖ Inverse of a periodic function does not exist.

Properties of Periodic Functions

(A) If f(x) has a period T, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also have a period T.

(B) If f(x) has a period T, then f (ax + b) has a period $\frac{T}{|a|}$.

(C) Every constant function defined for all real x, is always periodic, with no fundamental period.

(D) If f (x) has a period T_1 and g (x) also has a period T_2 then period of $f(x) \pm g(x)$ or $f(x) . g(x)$ or $\frac{f(x)}{g(x)}$ is L.C.M. of T_1 and T_2 provided their L.C.M. exists. However that L.C.M. (if exists) need not to be fundamental period. If L.C.M. does not exists then $f(x) \pm g(x)$ or $f(x) . g(x)$ or $\frac{f(x)}{g(x)}$ is nonperiodic.

$$\text{L.C.M. of} = \left(\frac{a}{b}, \frac{p}{q}, \frac{\ell}{m}\right) = \frac{\text{L.C.M.}(a, p, \ell)}{\text{H.C.F.}(b, q, m)}$$

e.g. |sinx| has the period π, | cosx | also has the period π

$\therefore$ |sinx| + |cosx| also has a period π. But the fundamental period of |sinx| + |cosx| is $\frac{\pi}{2}$.

(E) If g is a function such that gof is defined on the domain of f and f is periodic with T, then gof is also periodic with T as one of its periods.

Ex. Find period of the following functions

(i) $f(x) = \sin\frac{x}{2} + \cos\frac{x}{3}$

(ii) $f(x) = \{x\} + \sin x$, where {.}denotes fractional part function

(iii) $f(x) = \cos x . \cos 3x$

(iv) $f(x) = \sin\frac{3x}{2} - \cos\frac{x}{3} - \tan\frac{2x}{3}$

Sol. (i) Period of $\sin\frac{x}{2}$ is 4π while period of $\cos\frac{x}{3}$ is 6π. Hence period of $\sin\frac{x}{2} + \cos\frac{x}{3}$ is 12π

{L.C.M. of 4 and 6 is 12}

(ii) Period of $\sin x = 2\pi$

Period of $\{x\} = 1$

but L.C.M. of 2π and 1 is not possible as their ratio is irrational number

$\therefore$ it is aperiodic

(iii) $f(x) = \cos x \cdot \cos 3x$ period of f(x) is L.C.M. of $\left(2\pi, \frac{2\pi}{3}\right) = 2\pi$

but 2π may or may not be fundamental periodic, but fundamental period $= \frac{2\pi}{n}$, where $n \in N$.

Hence cross-checking for n = 1, 2, 3, ...we find π to be fundamental period $f(\pi + x) = (-\cos x)(-\cos 3x) = f(x)$

(iv) Period of f(x) is L.C.M. of $\frac{2\pi}{3/2}, \frac{2\pi}{1/3}, \frac{\pi}{2/3}$ = L.C.M. of $\frac{4\pi}{3}, 6\pi, \frac{3\pi}{2} = 12\pi$

If x, y are independent variables, then:

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$.

(ii) $f(xy) = f(x).\, f(y) \Rightarrow f(x) = x^n, n \in R$

(iii) $f(x+y) = f(x).\, f(y) \Rightarrow f(x) = a^{kx}$.

(iv) $f(x+y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

Ex. If f(x) is a polynomial function satisfying $f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ $\forall\ x \in R - \{0\}$ and $f(2) = 9$, then find f(3)

Sol. $f(x) = 1 \pm x^n$

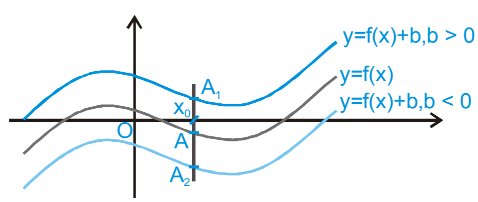
As $f(2) = 9$

$\therefore \quad f(x) = 1 + x^3$

Hence $f(3) = 1 + 3^3 = 28$

BASIC TRANSFORMATIONS ON GRAPHS

(i) Drawing the graph of $y = f(x) + b$, $b \in R$, from the known graph of $y = f(x)$



It is obvious that domain of f(x) and f(x) + b are the same. Let us take any point x_0 in the domain of f(x). $y|_{x=x_0} = f(x_0)$ The corresponding point on f(x) + b would be $f(x_0) + b$.

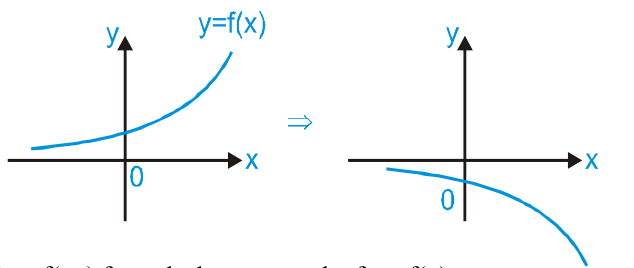
For $b > 0 \Rightarrow f(x_0) + b > f(x_0)$ it means that the corresponding point on f(x) + b would be lying at a distance 'b' units above the point on f(x).

For $b < 0 \Rightarrow f(x_0) + b < f(x_0)$ it means that the corresponding point on f(x) + b would be lying at a distance 'b' units below the point on f(x).

Accordingly the graph of f(x) + b can be obtained by translating the graph of f(x) either in the positive y-axis direction (if b > 0) or in the negative y-axis direction (if b < 0), through a distance |b| units.

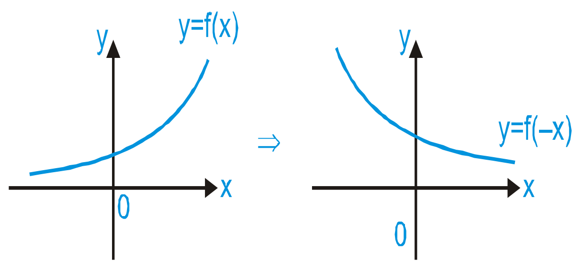
(ii) Drawing the graph of y = –f(x) from the known graph of y = f(x)

To draw y = –f(x), take the image of the curve y = f(x) in the x-axis as plane mirror.



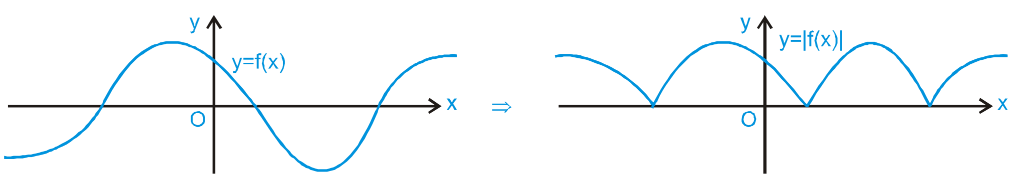
(iii) Drawing the graph of y = f(–x) from the known graph of y = f(x)

To draw y = f(–x), take the image of the curve y = f(x) in the y-axis as plane mirror.



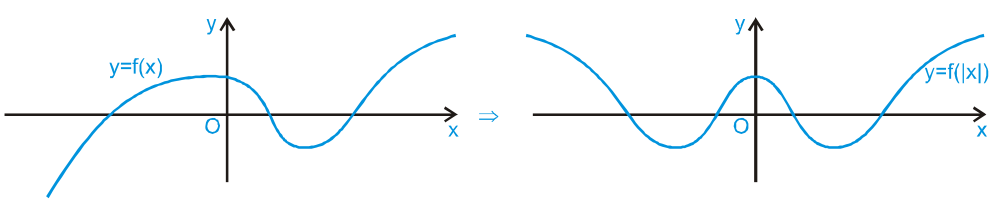
(iv) Drawing the graph of y = |f(x)| from the known graph of y = f(x)

$|f(x)| = f(x)$ if $f(x) \geq 0$ and $|f(x)| = -f(x)$ if $f(x) < 0$. It means that the graph of f(x) and |f(x)| would coincide if $f(x) \geq 0$ and for the portions where $f(x) < 0$ graph of |f(x)| would be image of y = f(x) in x-axis.



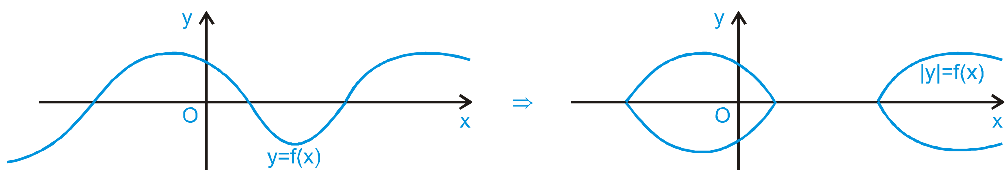
(v) Drawing the graph of y = f(|x|) from the known graph of y = f(x)

It is clear that, $f(|x|) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x < 0 \end{cases}$. Thus f(|x|) would be a even function, graph of f(|x|) and f(x) would be identical in the first and the fourth quadrants (ax $x \geq 0$) and as such the graph of f(|x|) would be symmetric about the y-axis (as (|x|) is even).

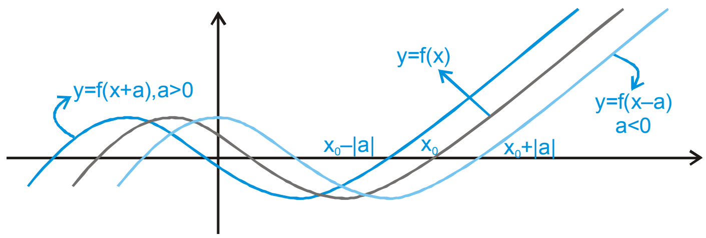


(vi) Drawing the graph of $|y| = f(x)$ from the known graph of $y = f(x)$

Clearly $|y| \geq 0$. If $f(x) < 0$, graph of $|y| = f(x)$ would not exist. And if $f(x) \geq 0$, $|y| = f(x)$ would give $y = \pm f(x)$. Hence graph of $|y| = f(x)$ would exist only in the regions where $f(x)$ is non-negative and will be reflected about the x-axis only in those regions.



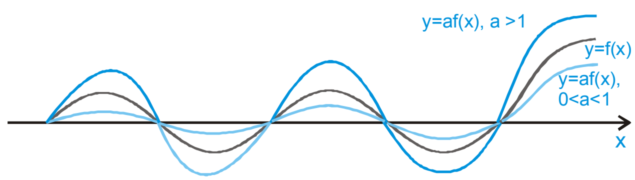
(vii) Drawing the graph of $y = f(x + a)$, $a \in R$ from the known graph of $y = f(x)$



(i) If $a > 0$, shift the graph of $f(x)$ through 'a' units towards left of $f(x)$.

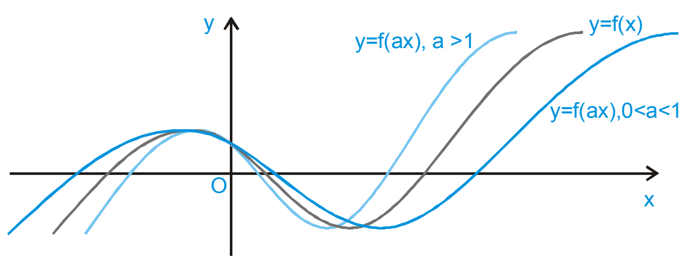
(ii) If $a < 0$, shift the graph of $f(x)$ through 'a' units towards right of $f(x)$.

(viii) Drawing the graph of $y = af(x)$ from the known graph of $y = f(x)$



It is clear that the corresponding points (points with same x co-ordinates) would have their ordinates in the ratio of 1 : a.

(ix) Drawing the graph of $y = f(ax)$ from the known graph of $y = f(x)$.

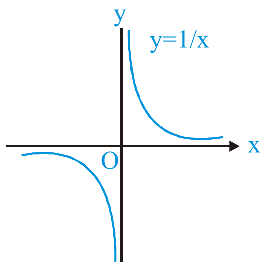


Let us take any point $x_0 \in$ domain of $f(x)$. Let $ax = x_0$ or $x = \frac{x_0}{a}$.

Clearly if $0 < a < 1$, then $x > x_0$ and $f(x)$ will stretch by $\frac{1}{a}$ units along the y-axis and if $a > 1$, $x < x_0$, then $f(x)$ will compress by 'a' units along the y-axis.

Ex. Draw the graph of $y = 2 - \dfrac{4}{|x-1|}$

Sol.



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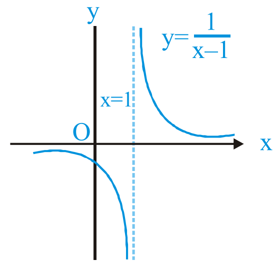

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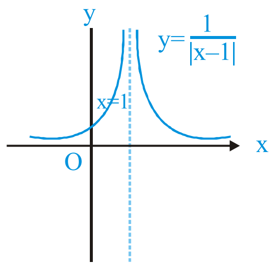

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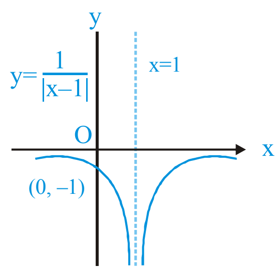

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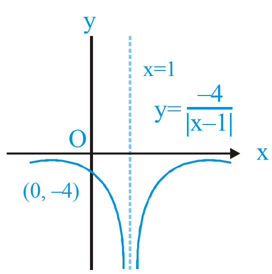
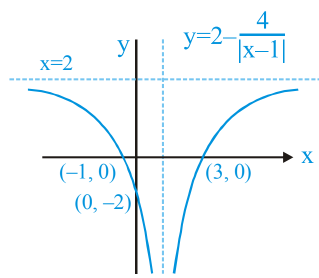
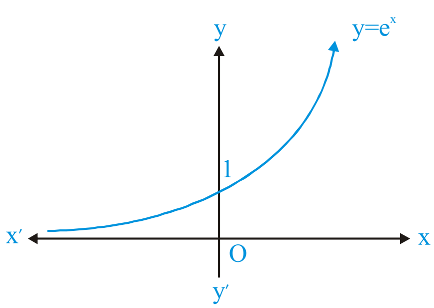




Ex. Draw the graph of $y = |e^{|x|} - 2|$

Sol.



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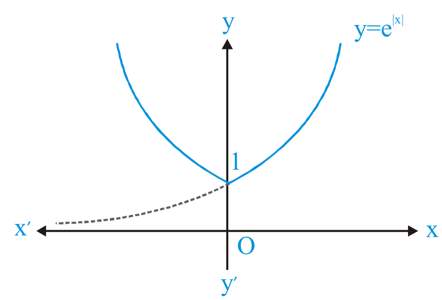

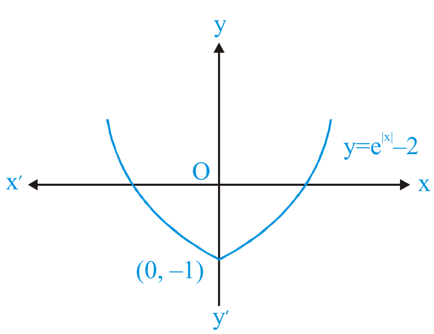

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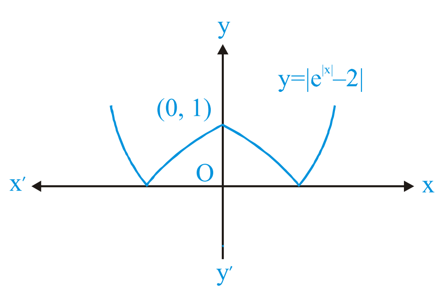


00TIPS & FORMULAS

1. Definition

If to every value (considered as real unless other-wise stated) of a variable x, which belongs to a set A, there corresponds one and only one finite value of the quantity y which belong to set B, then y is said to be a function of x and written as $f : A \to B$, $y = f(x)$, x is called argument or independent variable and y is called dependent variable.

Pictorially : $\xrightarrow[\text{input}]{x} \boxed{f} \xrightarrow[\text{output}]{f(x) = y}$

y is called the image of x & x is the pre-image of y, under f. Every function $f : A \to B$ satisfies the following conditions.

(i) $f \subset A \times B$ **(ii)** $\forall a \in A \quad \exists\ b \in B$ such that $(a,b) \in f$ and

(iii) If $(a,b) \in f$ & $(a,c) \in f \Rightarrow b = c$

2. Domain, Co-Domain & Range of a Function

Let $f : A \to B$, then the set A is known as the domain of 'f' & the set B is known as co-domain of 'f'. The set of all f images of elements of A is known as the range of 'f'. Thus

Domain of $f = \{ x \mid x \in A, (x, f(x)) \in f\}$

Range of $f = \{ f(x) \mid x \in A, f(x) \in B\}$

range is a subset of co-domain.

3. Important Types of Function

(A) Polynomial function :

If a function 'f' is called by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, ... a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

Note

(I) A polynomial of degree one with no constant term is called an odd linear function. i.e. $f(x) = ax, \neq 0$.

(ii) There are two polynomial functions, satisfying the relation ; f(x), f(1/x). They are :

(A) $f(x) = x^n + 1$ &

(B) $f(x) = 1 - x^n$, where n is a positive integer.

(iii) Domain of a polynofunction is R

(iv) Range of odd degree polynomial is R whereas range of an even degree polynomial is never R.

(B) Algebric function :

A function 'f' is called an algebric function if it can be constructed using algebric operations (such as addition, subtraction, multiplication, division and taking radicals) straight with polynomials

(C) Rational function :

A rational function is a function of the form $y = f(x) = \dfrac{g(x)}{h(x)}$, where g(x) & h(x) are polynomials & $h(x) \neq 0$,

Domain : $R - \{x \mid h(x) = 0\}$

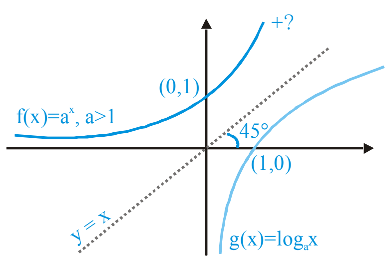
Any rational function is automatically an algebric function.

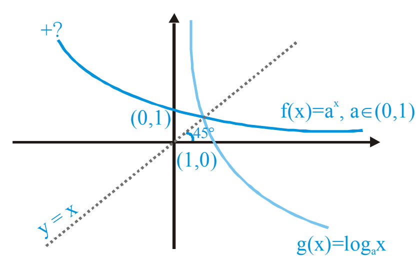
(D) Exponential and Logarithmic Function :

A function $f(x) = a^x$ $(a > 0)$, $a \neq 1$, $x \in R$ is called an exponential function. The inverse of the exponential function is called the logarithmic function, i.e. $g(x) = \log_a x$. Note that $f(x)$ & $g(x)$ are inverse of each other & their graphs are as shown. (Functions are mirror image of each other about the line $y = x$)

Domain of a^x is R **Range** R^+

Domain of $\log_a x$ is R^+ **Range** R



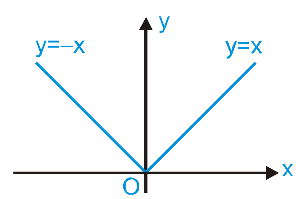


(E) Absolute value function

It is defined as : $y = |x|$

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Also defined as max $\{x, -x\}$



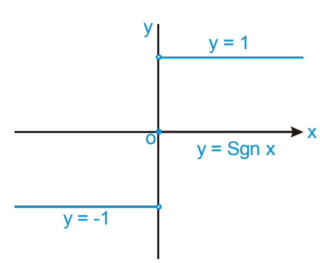
Domain : R
Range : $[0, \infty)$

Note : $f(x) = \dfrac{1}{|x|}$, **Domain :** $R - \{0\}$, **Range :** R^+

(F) Signum function

Signum function $y = \text{sgn}(x)$ is defined as follows

$$y = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$



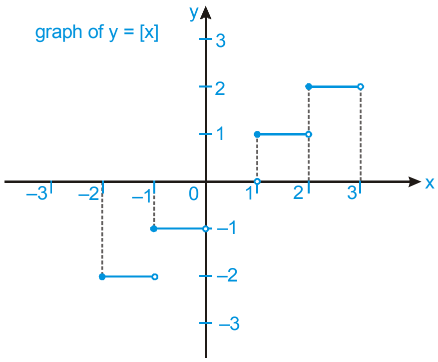
Domain : R

Range : $\{-1, 0, 1\}$

(G) Greatest integer or step up function

The function $y = f(x) = [x]$ is called the greatest integer function where $[x]$ denotes the greatest integer less than or equal to x. Note that for :

x	[x]
[–2,–1)	–2
[–1,0)	–1
[0,1)	0
[1,2)	1



Domain : R

Range : I

Properties of greatest integer function :

(i) $[x] \le x < [x] + 1$ and $x - 1 < [x] \le x,\ 0 \le x - [x] < 1$

(ii) $[x + m] = [x] + m$ if m is an integer.

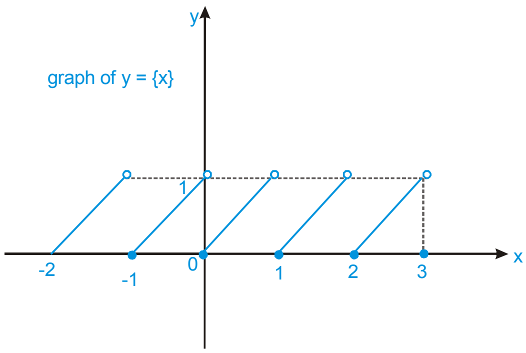
(iii) $[x] + [-x] = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$

Note : $f(x) = \dfrac{1}{[x]}$ **Domain :** $R - [0, 1)$ **Range :** $\{x \mid x = \dfrac{1}{n},\ n \in I_0\}$

(H) Fractional part function :

It is defined as : $g(x) = \{x\} = x - [x]$ e.g.

x	{x}
[–2,–1)	x+2
[–1,0)	x+1
[0,1)	x
[1,2)	x–1

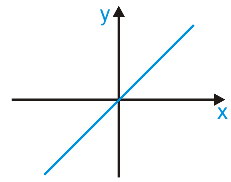


Domain : R **Range :** [0,1) **Period :** 1

The fractional part of the number 2.1 is 2.1–2 = 0.1 and the fractional part of –3.7 is 0.3 The period of this function is 1 and graph of this function is as shown.

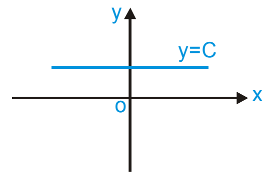
Note : $f(x) = \dfrac{1}{\{x\}}$ **Domain :** R – I **Range :** $(1, \infty)$

(I) Identity function :



The function $f : A \to A$ defined by $f(x) = x \ \forall\ x \in A$ is called the identity function on A and is denoted by I_A.

(J) Constant function :



$f : A \to B$ is said to be constant function if every element of A has the same f image in B. Thus $f : A \to B$; $f(x) = c,\ \forall\ x \in A,\ c \in B$ is constant function.

Domain : R **Range :** {C}

(K) Trigonometric functions :

(i) Sine function

$f(x) = \sin x$

Domain : R **Range :** $[-1, 1]$, period 2π

(ii) Cosine function

$f(x) = \cos x$

Domain : R **Range :** $[-1, 1]$, period 2π

(iii) Tangent function

$f(x) = \tan x$

Domain : $R - \left\{x \mid x = \frac{(2n+1)\pi}{2}, n \in I\right\}$ **Range :** R , period π

(iv) Cosecant function

$f(x) = \text{cosec } x$

Domain : $R - \{x \mid x = n\pi,\ n \in I\}$ **Range :** $R - (-1, 1)$, period 2π

(v) Secant function

$f(x) = \sec x$

Domain : $R - \{x \mid x = (2n + 1)\ \pi/2 : n \in I\}$ **Range :** $R - (-1, 1)$, period 2π

(vi) Cotangent function

$f(x) = \cot x$

Domain : $R - \{x \mid x = n\pi,\ n \in I\}$ **Range :** R, period π

(L) Inverse Trignometric function :

(I)	$f(x) = \sin^{-1} x$	**Domain :** $[-1, 1]$	**Range :** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(ii)	$f(x) = \cos^{-1} x$	**Domain :** $[-1, 1]$	**Range :** $[0, \pi]$
(iii)	$f(x) = \tan^{-1} x$	**Domain :** R	**Range :** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(iv)	$f(x) = \cot^{-1} x$	**Domain :** R	**Range :** $[0, \pi]$
(v)	$f(x) = \text{cosec}^{-1} x$	**Domain :** $R - (-1, 1)$	**Range :** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
(vi)	$f(x) = \sec^{-1} x$	**Domain :** $R - (-1, 1)$	**Range :** $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

4. Equal or Identical Function

Two function f & g are said to be equal if :

(A) The domain of f = the domain of g

(B) The range of f = range of g and

(C) $f(x) = g(x)$, for every x belonging to their common domain (i.e. should have the same graph)

5. Algebraic Operations on Functions

If f & g are real valued functions of x with domain set A, B respectively, f + g, f – g, (f. g) & (f/g) as follows :

(A) $(f \pm g)(x) = f(x) \pm g(x)$ domain in each case is $A \cap B$

(B) $(f.g)(x) = f(x).g(x)$ domain is $A \cap B$

(C) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain $A \cap B - \{x|g(x) = 0\}$

6. Classification of Functions

(A) One-One function (Injective mapping) :

A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B.

Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Note **(i)** Any continuous function which is entirely increasing or decreasing in whole domain is one-one.

(ii) If a function is one-one, any line parallel to x-axis cuts the graph of the function at atmost one point

(B) Many-one function :

A function $f : A \to B$ is said to be a many one function if two or more elements of A have the same f image in B.

Thus $f : A \to B$ is many one if $\exists\ x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$

Note : If a continuous function has local maximum or local minimum, then f(x) is many-one because atleast one line parallel to x-axis will intersect the graph of function atleast twice.

(C) Onto function (Surjective mapping) :

If range = co-domain, then f(x) is onto.

(D) Into function :

If $f : A \to B$ is such that there exists atleast one element in co-domain which is not the image of any element in domain, then f(x) is into.

Note :

(i) If 'f' is both injective & surjective, then it is called a **Bijective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains n distinct elements then the number of different functions defined from $A \to A$ is n^n & out of it n! are one one and rest are many one.

(iii) $f : R \to R$ is a polynomial

(A) Of even degree, then it will neither be injective nor surjective.

(B) Of odd degree, then it will always be surjective, no general comment can be given on its injectivity.

7. Composite of Uniformly & Non-Uniformly Defined Function

Let $f : A \to B$ & $g : B \to C$ be two functions. Then the function $gof : A \to C$ defined by $(gof)(x) = g(f(x))\ \forall\ x \in A$ is called the composite of the two functions f & g.

Hence in gof(x) the range of 'f' must be a subset of the domain of 'g'.

Properties of composite functions :

(A) In general composite of functions is not commutative i.e. $gof \neq fog$.

(B) The composite of functions is associative i.e. if f, g, h are three functions such that fo(goh) & (fog)oh are defined, then fo(goh) = (fog)oh.

(C) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

(D) If gof is one-one function then f is one-one but g may not be one-one.

8. Homogeneous Functions

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For examples $5x^2 + 3y^2 - xy$ is homogenous in x & y. Symbolically if, $f(tx, ty) = t^n f(x, y)$ then f(x, y) is homogeneous function of degree n.

9. Bounded Function

A function is said to be bounded if $|f(x)| \leq M$, where M is a finite quantity.

10. Implicit & Explicit Function

A function defined by an equation not solved for the dependent variable is called an **implicit function**. e.g. the equations $x^3 + y^3 = 1$ & $x^y = y^x$, defines y as an implicit function. If y has been expressed in terms of x alone then it is called an **Explicit function.**

11. Inverse of a Function

Let $f : A \to B$ be a one-one & onto function, then their exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x,\ \forall\ x \in A\ \&\ y \in B$. Then g is said to be inverse of f.

Thus $g = f^{-1} : B \to A = \{(f(x), x)) | (x, f(x)) \in f\}$.

Properties of inverse function :

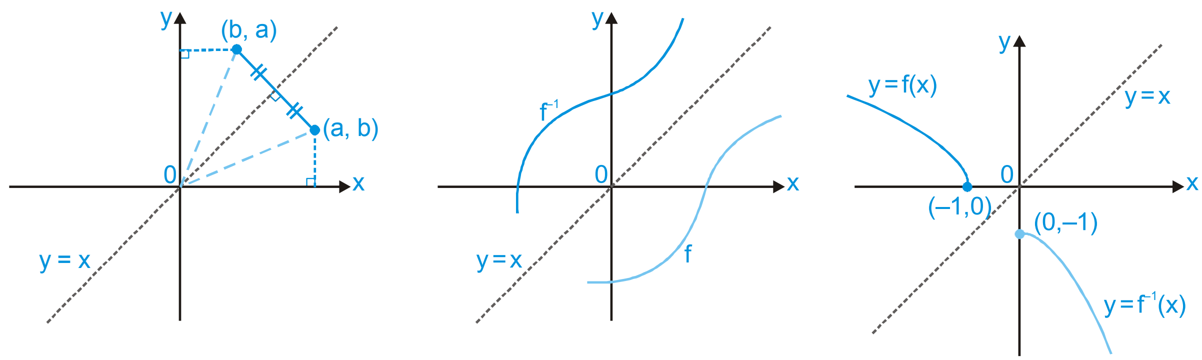
(A) The inverse of a bijection is unique.

(B) If $f : A \to B$ is a bijection & $g : B \to A$ is the inverse of f, then fog = I_B and gof = I_A, where I_A & I_B are identity functions on the sets A & B respectively. If fof = I, then f is inverse of itself.

(C) The inverse of a bijection is also a bijection.

(D) If f & g are two bijections $f : A \to B$, $g : B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} og^{-1}$.

(E) Since f**(A)** = b if and only if f^{-1}**(B)** = a, the point (a, b) is on the graph of 'f' if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line y = x.



$\boxed{\text{The graph of } f^{-1} \text{ is obtained by reflecting the graph of f about the line } y = x.}$

12. Odd & Even Functions

If a function is such that whenever 'x' is in it's domain '–x' is also in it's domain & it satisfies

f(–x) = f(x) it is an even function

f(–x) = –f(x) it is an odd function

Note

(i) A function may neither be odd nor even.

(ii) Inverse of an even function is not defined, as it is many – one function.

(iii) Every even function is symmetric about the y-axis & every odd function is symmetric about the origin.

(iv) Every function which has '–x' in it's domain whenever 'x' is in it's domain, can be expressed as the sum of an even & an odd function .

$$\text{e.g. } f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{ODD}}$$

(v) The only function which is defined on the entire number line & even and odd at the same time is f(x) = 0

(vi) If f(x) and g(x) both are even or both are odd then the function f(x) . g(x) will be even but if any one of them is odd & other is even, then f.g will be odd.

16. Periodic Function

A function f(x) is called periodic if there exists a least positive number T(T >0) called the period of the function such that f(x + T) = f(x), for all values of x within the domain of f(x).

Note :

(i) Inverse of a periodic function does not exist.

(ii) Every constant function is periodic, with no fundamental period.

(iii) If f(x) has a period T & g(x) also has a period T then it does not mean that f(x) + g(x) must have a period T. e.g. f(x) = |sin x| + |cos x|.

(iv) If f(x) has period p and g(x) has period q, then period of f(x) + g(x) will be LCM of p & q provided f (x) & g(x) are not interchangeable. If f(x) & g(x) can be interchanged by adding a least positive number r, then smaller of LCM & r will be the period.

(v) If f(x) has period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.

(vi) If f(x) has period T then f(ax + b) has a period T/a (a > 0).

(vii) |sinx|, |cosx|, |tanx|, |cotx|, |secx| & |cosecx| are periodic function with period π.

(viii) $\sin^n x$, $\cos^n x$, $\sec^n x$, $\text{cosec}^n x$, are periodic function with period 2π when 'n' is odd or π when n is even.

(ix) $\tan^n x$, $\cos^n x$ are periodic function with period π.

14. General

If x, y are independent variables, then :

(A) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k\ell n\ x$

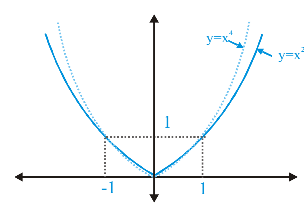
(B) $f(xy) = f(x)\ .\ f(y) \Rightarrow f(x) = x^n,\ n \in R$ or $f(x) = 0$

(C) $f(x + y) = f(x)\ .\ f(y) \Rightarrow f(x) = a^{kx}$ or $f(x) = 0$

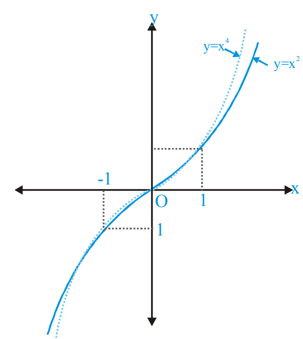
(D) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

15. Some Basic Function & their Graph

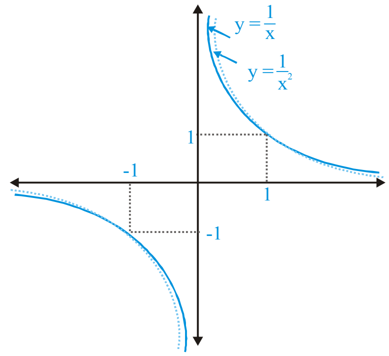
(A) $y = x^{2n}$, where $n \in N$



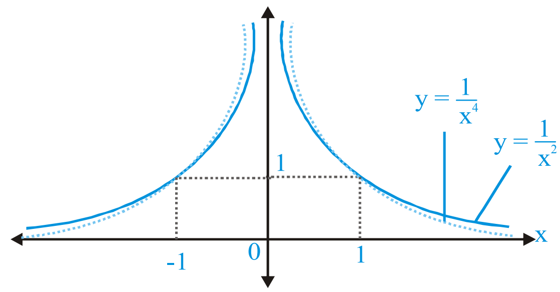
(B) $y = x^{2n+1}$, where $n \in N$



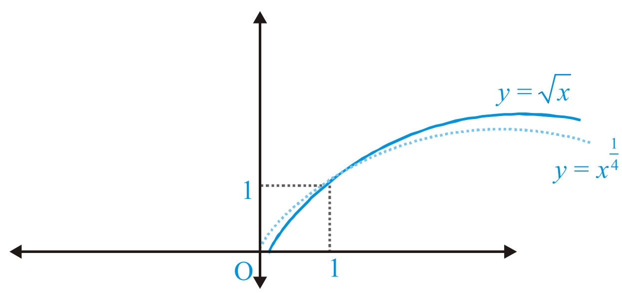
(C) $y = \frac{1}{x^{2n-1}}$, where $n \in N$



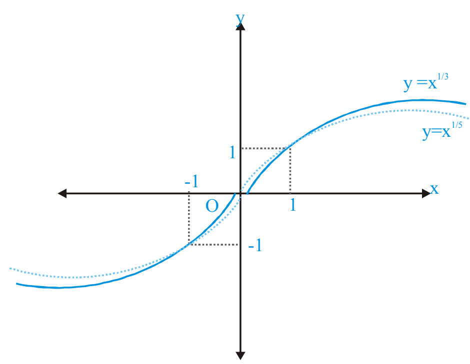
(D) $y = \frac{1}{x^{2n}}$, where $n \in N$



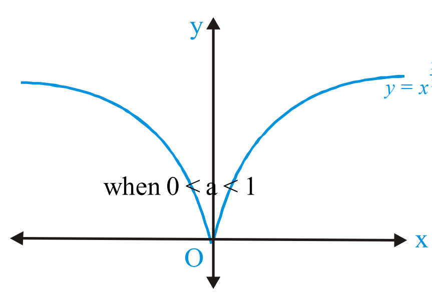
(E) $y = x^{\frac{1}{2n}}$, where $n \in N$



(F) $y = x^{\frac{1}{2n+1}}$, where $n \in N$



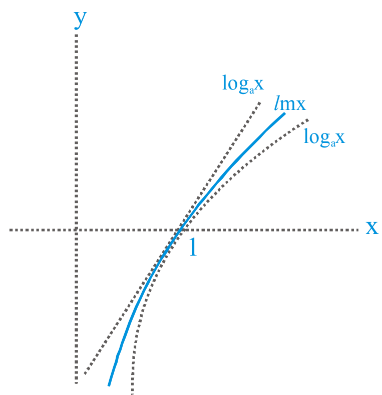
Note : $y = x^{2/3}$

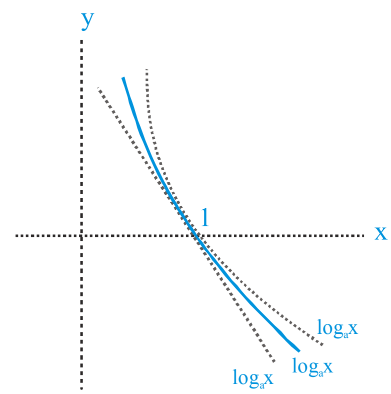


(G) $y = \log_a x$

when $a > 1$

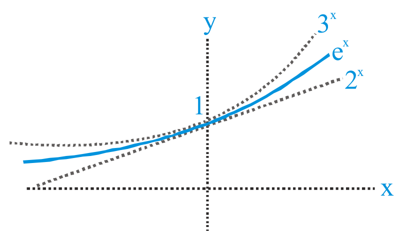
when $0 < a < 1$

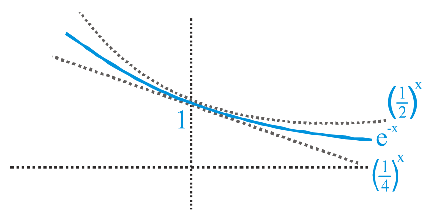




(H) $y = a^x$

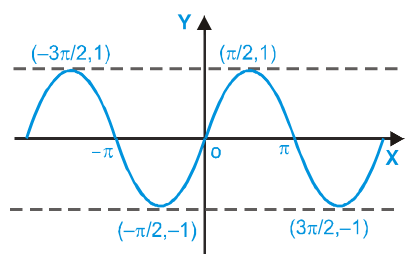
$a > 1$ $\qquad\qquad$ $0 < a < 1$



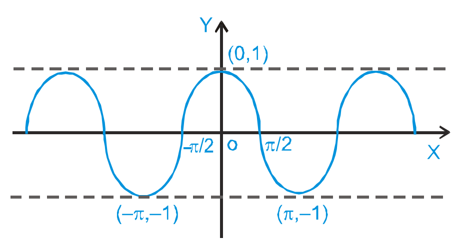


(I) **Trigonometric functions**

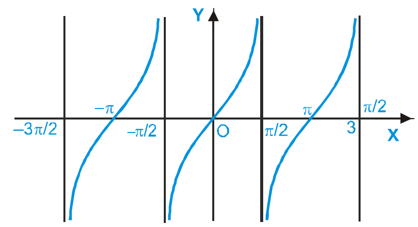
$y = \sin x$



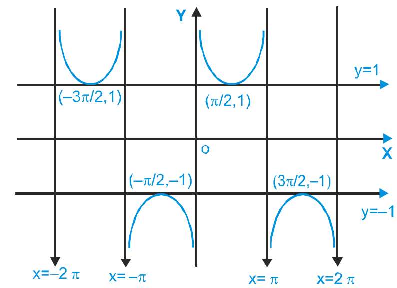
$y = \cos x$



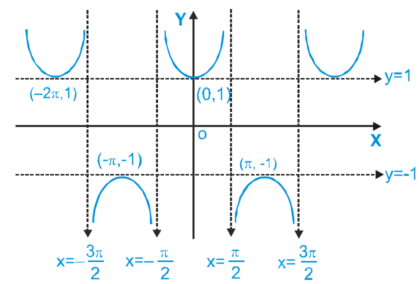
$y = \tan x$



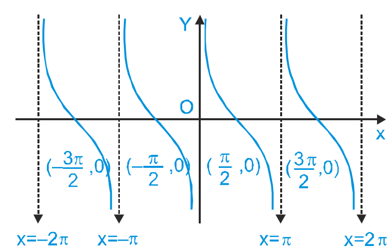
$y = \operatorname{cosec} x$



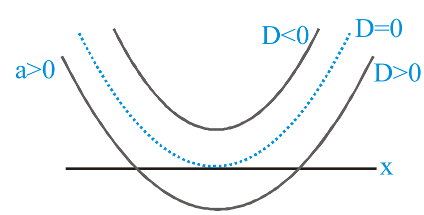
$y = \sec x$



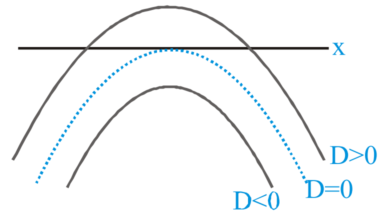
$y = \cot x$



(J) $y = ax^2 + bx + c$



vertex $\left(-\dfrac{b}{2a}, -\dfrac{D}{4a}\right)$



where $D = b^2 - 4ac$

16. Transformation of Graph

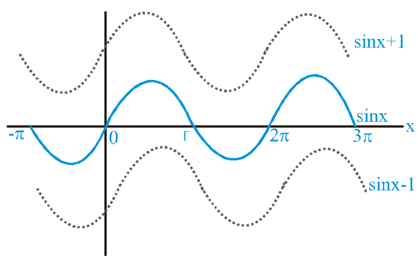
(A) when f (x) transforms to f(x) + k

if k > 0 then shift graph of f(x) upward through k

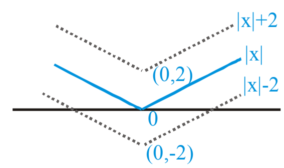
if k < 0 then shift graph of f(x) downward through k

Examples :

1.



2.



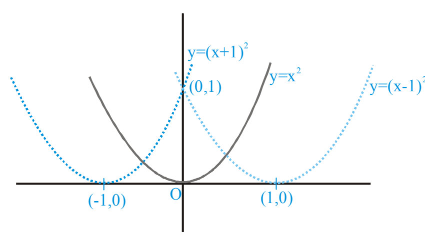
(B) f(x) transforms to f (x + k) :

if k > 0 then shift graph of f(x) through k towards left.

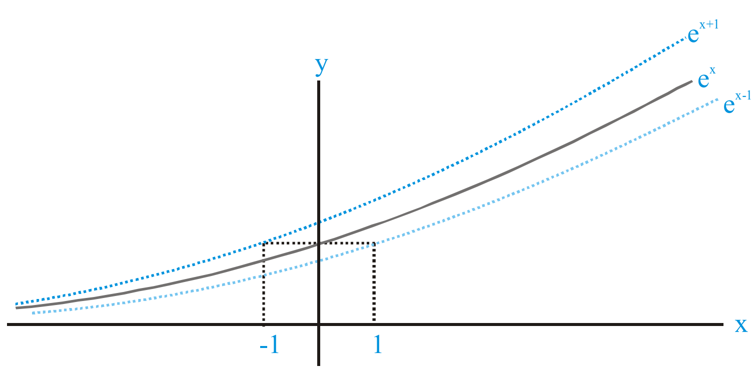
if k < 0 then shift graph of f (x) through k towards right.

Examples :

1.



2.

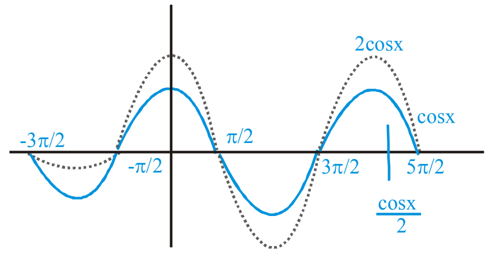


(C) f (x) transforms to kf (x)

if k > 1 then strech graph of k (x) k times along y-axis

if 0 < k < 1 then shrink graph of f(x), k times along y-axis

Examples :

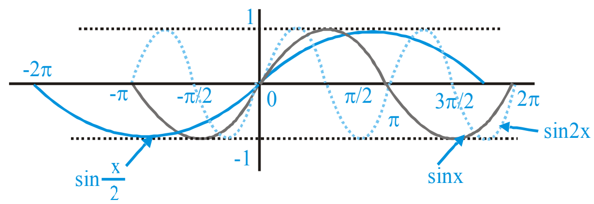


(D) f(x) transforms to f (kx) :

if k > 1 then shrink graph of f (x), 'k' times along x-axis.

if 0 < k < 1 then strech graph of f(x), 'k' times along x-axis.

Examples :

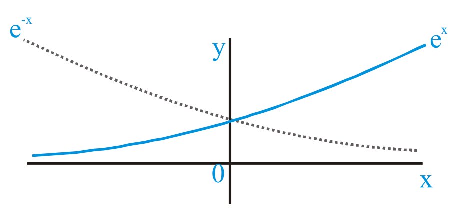


(E) f (x) transforms to f (–x) :

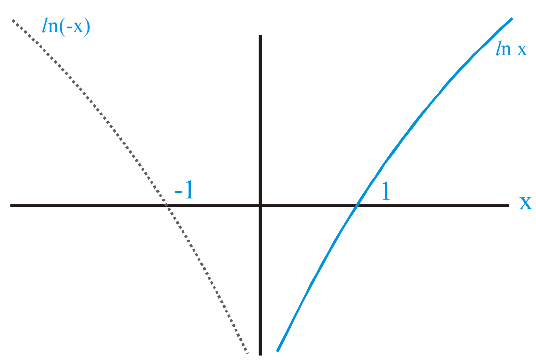
Take mirror image of the curve y - f(x) in y-axis as plane mirror.

Examples :

1.



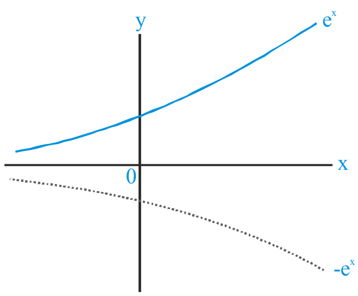
2.

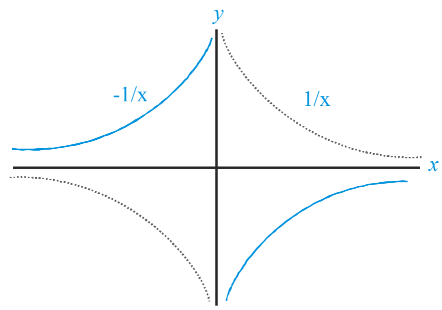


(F) f (x) transforms to – f (x) :

Take image of y = f(x) in the x axis as plane mirror.

Examples :

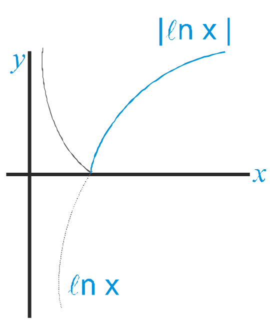


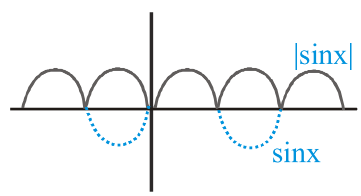


(G) f (x) tansforms to |f (x) | :

Take mirror image (in a axis) of the portion of the graph of f(x) which lies below x-axis.

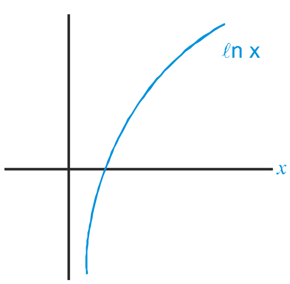
Examples :

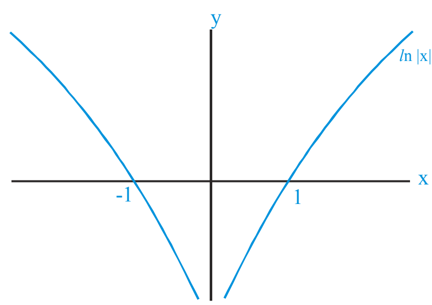


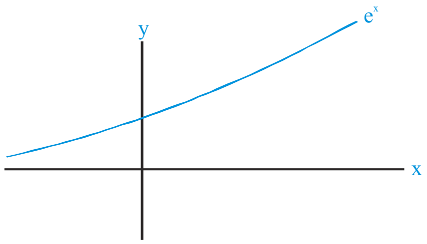


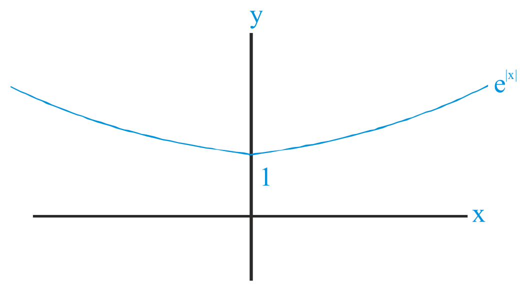
(H) f (x) transforms to f (|x|) :

Neglect the curve for x < 0 and take the image of curve for ≥ 0 about y-axis.









(l) y = f (x) transforms to |y| = f (x) :

Remove the portion of graph which lies below x-axis & then take mirror image (in x-axis) of remaining portion of graph.

Examples :