Definite Integrals

Let $f(x)$ be a function defined on the interval $[a, b]$ and $F(x)$ be its anti-derivative. Then, $\int_a^b f(x)\,dx = F(b) - F(a)$ is defined as the **definite integral** of $f(x)$ from $x = a$ to $x = b$.

The numbers a and b are called upper and lower limits of integration, respectively.

Fundamental Theorem of Calculus

There is a connection between indefinite integral and definite integral is known as fundamental theorem of calculus.

First Fundamental Theorem

Let f be a continuous function defined on the closed interval $[a, b]$ and let $A(x)$ be the area of function i.e. $A(x) = \int_a^x f(x)dx$. Then, $A'(x) = f(x)$ for all $x \in [a, b]$.

Second Fundamental Theorem

Let f be a continuous function defined on the closed integral $[a, b]$ and F be an anti-derivative of f. Then,

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a).$$

Evaluation of Definite Integrals by Substitution

Consider a definite integral of the following form

$$\int_a^b f[g(x)] \cdot g'(x)\,dx$$

***Step* I** Substitute $g(x) = t \Rightarrow g'(x)\,dx = dt$

***Step* II** Find the limits of integration in new system of variable *i.e.*, the lower limit is $g(a)$ and the upper limit is $g(b)$ and the new integral will be $\int_{g(a)}^{g(b)} f(t)\,dt$.

***Step* III** Evaluate the integral, so obtained by usual method.

Properties of Definite Integral

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^a f(x)\,dx = 0$

4. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$

   Generalisation

   If $a < c_1 < c_2 < \ldots < c_{n-1} < c_n < b$, then

   $$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \ldots + \int_{c_{n-1}}^{c_n} f(x)\,dx + \int_{c_n}^{b} f(x)\,dx$$

5. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

   Deduction $\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \frac{a}{2}$

6. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

   Deduction $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$

7. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

8. $\int_{-a}^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(-x)\,dx$

9. $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if, } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

10. $\int_a^b f(x)\,dx = \begin{cases} 0, & \text{if } f(a+x) = -f(b-x) \\ 2\int_a^{\frac{a+b}{2}} f(x)\,dx, & \text{if } f(a+x) = f(b-x) \end{cases}$

11. $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even i.e. } f(-x) = f(x) \\ 0, & \text{if } f(x) \text{ is odd i.e. } f(-x) = -f(x) \end{cases}$

12. If $\int_a^b f(x)\,dx = (b-a)\int_0^1 f\,[(b-a)x+a]\,dx$

13. If $f(x)$ is periodic function with period T [i.e. $f(x+T) = f(x)$]. Then, $\int_a^{a+T} f(x)\,dx$ is independent of a.

(a) $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx, n \in I$

(b) $\int_a^{a+nT} f(x)\,dx = n\int_0^T f(x)\,dx, n \in I$

(c) $\int_{a+mT}^{a+nT} f(x)dx = \int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^T f(x)\,dx, m, n \in I$

(d) $\int_{a+mT}^{b+mT} f(x)\,dx = \int_a^b f(x)\,dx, n \in I$

(e) $\int_{nT}^{a+nT} f(x)\,dx = \int_0^a f(x)\,dx$

14. **Leibnitz Rule for Differentiation under Integral Sign**

If $\phi(x)$ and $\psi(x)$ are defined on $[a, b]$ and differentiable at point $x \in (a, b)$ and $f(t)$ is continuous, then

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)\,dt\right] = f[\psi(x)]\cdot\frac{d}{dx}\psi(x) - f\,[\phi(x)]\cdot\frac{d}{dx}\phi(x).$$

15. If $f(x) \geq 0$ on the interval $[a, b]$, then $\int_a^b f(x) \geq 0$.

16. If $f(x) \leq \phi(x)$ for $x \in [a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b \phi(x)\,dx$.

17. If at every point x of an interval $[a, b]$ the inequalities

$$g(x) \leq f(x) \leq h(x)$$

are fulfilled, then

$$\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b h(x)\,dx.$$

18. $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$

19. If m is the least value and M is the greatest value of the function $f(x)$ on the interval $[a, b]$ (estimation of an integral), then

$$m\,(b-a) \leq \int_a^b f(x)\,dx \leq M\,(b-a).$$

20. If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ at which

$$f(c) = \frac{1}{(b-a)}\int_a^b f(x)\,dx$$

is called the mean value of the function $f(x)$ on the interval $[a, b]$.

21. If $f^2(x)$ and $g^2(x)$ are integrable on $[a, b]$, then

$$\left|\int_a^b f(x)g(x)\,dx\right| \le \left(\int_a^b f^2(x)dx\right)^{1/2}\left(\int_a^b g^2(x)dx\right)^{1/2}$$

22. If $f(t)$ is an odd function, then $\phi(x) = \int_a^x f(t)\,dt$ is an even function.

23. If $f(t)$ is an even function, then $\phi(x) = \int_0^x f(t)dt$ is an odd function.

24. If $f(t)$ is an even function, then for non-zero a, $\int_a^x f(t)\,dt$ is not necessarily an odd function. It will be an odd function, if $\int_0^a f(t)\,dt = 0$.

25. If $f(x)$ is continuous on $[a, \infty)$, then $\int_a^\infty f(x)\,dx$ is called an improper integral and is defined as $\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$.

26. $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$ and

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx$$

27. Geometrically, for $f(x) > 0$, the improper integral $\int_a^\infty f(x)\,dx$ gives area of the figure bounded by the curve $y = f(x)$, the axis and the straight line $x = a$.

Integral Function

Let $f(x)$ be a continuous function defined on $[a, b]$, then a function $\phi(x)$ defined by $\phi(x) = \int_a^x f(t)\,dt, x \in [a, b]$ is called the integral function of the function f.

Properties of Integral Function

(i) The integral function of an integrable function is continuous.

(ii) If $\phi(x)$ is the integral function of continuous function, then $\phi(x)$ is derivable and $\phi'(x) = f(x), \forall\, x \in [a, b]$.

Walli's Formula

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx$$

$$= \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \dots \dfrac{2}{3}, \text{ when } n \text{ is odd.} \\ \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \dots \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2}, \text{ when } n \text{ is even.} \end{cases}$$

Some Important Deduction

(v) $\int_0^{\pi/2} \sin^m x \cos^n x\, dx$

$$= \frac{[(m-1)(m-3)\dots 2 \text{ or } 1]\,[(n-1)(n-3)\dots 2 \text{ or } 1]}{[(m+n)(m+n-2)\dots 2 \text{ or } 1]}$$

On multiplying the above by $\frac{\pi}{2}$, when both m and n are even.

(a) $\int_0^{\pi/2} \sin^6 x \cos^3 x\, dx = \dfrac{(5 \cdot 3 \cdot 1)(2)}{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1} = \dfrac{2}{63}$

(b) $\int_0^{\pi/2} \sin^8 x \cos^2 x\, dx = \dfrac{(7 \cdot 5 \cdot 3 \cdot 1)}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \dfrac{\pi}{2} = \dfrac{7\pi}{512}$

(vi) Particular case when m or $n = 1$

(a) $\int_0^{\pi/2} \sin^m x \cos x\, dx = \left[\dfrac{\sin^{m+1} x}{m+1}\right]_0^{\pi/2} = \dfrac{1}{m+1}$

(b) $\int_0^{\pi/2} \cos^m x \sin x\, dx = \left[\dfrac{-\cos^{m+1} x}{m+1}\right]_0^{\pi/2} = \dfrac{1}{m+1}$

Summation of Series by Definite Integral

Let $f(x)$ be a continuous function in $[a, b]$ and h be the length of n equal subintervals, then

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} h \sum_{r=0}^{n} f(a + rh)$$

where, $nh = b - a$

Now, put $a = 0, b = 1$

$\therefore$ $nh = 1 - 0 = 1$ or $h = \dfrac{1}{n}$

$\therefore$ $\int_0^1 f(x)\, dx = \lim_{n \to \infty} \dfrac{1}{n} \sum_{r=0}^{n-1} f\left(\dfrac{r}{n}\right)$

Method Express the given series in the form of

$$\lim_{n \to \infty} \sum \frac{1}{n} f\left(\frac{r}{n}\right)$$

Replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx and the limit of the sum is $\int_0^1 f(x)\, dx$.

Note $\lim_{n\to\infty} \sum_{r=1}^{pn} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_\alpha^\beta f(x) dx$

where, $\alpha = \lim_{n\to\infty} \frac{r}{n} = 0$ (as $r = 1$)

and $\beta = \lim_{n\to\infty} \frac{r}{n} = p$ (as $r = pn$)

The method to evaluate the integral, as limit of the sum of an infinite series is known as integration by first principle.

Some Important Results

(i) (a) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x} dx$

(b) $\int_0^{\pi/2} \frac{\tan^n x}{1 + \tan^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{dx}{1 + \tan^n x}$

(c) $\int_0^{\pi/2} \frac{dx}{1 + \cot^n x} = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cot^n x}{1 + \cot^n x} dx$

(d) $\int_0^{\pi/2} \frac{\tan^n x}{\tan^n x + \cot^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\cot^n x}{\tan^n x + \cot^n x} dx$

(e) $\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x + \operatorname{cosec}^n x} dx = \frac{\pi}{4} = \int_0^{\pi/2} \frac{\operatorname{cosec}^n x}{\sec^n x + \operatorname{cosec}^n x} dx$ where, $n \in R$

(ii) $\int_0^{\pi/2} \frac{a^{\sin^n x}}{a^{\sin^n x} + a^{\cos^n x}} dx = \int_0^{\pi/2} \frac{a^{\cos^n x}}{a^{\sin^n x} + a^{\cos^n x}} dx = \frac{\pi}{4}$

(iii) (a) $\int_0^{\pi/2} \log \sin x\, dx = \int_0^{\pi/2} \log \cos x\, dx = -\frac{\pi}{2} \log 2$

(b) $\int_0^{\pi/2} \log \tan x\, dx = \int_0^{\pi/2} \log \cot x\, dx = 0$

(c) $\int_0^{\pi/2} \log \sec x\, dx = \int_0^{\pi/2} \log \operatorname{cosec} x\, dx = \frac{\pi}{2} \log 2$

(iv) (a) $\int_0^{\infty} e^{-ax} \sin bx\, dx = \frac{b}{a^2 + b^2}$ (b) $\int_0^{\infty} e^{-ax} \cos bx\, dx = \frac{a}{a^2 + b^2}$

(c) $\int_0^{\infty} e^{-ax} x^n\, dx = \frac{n!}{a^n + 1}$