Derivatives

Derivative or Differential Coefficient

The rate of change of a quantity y with respect to another quantity x is called the **derivative** or differential coefficient of y with respect to x.

Differentiation

The process of finding derivative of a function is called differentiation.

Differentiation using First Principle

Let $f(x)$ is a function, differentiable at every point on the real number line, then its derivative is given by

$$f'(x) = \frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Derivatives of Standard Functions

(i) $\frac{d}{dx}(x^n) = nx^{n-1},\ n \in R$

(ii) $\frac{d}{dx}(k) = 0$, where k is constant.

(iii) $\frac{d}{dx}(e^x) = e^x$

(iv) $\frac{d}{dx}(a^x) = a^x \log_e a$, where $a > 0, a \neq 1$

(v) $\frac{d}{dx}(\log_e x) = \frac{1}{x},\ x > 0$

(vi) $\frac{d}{dx}(\log_a x) = \frac{1}{x}(\log_a e) = \frac{1}{x \log_e a},\ x > 0$

(vii) $\frac{d}{dx}(\sin x) = \cos x$

(viii) $\frac{d}{dx}(\cos x) = -\sin x$

(ix) $\frac{d}{dx}(\tan x) = \sec^2 x, x \neq (2n+1)\frac{\pi}{2}, n \in I$

(x) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x, x \neq n\pi, n \in I$

(xi) $\frac{d}{dx}(\sec x) = \sec x \tan x, x \neq (2n+1)\frac{\pi}{2}, n \in I$

(xii) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x, x \neq n\pi, n \in I$

(xiii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

(xiv) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

(xv) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(xvi) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

(xvii) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

(xviii) $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

(xix) $\frac{d}{dx}(\sinh x) = \cos h\ x$

(xx) $\frac{d}{dx}(\cosh x) = \sin h\ x$

(xxi) $\frac{d}{dx}(\tanh x) = \text{sec}h^2 x$

(xxii) $\frac{d}{dx}(\coth x) = -\text{cosec}h^2 x$

(xxiii) $\frac{d}{dx}(\text{sech } x) = -\sec h\ x \tan h\ x$

(xxiv) $\frac{d}{dx}(\text{cosech } x) = -\text{cosec}h\ x \cot h\ x$

(xxv) $\frac{d}{dx}(\sinh^{-1} x) = 1/\sqrt{(x^2+1)}$

(xxvi) $\frac{d}{dx}(\cosh^{-1} x) = 1/\sqrt{(x^2-1)},\ x > 1$

(xxvii) $\frac{d}{dx}(\tanh^{-1} x) = 1/(1-x^2),\ |x| < 1$

(xxviii) $\frac{d}{dx}(\coth^{-1} x) = 1/(1-x^2),\ |x| > 1$

(xxix) $\frac{d}{dx}(\text{sech}^{-1} x) = -1/x\sqrt{(1-x^2)},\ x \in (0, 1)$

(xxx) $\frac{d}{dx}(\text{cosech}^{-1} x) = -1/|x|\sqrt{(1+x^2)},\ x \neq 0$

Fundamental Rules for Derivatives

(i) $\frac{d}{dx}\{cf(x)\} = c\frac{d}{dx}f(x)$, where c is a constant.

(ii) $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$ [sum and difference rule]

(iii) $\frac{d}{dx}\{f(x)\,g(x)\} = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$

[leibnitz product rule or product rule]

Generalisation If $u_1, u_2, u_3, \ldots, u_n$ are functions of x, then

$$\frac{d}{dx}(u_1 u_2 u_3 \ldots u_n) = \left(\frac{du_1}{dx}\right)[u_2 u_3 \ldots u_n] + u_1\left(\frac{du_2}{dx}\right)[u_3 \ldots u_n] + u_1 u_2\left(\frac{du_3}{dx}\right)[u_4 u_5 \ldots u_n] + \ldots + [u_1 u_2 \ldots u_{n-1}]\left(\frac{du_n}{dx}\right)$$

(iv) $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{\{g(x)\}^2}$ [quotient rule]

(v) If $\frac{d}{dx}f(x) = \phi(x)$, then $\frac{d}{dx}f(ax+b) = a\,\phi(ax+b)$

Derivatives of Different Types of Function

1. **Derivatives of Composite Functions** (Chain Rule)

If f and g are differentiable functions in their domain, then fog is also differentiable

Also, $(fog)'(x) = f'\{g(x)\}\, g'(x)$

More easily, if $y = f(u)$ and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$.

Extension of Chain Rule

If y is a function of u, u is a function of v and v is a function of x. Then,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}.$$

2. **Derivatives of Inverse Trigonometric Functions**

Sometimes, it becomes very tedious to differentiate inverse trigonometric function. It can be made easy by using trigonometrical transformations and standard substitution.

Some Standard Substitution

S. No.	Expression	Substitution
(i)	$a^2 - x^2$	$x = a \sin\theta$ or $a \cos\theta$
(ii)	$a^2 + x^2$	$x = a \tan\theta$ or $a \cot\theta$
(iii)	$x^2 - a^2$	$x = a \sec\theta$ or $a \operatorname{cosec}\theta$
(iv)	$\sqrt{\dfrac{a-x}{a+x}}$ or $\sqrt{\dfrac{a+x}{a-x}}$	$x = a \cos 2\theta$
(v)	$\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$ or $\sqrt{\dfrac{a^2+x^2}{a^2-x^2}}$	$x^2 = a^2 \cos 2\theta$
(vi)	$\sqrt{\dfrac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(x-\beta)}$	$x = \alpha \cos^2\theta + \beta \sin^2\theta$
(vii)	$a \sin x + b \cos x$	$a = r\cos\alpha,\ b = r\sin\alpha$

3. **Derivatives of Implicit Functions**

To find $\dfrac{dy}{dx}$ of a function $f(x, y) = 0$, which can not be expressed in the form $y = \phi(x)$, we differentiate both sides of the given relation with respect to x and collect the terms containing $\dfrac{dy}{dx}$ at one side and find $\dfrac{dy}{dx}$.

4. **Derivatives of Parametric Functions**

If the given function is of the form $x = f(t), y = g(t)$, where t is parameter, then

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{d}{dt}g(t)}{\frac{d}{dt}f(t)} = \frac{g'(t)}{f'(t)}$$

Derivative of a Function with Respect to Another Function

If $y = f(x)$ and $z = g(x)$, then the differentiation of y with respect to z is

$$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{f'(x)}{g'(x)}$$

Logarithmic Differentiation

(i) If a function is the product or quotient of functions such as $y = f_1(x)\, f_2(x) \dots f_n(x)$ or $\frac{f_1(x)\, f_2(x)\, f_3(x) \dots}{g_1(x)\, g_2(x)\, g_3(x) \dots}$, we first take logarithm and then differentiate it.

(ii) If a function is in the form of $[f(x)]^{g(x)}$, we first take logarithm and then differentiate it.

Note If $\{f(x)\}^{g(y)} = \{g(y)\}^{f(x)}$, then

$$\frac{dy}{dx} = \frac{g(y)}{f(x)} \cdot \frac{f'(x)}{g'(y)} \left[\frac{f(x) \log g(y) - g(y)}{g(y) \log f(x) - f(x)}\right]$$

Differentiation of Infinite Series

Sometimes, the function is given in the form of an infinite series, e.g. $y = \sqrt{f(x) + \sqrt{f(x) + \dots \infty}}$, then the process to find the derivative of such infinite series is called differentiation of infinite series.

e.g. Suppose $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \dots \infty}}}$

Then, $y = \sqrt{\log x + y} \Rightarrow y^2 = \log x + y$

Now, differentiate it by usual method.

Note

(i) If $y = f(x)^{\{f(x)\}^{\cdots\infty}}$, then $\dfrac{dy}{dx} = \dfrac{y^2 f'(x)}{f(x)\{1 - y\log f(x)\}}$

(ii) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \ldots\infty}}}$, then $\dfrac{dy}{dx} = \dfrac{f'(x)}{2y - 1}$

Differentiation of a Determinant

If $y = \begin{vmatrix} p & q & r \\ u & v & w \\ l & m & n \end{vmatrix}$, where all elements of determinant are differentiable functions of x, then

$$\frac{dy}{dx} = \begin{vmatrix} \frac{dp}{dx} & \frac{dq}{dx} & \frac{dr}{dx} \\ u & v & w \\ l & m & n \end{vmatrix} + \begin{vmatrix} p & q & r \\ \frac{du}{dx} & \frac{dv}{dx} & \frac{dw}{dx} \\ l & m & n \end{vmatrix} + \begin{vmatrix} p & q & r \\ u & v & w \\ \frac{dl}{dx} & \frac{dm}{dx} & \frac{dn}{dx} \end{vmatrix}$$

Successive Differentiations

If the function $y = f(x)$ is differentiated with respect to x, then the result $\dfrac{dy}{dx}$ or $f'(x)$, so obtained, is a function of x (may be a constant).

Hence, $\dfrac{dy}{dx}$ can again be differentiated with respect to x.

The differential coefficient of $\dfrac{dy}{dx}$ with respect to x is written as $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$ or $f''(x)$. Again, the differential coefficient of $\dfrac{d^2y}{dx^2}$ with respect to x is written as

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} \text{ or } f'''(x) \ldots$$

Here, $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}, \ldots$ are respectively known as first, second, third, … order differential coefficients of y with respect to x. These are alternatively denoted by $f'(x), f''(x), f'''(x), \ldots$ or $y_1, y_2, y_3, \ldots$, respectively.

Note $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ but $\dfrac{d^2y}{dx^2} \neq \dfrac{\frac{d^2y}{d\theta^2}}{\frac{d^2x}{d\theta^2}}$

*n*th Derivative of Some Functions

(i) $$\frac{d^n}{dx^n}[\sin(ax+b)] = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$$

(ii) $$\frac{d^n}{dx^n}[\cos(ax+b)] = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$$

(iii) $$\frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}$$

(iv) $$\frac{d^n}{dx^n}[\log(ax+b)] = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$$

(v) $$\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$$

(vi) $$\frac{d^n}{dx^n}(a^x) = a^x (\log a)^n$$

(vii) (a) $$\frac{d^n}{dx^n}[e^{ax}\sin(bx+c)] = r^n e^{ax}\sin(bx+c+n\phi)$$

(b) $$\frac{d^n}{dx^n}[e^{ax}\cos(bx+c)] = r^n e^{ax}\cos(bx+c+n\phi)$$

where, $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

Partial Differentiation

The partial differential coefficient of $f(x, y)$ with respect to x is the ordinary differential coefficient of $f(x, y)$ when y is regarded as a constant. It is written as $\frac{\partial f}{\partial x}$ or f_x.

Thus, $$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Similarly, the differential coefficient of $f(x, y)$ with respect to y is $\frac{\partial f}{\partial y}$ or f_y, where $$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

e.g. If $z = f(x, y) = x^4 + y^4 + 3xy^2 + x^2y + x + 2y$,

then $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or $f_x = 4x^3 + 3y^2 + 2xy + 1$

[here, y is consider as constant]

and $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or $f_y = 4y^3 + 6xy + x^2 + 2$ [here, x is consider as constant]

Higher Partial Derivatives

Let $f(x, y)$ be a function of two variables such that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ both exist.

(i) The partial derivative of $\frac{\partial f}{\partial x}$ w.r.t. x is denoted by $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}.

(ii) The partial derivative of $\frac{\partial f}{\partial y}$ w.r.t. y is denoted by $\frac{\partial^2 f}{\partial y^2}$ or f_{yy}.

(iii) The partial derivative of $\frac{\partial f}{\partial x}$ w.r.t. y is denoted by $\frac{\partial^2 f}{\partial y\, \partial x}$ or f_{xy}.

(iv) The partial derivative of $\frac{\partial f}{\partial y}$ w.r.t. x is denoted by $\frac{\partial^2 f}{\partial x\, \partial y}$ or f_{yx}.

Euler's Theorem on Homogeneous Function

If $f(x, y)$ is a homogeneous function of x, y of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$