DCAM classes

Dynamic Classes for Academic Mastery

Indefinite Integrals

Let $f(x)$ be a function. Then, the collection of all its primitives is called the **indefinite integral** (or anti-derivative) of $f(x)$ and is denoted by $\int f(x)dx$. Integration as an inverse process of differentiation.

If $\frac{d}{dx}\{\phi(x)\} = f(x)$, then $\int f(x)\,dx = \phi(x) + C$, where C is called the **constant of integration** or **arbitrary constant**.

Symbols $f(x) \to$ Integrand

$f(x)dx \to$ Element of integration

$\int \to$ Sign of integral

$\phi(x) \to$ Anti-derivative or primitive or integral of function $f(x)$

The process of finding functions whose derivative is given, is called anti-differentiation or integration.

Note The derivative of function is unique but integral of a function is not unique.

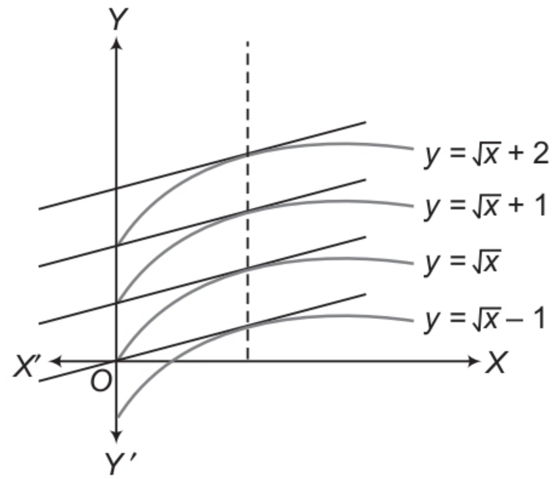
Some Standard Integral Formulae

	Derivatives	Indefinite Integrals
(i)	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, n \neq -1$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
(ii)	$\frac{d}{dx}(\log_e x) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e \lvert x \rvert + C$
(iii)	$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + C$
(iv)	$\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x, a > 0, a \neq 1$	$\int a^x dx = \frac{a^x}{\log_e a} + C$
(v)	$\frac{d}{dx}(-\cos x) = \sin x$	$\int \sin x\, dx = -\cos x + C$

	Derivatives	Indefinite Integrals
(vi)	$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x \, dx = \sin x + C$
(vii)	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$
(viii)	$\frac{d}{dx}(-\cot x) = \text{cosec}^2 x$	$\int \text{cosec}^2 x \, dx = -\cot x + C$
(ix)	$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$
(x)	$\frac{d}{dx}(-\text{cosec } x) = \text{cosec } x \cot x$	$\int \text{cosec } x \cot x \, dx = -\text{cosec } x + C$
(xi)	$\frac{d}{dx}(\log \sin x) = \cot x$	$\int \cot x \, dx = \log\lvert \sin x \rvert + C$ $= -\log\lvert \text{cosec } x \rvert + C$
(xii)	$\frac{d}{dx}(-\log \cos x) = \tan x$	$\int \tan x \, dx = -\log\lvert \cos x \rvert + C$ $= \log\lvert \sec x \rvert + C$
(xiii)	$\frac{d}{dx}[\log(\sec x + \tan x)] = \sec x$	$\int \sec x \, dx = \log\lvert \sec x + \tan x \rvert + C$ $= \log\left\lvert \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\rvert + C$
(xiv)	$\frac{d}{dx}[\log(\text{cosec } x - \cot x)]$	$\int \text{cosec } x \, dx = \log = \text{cosec } x$ $\lvert \text{cosec } x - \cot x \rvert + C = \log\left\lvert \tan\frac{x}{2}\right\rvert + C$
(xv)	$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$	$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
(xvi)	$\frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right) = \int \frac{-1}{\sqrt{a^2 - x^2}}$	$\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + C$
(xvii)	$\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right) = \frac{1}{a^2 + x^2}$	$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
(xviii)	$\frac{d}{dx}\left(\frac{1}{a}\cot^{-1}\frac{x}{a}\right) = \frac{-1}{a^2 + x^2}$	$\int \frac{-1}{a^2 + x^2} dx = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$
(xix)	$\frac{d}{dx}\left(\frac{1}{a}\sec^{-1}\frac{x}{a}\right) = \frac{1}{x\sqrt{x^2 - a^2}}$	$\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$
(xx)	$\frac{d}{dx}\left(\frac{1}{a}\text{cosec}^{-1}\frac{x}{a}\right) = \frac{-1}{x\sqrt{x^2 - a^2}}$	$\int \frac{-1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a}\text{cosec}^{-1}\left(\frac{x}{a}\right) + C$

Geometrical Interpretation of Indefinite Integral

If $\frac{d}{dx}\{\phi(x)\} = f(x)$, then $\int f(x)\,dx = \phi(x) + C$. For different values of C, we get different functions, differing only by a constant. The graphs of these functions give us an infinite family of curves such that at the points on these curves with the same x-coordinate, the tangents are parallel as they have the same slope $\phi'(x) = f(x)$.



Consider the integral of $\frac{1}{2\sqrt{x}}$,

i.e. $$\int \frac{1}{2\sqrt{x}}\,dx = \sqrt{x} + C, C \in R$$

Above figure shows some members of the family of curves given by $y = \sqrt{x} + C$ for different $C \in R$.

Properties of Integration

(i) $\frac{d}{dx}\{\int f(x)dx\} = f(x)$

(ii) $\int k \cdot f(x)dx = k\int f(x)dx$

(iii) $\int \{f_1(x) \pm f_2(x) \pm f_3(x) \pm \ldots \pm f_n(x)\}dx$

$$= \int f_1(x)\,dx \pm \int f_2(x)\,dx \pm \int f_3(x)\,dx \pm \ldots \pm \int f_n(x)\,dx$$

Comparison between Differentiation and Integration

(i) Both differentiation and integration are linear operator on functions as $\frac{d}{dx}\{af(x) \pm bg(x)\} = a\frac{d}{dx}\{f(x)\} \pm b\frac{d}{dx}\{g(x)\}$

and $\int [a \cdot f(x) \pm b \cdot g(x)]dx = a\int f(x)dx \pm b\int g(x)dx.$

(ii) All functions are not differentiable, similarly there are some function which are not integrable.

e.g. Let $f(x) = \frac{1}{x-1}$ and $g(x) = \frac{1}{x-4}$.

Then, $f(x)$ is not differentiable at $x = 1$ and $g(x)$ is not integrable at $x = 4$

(iii) Integral of a function is always discussed in an interval but derivative of a function can be discussed in a interval as well as at a point.

(iv) Geometrically derivative of a function represents slope of the tangent to the graph of function at the point. On the other hand, integral of a function represents an infinite family of curves placed parallel to each other having parallel tangents at points of intersection of the curves with a line parallel to Y-axis.

Method of Integration

Some integrals are not in standard form, to reduce them into standard forms, we use the following methods

1. Integration by Substitution

For integral $\int f'\{g(x)\}g'(x)\,dx$, we create a new variable $t = g(x)$, so that $g'(x) = \frac{dt}{dx}$ or $g'(x)dx = dt$.

Hence, $\int f'\{g(x)\}g'(x)dx = \int f'(t)dt = f(t) + C = f\{g(x)\} + C$

Note

(i) $\int \{f(x)\}^n \cdot f'(x)dx = \frac{\{f(x)\}^{n+1}}{n+1} + C, n \neq -1$

(ii) $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C, f(x) \neq 0$

Basic Formulae Using Method of Substitution

If $\int f(x)\,dx = \phi(x) + C$, then $\int f(ax+b)dx = \frac{1}{a}\phi(ax+b) + C$.

(i) $\int (ax+b)^n\,dx = \frac{1}{a} \cdot \frac{(ax+b)^{n+1}}{n+1} + C, n \neq -1$

(ii) $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log|ax+b| + C$

(iii) $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$

(iv) $\int a^{bx+c}\,dx = \frac{1}{b} \cdot \frac{a^{bx+c}}{\log a} + C, a > 0$ and $a \neq 1$

(v) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$

(vi) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$

(vii) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + C$

(viii) $\int \text{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + C$

(ix) $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + C$

(x) $\int \text{cosec}(ax+b)\cot(ax+b)dx = -\frac{1}{a}\text{cosec}(ax+b) + C$

(xi) $\int \tan(ax+b)\,dx = -\frac{1}{a}\log|\cos(ax+b)| + C$

(xii) $\int \cot(ax+b)\,dx = \frac{1}{a}\log|\sin(ax+b)| + C$

(xiii) $\int \sec(ax+b)\,dx = \frac{1}{a}\log|\sec(ax+b) + \tan(ax+b)| + C$

(xiv) $\int \text{cosec}(ax+b)\,dx = \frac{1}{a}\log|\text{cosec}(ax+b) - \cot(ax+b)| + C$

Trigonometric Identities, Used for Conversion of Integrals into the Standard Integrable Forms

(i) $\sin^2 nx = \frac{1-\cos 2nx}{2}$

(ii) $\cos^2 nx = \frac{1+\cos 2nx}{2}$

(iii) $\sin nx = 2\sin\frac{nx}{2}\cos\frac{nx}{2}$

(iv) $\sin^3 nx = \frac{3}{4}\sin nx - \frac{1}{4}\sin 3nx$

(v) $\cos^3 nx = \frac{3}{4}\cos nx + \frac{1}{4}\cos 3nx$

(vi) $\tan^2 nx = \sec^2 nx - 1$

(vii) $\cot^2 nx = \text{cosec}^2 nx - 1$

(viii) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Standard Substitutions

S.No.	Functions	Substitution
(i)	(a^2+x^2), $\sqrt{x^2+a^2}$, $\frac{1}{\sqrt{x^2+a^2}}$	$x = a\tan\theta$ or $a\cot\theta$ or $a\sinh\theta$
(ii)	(a^2-x^2), $\sqrt{a^2-x^2}$, $\frac{1}{\sqrt{a^2-x^2}}$	$x = a\sin\theta$ or $a\cos\theta$
(iii)	$(x \pm \sqrt{x^2 \pm a^2})^n$	expression inside the bracket $= t$
(iv)	$\frac{2x}{a^2-x^2}$, $\frac{2x}{a^2+x^2}$, $\frac{a^2-x^2}{a^2+x^2}$	$x = a\tan\theta$
(v)	$\frac{1}{(x+a)^{1-\frac{1}{n}}(x+b)^{1+\frac{1}{n}}}$ $(n \in N, n > 1)$	$\frac{x+a}{x+b} = t$
(vi)	(x^2-a^2), $\sqrt{x^2-a^2}$, $\frac{1}{\sqrt{x^2-a^2}}$	$x = a\sec\theta$ or $a\,\mathrm{cosec}\,\theta$ or $a\cosh\theta$
(vii)	$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
(viii)	$\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(\beta-x)}$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$
(ix)	$\sqrt{2ax-x^2}$	$x = a(1-\cos\theta)$
(x)	$\sqrt{\frac{x}{a+x}}$, $\sqrt{\frac{a+x}{x}}$, $\sqrt{x(a+x)}$,	$x = a\tan^2\theta$ or $a\cot^2\theta$
(xi)	$\sqrt{\frac{x}{a-x}}$; $\sqrt{\frac{a-x}{x}}$, $\sqrt{x(a-x)}$, $\frac{1}{\sqrt{x(a-x)}}$	$x = a\sin^2\theta$ or $a\cos^2\theta$
(xii)	$\sqrt{\frac{x}{x-a}}$; $\sqrt{\frac{x-a}{x}}$, $\sqrt{x(x-a)}$, $\frac{1}{\sqrt{x(x-a)}}$	$x = a\sec^2\theta$ or $a\,\mathrm{cosec}^2\,\theta$

Special Integrals

(i) $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(ii) $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C$

(iii) $\int \frac{dx}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C = -\frac{1}{a} \coth^{-1}\left(\frac{x}{a}\right) + C$

(iv) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C = -\cos^{-1}\frac{x}{a} + c$

(v) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log|x + \sqrt{x^2 - a^2}| + C = \cosh^{-1}\left(\frac{x}{a}\right) + C$

(vi) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log|x + \sqrt{x^2 + a^2}| + C = \sinh^{-1}\left(\frac{x}{a}\right) + C$

Important Forms to be Converted into Special Integrals

(i) **Form I** $\int \frac{1}{ax^2 + bx + c} dx$ or $\int \frac{1}{\sqrt{ax^2 + bx + c}} dx$

Express $ax^2 + bx + c$ as sum or difference of two squares.

For this write

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

(ii) **Form II** $\int \frac{px + q}{ax^2 + bx + c} dx$ or $\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx$

Put $px + q = \lambda \cdot \frac{d}{dx}(ax^2 + bx + c) + \mu.$

Now, find values of λ and μ and then integrate it.

(iii) **Form III** $\int \frac{P(x)}{ax^2 + bx + c} dx$, when $P(x)$ is a polynomial of degree 2 or more carry out the dimension and express in the form $\frac{P(x)}{ax^2 + bx + c} = Q(x) + \frac{R(x)}{ax^2 + bx + c}$, where $R(x)$ is a linear expression or constant, then integral reduces to the forms discussed earlier.

Note If degree of the numerator of the integrand is equal to or greater than that of denominator divide the numerator by the denominator until the degree of the remainder is less than that of denominator i.e.

$$\frac{\text{Numerator}}{\text{Denominator}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}}$$

(iv) **Form IV** $\int \frac{dx}{a + b\sin^2 x}, \int \frac{dx}{a + b\cos^2 x}, \int \frac{dx}{a\sin^2 x + b\cos^2 x},$

$\int \frac{dx}{a\sin^2 x + b\cos^2 x + c}, \int \frac{dx}{(a\sin x + b\cos x)^2}$

To evaluate the above type of integrals, we proceed as follows

(a) Divide numerator and denominator by $\cos^2 x$.

(b) Replace $\sec^2 x$, if any in denominator by $1 + \tan^2 x$.

(c) Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

(v) **Form V** $\int \frac{dx}{a + b\sin x}, \int \frac{dx}{a + b\cos x}, \int \frac{dx}{a\sin x + b\cos x},$

$\int \frac{dx}{a\sin x + b\cos x + c}$

To evaluate the above type of integrals, we proceed as follows

(a) Put $\sin x = \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$ and $\cos x = \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$

(b) Replace $1 + \tan^2\frac{x}{2}$ by $\sec^2\frac{x}{2}$.

(c) Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\, dx = dt$

(vi) **Form VI** $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\, dx,$

Write numerator

$= \lambda$ (differentiation of denominator) $+ \mu$ (denominator)

i.e. $a\sin x + b\cos x = \lambda(c\cos x - d\sin x) + \mu(c\sin x + d\cos x)$

$$\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\, dx = \lambda \int \frac{c\cos x - d\sin x}{c\sin x + d\cos x}\, dx$$

$$+ \mu \int \frac{c\sin x + d\cos x}{c\sin x + d\cos x}\, dx$$

$$= \lambda \log|c\sin x + d\cos x| + \mu x + C$$

(vii) **Form VII** $\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\, dx$

Write numerator $= \lambda$ (differentiation of denominator)

$+ \mu$(denominator) $+ \gamma$

i.e. $a\sin x + b\cos x + c = \lambda(p\cos x - q\sin x)$

$+ \mu(p\sin x + q\cos x + r) + \gamma$

$$\therefore \int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx = \lambda \int \frac{p\cos x - q\sin x}{p\sin x + q\cos x + r}\,dx$$

$$+\mu \int \frac{p\sin x + q\cos x + r}{p\sin x + q\cos x + r}\,dx + \gamma \int \frac{1}{p\sin x + q\cos x + r}\,dx$$

$$= \lambda \log|\, p\sin x + q\cos x + r\,|$$

$$+ \mu x + \gamma \int \frac{1}{p\sin x + q\cos x + r}\,dx$$

(viii) **Form VIII** $\int \frac{x^2+1}{x^4+\lambda x^2+1}\,dx,\ \int \frac{x^2-1}{x^4+\lambda x^2+1}\,dx,$

$\int \frac{1}{x^4+\lambda x^2+1}\,dx,\ \int \frac{x^2}{x^4+\lambda x^2+1}\,dx$

To evaluate this type of integrals we proceed as follows:

(a) Divide numerator and denominator by x^2.

(b) Express the denominator of integrands in the form of $\left(x+\frac{1}{x}\right)^2 \pm k^2$.

(c) Introduce $d\left(x+\frac{1}{x}\right)$ or $d\left(x-\frac{1}{x}\right)$ or both in numerator.

(d) Put $x+\frac{1}{x}=t$ or $x-\frac{1}{x}=t$ as the case may be.

(e) Integral reduced to the form of $\int \frac{1}{x^2+a^2}\,dx$ or $\int \frac{1}{x^2-a^2}\,dx$.

(ix) **Form IX** $\int \sqrt{\tan x}\,dx, \int \sqrt{\cot x}\,dx,\ \int \frac{dx}{\sin^4 x + \cos^4 x}$

To evaluate this type of integrals

put $\tan x = t^2 \ \Rightarrow \sec^2 x\,dx = 2t\,dt$

$\Rightarrow$ Then do same as in Form VIII.

2. Integration by Parts

This method is used to integrate the product of two functions.

If $f(x)$ and $g(x)$ be two integrable functions, then

$$\int \underset{\text{I}}{f(x)} \cdot \underset{\text{II}}{g(x)}\,dx = f(x)\int g(x)\,dx - \int \left\{\frac{d}{dx} f(x) \int g(x)\,dx\right\} dx$$

(i) We use the following preferential order for taking the first function.
Inverse → **L**ogarithm → **A**lgebraic → **T**rigonometric → **E**xponential. In short, we write it **ILATE**.

(ii) If one of the function is not directly integrable, then we take it as the first function.

(iii) If only one function is there, e.g. $\int \log x \, dx$ or $\int \sin^{-1} x \, dx$ etc.
then 1 (unity) can be taken as second function.

(iv) If both the functions are directly integrable, then the first function is chosen in such a way that its derivative vanishes easily or the function obtained in integral sign is easily integrable.

Note

(i) Integration by parts is not applicable to product of functions in all cases e.g. $\int \sqrt{x} \sin x \, dx$

(ii) Normally, if any function is a polynomial in x, then we take it as the first function.

Integral of the Form $\int e^x \{f(x) + f'(x)\} dx$

$$\int e^x \{f(x) + f'(x)\} dx = \int \underset{\text{II}}{e^x} \underset{\text{I}}{f(x)} dx + \int e^x f'(x) dx$$

$$= f(x) \int e^x dx - \int \{f'(x) \int e^x dx\} dx + \int e^x f'(x) dx$$

$$= f(x) e^x - \int f'(x) e^x dx + \int e^x f'(x) dx$$

$$= e^x \cdot f(x) + C$$

Note $\int \{xf'(x) + f(x)\} dx = xf(x) + C.$

Integral of the Form $\int e^{ax} \sin(bx + c) dx$ or $\int e^{ax} \cos(bx + c) dx$

(i) $\int e^{ax} \sin(bx + c) \, dx = \dfrac{e^{ax}}{a^2 + b^2} \{a \sin(bx + c) - b \cos(bx + c)\} + k$

(ii) $\int e^{ax} \cos(bx + c) dx = \dfrac{e^{ax}}{a^2 + b^2} \{a \cos(bx + c) + b \sin(bx + c)\} + k$

Some More Special Integral based on Integration by Parts

(i) $\int \sqrt{x^2 + a^2}\, dx = \frac{1}{2}\left[x\sqrt{x^2 + a^2} + a^2 \log|x + \sqrt{x^2 + a^2}|\right] + C$

(ii) $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right] + C$

(iii) $\int \sqrt{x^2 - a^2}\, dx = \frac{1}{2}\left[x\sqrt{x^2 - a^2} - a^2 \log|x + \sqrt{x^2 - a^2}|\right] + C$

Important Forms to be converted into special Integrals

Form I $\int \sqrt{ax^2 + bx + c}\, dx$

Express $ax^2 + bx + c$ as sum or difference of two squares. For this write

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] \text{ or } a\left[\left(x + \frac{b}{2a}\right)^2 \pm k^2\right],$$

where $k^2 = \frac{4ac - b^2}{4a^2}$

Form II $\int (px + q)\sqrt{ax^2 + bx + c}\, dx$

Put $px + q = A\left[\frac{d}{dx}(ax^2 + bx + c)\right] + B = A(2ax + b) + B$

Now, find the values of A and B and then integrate it.

3. Integration by Partial Fractions

Sometimes, an integral of the form $\int \frac{P(x)}{Q(x)}\, dx$, where $P(x)$ and $Q(x)$ are polynomials in x and $Q(x) \neq 0$, also $Q(x)$ has only linear and quadratic factors. For solving such types of integrals, we use the partial fractions.

Partial Fraction Decomposition

(i) If $f(x)$ and $g(x)$ are two polynomials, then $\frac{f(x)}{g(x)}$ defines a rational algebraic function of x. If degree of $f(x) <$ degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called a proper rational function.

(ii) If degree of $f(x) \geq$ degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called an improper rational function.

(iii) If $\frac{f(x)}{g(x)}$ is an improper rational function, then we divide $f(x)$ by $g(x)$ and convert it into a proper rational function as $\frac{f(x)}{g(x)} = \phi(x) + \frac{h(x)}{g(x)}$.

(iv) Any proper rational function $\frac{f(x)}{g(x)}$ can be expressed as the sum of rational functions each having a simple factor of $g(x)$. Each such fraction is called a partial fraction and the process of obtaining them, is called the resolution or decomposition of $\frac{f(x)}{g(x)}$ into partial fraction.

S.No.	Type of proper rational function	Partial fraction
(i)	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
(ii)	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}, a \neq b \neq c$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
(iii)	$\frac{px+q}{(x-a)^3}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$
(iv)	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
(v)	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$, where x^2+bx+c cannot be factorised.	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$
(vi)	$\frac{px^3+qx^2+rx+s}{(x^2+ax+b)(x^2+cx+d)}$, where (x^2+ax+b) and (x^2+cx+d) can not be factorised.	$\frac{Ax+B}{x^2+ax+b} + \frac{Cx+D}{x^2+cx+d}$

Shortcut for Finding Values of *A*, *B* and *C* etc.

Suppose rational function in the form of $\frac{f(x)}{g(x)}$.

***Case* I** When $g(x)$ is expressible as the product of non-repeated linear factors.

Let $g(x) = (x-a_1)(x-a_2)(x-a_3)\ldots(x-a_n)$,

then $$\frac{f(x)}{g(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \frac{A_3}{x-a_3} + \ldots + \frac{A_n}{x-a_n}$$

Now, $$A_1 = \frac{f(a_1)}{(a_1 - a_2)(a_1 - a_3)(a_1 - a_4)\dots(a_1 - a_n)}$$

$$A_2 = \frac{f(a_2)}{(a_2 - a_1)(a_2 - a_3)(a_2 - a_4)\dots(a_2 - a_n)} \dots$$

$$A_n = \frac{f(a_n)}{(a_n - a_1)(a_n - a_2)(a_n - a_3)\dots(a_n - a_{n-1})}$$

Trick To find A_p, put $x = a_p$ in numerator and denominator after deleting the factor $(x - a_p)$.

***Case* II** When $g(x)$ is expressible as product of repeated linear factors.

Let $g(x) = (x - a)^k (x - a_1)(x - a_2)\dots\ (x - a_n)$,

then $$\frac{f(x)}{g(x)} = \frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \dots + \frac{A_k}{(x - a)^k} + \frac{B_1}{(x - a_1)} + \frac{B_2}{(x - a_2)} + \dots + \frac{B_n}{(x - a)^n}$$

Here, all the constant cannot be calculated by using the method in Case I. However, $B_1, B_2, B_3, \dots, B_n$ can be found using the same method i.e. shortcut can be applied only in the case of non-repeated linear factors.

Integration of Irrational Algebraic Function

Irrational function of the form of $(ax + b)^{1/n}$ and x can be evaluated by substitution $(ax + b) = t^n$, thus

$$\int f\{x, (ax + b)^{1/n}\} dx = \int f\left(\frac{t^n - b}{a}, t\right) \frac{nt^{n-1}}{a} dt.$$

(i) $\int \frac{dx}{(Ax + B)\sqrt{Cx + D}}$, substitute $Cx + D = t^2$, then the given integral reduces into $\int \frac{2\,dt}{At^2 - AD + BC}$.

(ii) $\int \frac{dx}{(Ax^2 + B)\sqrt{Cx + D}}$, substitute $Cx + D = t^2$, then the given integral reduces into $\int \frac{2C\,dt}{At^4 - 2DAt^2 + (AD^2 + BC^2)}$.

(iii) $\int \frac{dx}{(x - k)^r \sqrt{Ax^2 + Bx + C}}$, substitute $x - k = \frac{1}{t}$, then the given integral reduces into $\int \frac{t^{r-1}}{At^2 + Bt + C} dt$.

(iv) $\int \frac{1}{(Ax^2 + B)\sqrt{Cx^2 + D}} dx$, substitute $x = \frac{1}{t}$, then the given integral reduces into $\int \frac{-t}{(A + Bt^2)\sqrt{C + Dt^2}} dt$.

Again substitute $C + Dt^2 = u^2$, then it reduces into the form $\int \frac{1}{u^2 \pm a^2} du$.

(v) $\int \frac{ax^2 + bx + c}{(dx + e)\sqrt{fx^2 + gx + h}} dx$

Here, we write

$$ax^2 + bx + c = A_1 (dx + e) \frac{d}{dx}(fx^2 + gx + h) + B_1 (dx + e) + C_1$$

where, A_1, B_1 and C_1 are constants.

Integrals of the Type $x^m (a + bx^n)^p$, $p \neq 0$

***Case* I** If $p \in N$ (natural number) we expand the integral using binomial theorem and integrate it.

***Case* II** If $p \in$ negative integer and m and n are rational numbers put $x = t^k$, where k is the LCM of denominator of m and n.

***Case* III** If $\frac{m+1}{n}$ is an integer and p is rational number, we put $(a + bx^n) = t^k$, where k is the denominator of the fraction p.

***Case* IV** If $\frac{m+p}{n}$ is an integer and p is a rational number, we put $\frac{a + bx^n}{x^n}$, where k is the denominator of the fraction p.

Integration of Hyperbolic Functions

(i) $\int \sinh x \, dx = \cosh x + C$

(ii) $\int \cosh x \, dx = \sinh x + C$

(iii) $\int \text{sech}^2 x \, dx = \tanh x + C$

(iv) $\int \text{cosech}^2 x \, dx = -\coth x + C$

(v) $\int \text{sech}\, x \tanh x \, dx = -\text{sech}\, x + C$

(vi) $\int \text{cosech}\, x \coth x \, dx = -\text{cosech}\, x + C$

Important Results of Integration

(i) (a) Anti-derivative of signum exists in that interval in which $x=0$ is not included.

(b) Anti-derivative of odd function is always even and of even function is always odd.

(ii) If $I_n = \int x^n e^{ax} dx$, then $I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$

(iii) (a) $\int (\log x) dx = x \log x - x + C$

(b) $\int \frac{1}{\log x} dx = \log(\log x) + \log x + \frac{(\log x)^2}{2(2!)} + \frac{(\log x)^3}{3(3!)} + \ldots$

(iv) $\int \frac{a \cos x + b \sin x}{c \cos x + d \sin x} dx = \frac{ac+bd}{c^2+d^2} x + \frac{ad-bc}{c^2+d^2} \log|c \cos x + d \sin x| + k$

(v) $\int \frac{\sin^n x}{\cos^m x} dx = \frac{1}{m-1} \cdot \frac{\sin^{n-1} x}{\cos^{m-1} x} - \frac{n-1}{m-1} \int \frac{\sin^{n-2} x}{\cos^{m-2} x} dx$

(vi) (a) $\int a^x \cos(bx+c)\, dx = \frac{a^x}{(\log a)^2 + b^2} [(\log a)\cos(bx+c) + b \sin(bx+c)] + k$

(b) $\int a^x \sin(bx+c)\, dx = \frac{a^x}{(\log a)^2 + b^2} [(\log a)\sin(bx+c) - b \cos(bx+c)] + k$

(vii) (a) $\int x e^{ax} \cos(bx+c) dx = \frac{x e^{ax}}{a^2+b^2} [a \cos(bx+c) + b \sin(bx+c)]$

$$- \frac{e^{ax}}{(a^2+b^2)^2} [(a^2-b^2)\cos(bx+c) + 2ab \sin(bx+c)] + k$$

(b) $\int x e^{ax} \sin(bx+c) dx = \frac{x e^{ax}}{a^2+b^2} [a \sin(bx+c) - b \cos(bx+c)]$

$$- \frac{e^{ax}}{(a^2+b^2)^2} [(a^2-b^2)\sin(bx+c) - 2ab \cos(bx+c)] + k$$

(viii) (a) $\sin^n x\, dx = \frac{-\cos x \cdot \sin^{n-1} x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\, dx$

(b) $\int \cos^n x\, dx = \frac{\sin x \cdot \cos^{n-1} x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\, dx$

(c) $\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx$