DCAM classes
Dynamic Classes for Academic Mastery

# Limits, Continuity & Differentiability

## Limit

Let $y = f(x)$ be a function of $x$. If at $x = a, f(x)$ takes indeterminate form $\left(\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, 1^{\infty}, 0^0 \text{ and } \infty^0\right)$, then we consider the values of the function at the points which are very near to $a$. If these values tend to a definite unique number as $x$ tends to $a$, then the unique number, so obtained is called the limit of $f(x)$ at $x = a$ and we write it as $\lim_{x \to a} f(x)$.

## Left Hand and Right Hand Limits

If values of the function, at the points which are very near to the left of $a$, tends to a definite unique number, then the unique number so obtained is called the left hand limit of $f(x)$ at $x = a$. We write it as

$$f(a-0) = \lim_{x \to a^-} f(x) = \lim_{h \to 0^+} f(a-h)$$

Similarly, right hand limit is written as

$$f(a+0) = \lim_{x \to a^+} f(x) = \lim_{h \to 0^+} f(a+h)$$

### Existence of Limit

$\lim_{x \to a} f(x)$ exists, if

(i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist

(ii) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

### Uniqueness of Limit

If $\lim_{x \to a} f(x)$ exists, then it is unique, i.e. there cannot be two distinct numbers $l_1$ and $l_2$ such that when $x$ tends to $a$, the function $f(x)$ tends to both $l_1$ and $l_2$.

## Fundamental Theorems on Limits

If $f(x)$ and $g(x)$ are two functions of $x$ such that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist, then

(i) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

(ii) $\lim_{x \to a} [kf(x)] = k \lim_{x \to a} f(x)$, where $k$ is a fixed real number.

(iii) $\lim_{x \to a} [f(x)\, g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$

(iv) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$

(v) $\lim_{x \to a} [f(x)]^{g(x)} = \left[\lim_{x \to a} f(x)\right]^{\lim_{x \to a} g(x)}$

(vi) $\lim_{x \to a} (gof)(x) = \lim_{x \to a} g[f(x)] = g\left[\lim_{x \to a} f(x)\right]$

(vii) $\lim_{x \to a} \log f(x) = \log\left[\lim_{x \to a} f(x)\right]$, provided $\lim_{x \to a} f(x) > 0$.

(viii) $\lim_{x \to a} e^{f(x)} = e^{\lim_{x \to a} f(x)}$

(ix) If $f(x) \leq g(x)$ for every $x$ excluding $a$, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.

(x) $\lim_{x \to a} |f(x)| = \left|\lim_{x \to a} f(x)\right|$

(xi) If $\lim_{x \to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x \to a} \dfrac{1}{f(x)} = 0$

## Important Results on Limits

1. **Algebraic Limits**

(i) $\lim_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1},\ n \in Q$

(ii) $\lim_{x \to 0} \dfrac{(1 + x)^n - 1}{x} = n,\ n \in Q$

2. **Trigonometric Limits**

(i) $\lim_{x\to 0} \frac{\sin x}{x} = 1 = \lim_{x\to 0} \frac{x}{\sin x}$

(ii) $\lim_{x\to 0} \frac{\tan x}{x} = 1 = \lim_{x\to 0} \frac{x}{\tan x}$

(iii) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1 = \lim_{x\to 0} \frac{x}{\sin^{-1} x}$

(iv) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1 = \lim_{x\to 0} \frac{x}{\tan^{-1} x}$

(v) $\lim_{x\to 0} \frac{\sin x^\circ}{x} = \frac{\pi}{180}$

(vi) $\lim_{x\to 0} \cos x = 1$

(vii) $\lim_{x\to a} \frac{\sin(x-a)}{x-a} = 1$

(viii) $\lim_{x\to a} \frac{\tan(x-a)}{x-a} = 1$

(ix) $\lim_{x\to a} \sin^{-1} x = \sin^{-1} a, |a| \le 1$

(x) $\lim_{x\to a} \cos^{-1} x = \cos^{-1} a, |a| \le 1$

(xi) $\lim_{x\to a} \tan^{-1} x = \tan^{-1} a, -\infty < a < \infty$

(xii) $\lim_{x\to \infty} \frac{\sin x}{x} = \lim_{x\to \infty} \frac{\cos x}{x} = 0$

(xiii) $\lim_{x\to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = 1$

(xiv) $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

(where, $x$ is measured in radian)

3. **Exponential Limits**

(i) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(ii) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a, \ a > 0$

(iii) $\lim_{x\to 0} \frac{e^{\lambda x}-1}{x} = \lambda$, where ($\lambda \neq 0$).

(iv) $$\lim_{x\to\infty} a^x = \begin{cases} 0, & 0 \le a < 1 \\ 1, & a = 1 \\ \infty, & a > 1 \\ \text{does not exist}, & a < 0 \end{cases}$$

## 4. Logarithmic Limits

(i) $\lim_{x\to 0} \frac{\log_e(1+x)}{x} = 1$

(ii) $\lim_{x\to e} \log_e x = 1$

(iii) $\lim_{x\to 0} \frac{\log_e(1-x)}{x} = -1$

(iv) $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e, a > 0, \neq 1$

## 5. Limits of the Form $\lim_{x\to a} (f(x))^{g(x)}$

If $\lim_{x\to a} f(x)$ exists and positive, then $\lim_{x\to a} [f(x)]^{\phi(x)} = e^{\lim_{x\to a} \phi(x)\log f(x)}$

## 6. Limits of the Form $1^\infty$

To evaluate the exponential form $1^\infty$, we use following results.

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then, $\lim_{x\to a} \{1 + f(x)\}^{1/g(x)} = e^{\lim_{x\to a} \frac{f(x)}{g(x)}}$

Or If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$,

Then, $\lim_{x\to a} \{f(x)\}^{g(x)} = \lim_{x\to a} \{1 + f(x) - 1\}^{g(x)} = e^{\lim_{x\to a} \{f(x)-1\} g(x)}$

### Particular Cases

(i) $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

(ii) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

(iii) $\lim_{x\to 0} (1+\lambda x)^{\frac{1}{x}} = e^\lambda$

(iv) $\lim_{x\to\infty} \left(1+\frac{\lambda}{x}\right)^x = e^\lambda$

## Methods of Evaluating Limits

### 1. **Determinate Forms** (Limits by Direct Substitution)

To find $\lim_{x \to a} f(x)$, we substitute $x = a$ in the function. If the value comes out to be a definite value, then it is the limit.

Thus, $\lim_{x \to a} f(x) = f(a)$ provided it exists.

### 2. **Indeterminate Forms**

While evaluating $\lim_{x \to a} f(x)$, if direct substitution of $x = a$ leads to one of the following form $\frac{0}{0}; \frac{\infty}{\infty}; \infty - \infty; 0 \times \infty; 1^{\infty}, 0^0$ and $\infty^0$, then these limits can be determined by using L' Hospital's rule or by some other method given below.

#### (i) **Limits by Factorisation**

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ attains $\frac{0}{0}$ form, then $x - a$ must be a factor of numerator and denominator which can be cancelled out.

#### (ii) **Limits by Rationalisation**

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ attains $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form and either $f(x)$ or $g(x)$ or both involve expression consisting of square root, then this can be evaluated by rationalising.

#### (iii) **Limits by Substitution**

In order to evaluate $\lim_{x \to a} f(x)$, we may substitute $x = a + h$ (or $x = a - h$), so that $x \to a$ changes to $h \to 0$.

Thus, $\lim_{x \to a} f(x) = \lim_{h \to 0} f(a \pm h)$

#### (iv) **Limits when $x \to \infty$**

If $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ is of the form $\frac{\infty}{\infty}$ and both $f(x)$ and $g(x)$ are polynomial of $x$.

Then, we divide numerator and denominator by the highest power of $x$ and put 0 for $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}$, etc.

**Note** If $m$ and $n$ are positive integers and $a_0$, $b_0 \neq 0$ are real numbers, then

$$\lim_{x \to \infty} \frac{a_0 x^m + a_1 x^{m-1} + \ldots + a_{m-1} x + a_m}{b_0 x^n + b_1 x^{n-1} + \ldots + b_{n-1} x + b_n} = \begin{cases} \dfrac{a_0}{b_0}, & \text{if } m = n \\ 0, & \text{if } m < n \\ \infty, & \text{if } m > n, a_0 b_0 > 0 \\ -\infty, & \text{if } m > n, a_0 b_0 < 0. \end{cases}$$

**(v) L'Hospital's Rule**

If $f(x)$ and $g(x)$ be two functions of $x$ such that

(i) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

(ii) both are continuous at $x = a$.

(iii) both are differentiable at $x = a$.

(iv) $f'(x)$ and $g'(x)$ are continuous at the point $x = a$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

Above rule is also applicable, if $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$.

**Note** If $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $f'(x), g'(x)$ satisfy all the condition embeded in L'Hospital's rule, then we can repeat the application of this rule on $\frac{f'(x)}{g'(x)}$ to get $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ i.e. $\lim_{x \to a} \frac{f''(x)}{g''(x)}$.

## Limit Using Expansions

Many limits can be evaluated very easily by applying expansion of expressions involving in it. Some of the standard expansions are

(i) $(1 + x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n, n \in N, x \in R$

(ii) $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{2!} x^2 + \ldots \infty, -1 < x < 1, n \in Q$

(iii) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots \infty, x \in R$

(iv) $a^x = e^{x \log a} = 1 + x \log_e a + \dfrac{(x \log_e a)^2}{2!} + \ldots \infty, x \in R, a > 0, a \neq 1$

(v) $\log_e (1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots \infty, -1 < x \leq 1$

(vi) $\log_e (1 - x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \ldots \infty, -1 \leq x < 1$

(vii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \infty, x \in R$

(viii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \infty, x \in R$

(ix) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

(x) $\sin^{-1} x = x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \ldots$

(xi) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

## Some Important Results

(i) $\lim_{x \to 0} \frac{1 - \cos mx}{1 - \cos nx} = \frac{m^2}{n^2}$

(ii) $\lim_{x \to 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx} = \frac{a^2 - b^2}{c^2 - d^2}$

(iii) $\lim_{x \to 0} \frac{\cos mx - \cos nx}{x^2} = \frac{n^2 - m^2}{2}$

(iv) $\lim_{x \to 0} \frac{\sin^p mx}{(nx)^p} = \left(\frac{m}{n}\right)^p$

(v) $\lim_{x \to 0} \frac{\tan^p mx}{\tan^p nx} = \left(\frac{m}{n}\right)^p$

(vi) $\lim_{x \to a} \frac{x^a - a^x}{x^x - a^a} = \frac{1 - \log a}{1 + \log a}$

(vii) $\lim_{x \to 0} \frac{(1+x)^m - 1}{(1+x)^n - 1} = \frac{m}{n}$

(viii) $\lim_{x \to 0} \frac{(1+bx)^m - 1}{(1+ax)^n - 1} = \frac{mb}{na}$

(ix) $\lim_{x \to 0} (1+ax)^{b/x} = \lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{bx} = e^{ab}$

(x) $\lim_{n \to \infty} (x^n + y^n)^{1/n} = y, (0 < x < y)$

(xi) $\lim_{x \to 0} (\cos x + a \sin bx)^{1/x} = e^{ab}$

(xii) $\lim_{x \to \infty} \frac{x^n}{e^x} = 0, \forall\, n$

(xiii) $\lim_{m \to \infty} \left(\cos \frac{x}{m}\right)^m = 1$

(xiv) $\lim_{n \to \infty} \cos \frac{x}{2} \cos \frac{x}{4} \cos \frac{x}{8} \dots \cos \frac{x}{2^n} = \frac{\sin x}{x}$

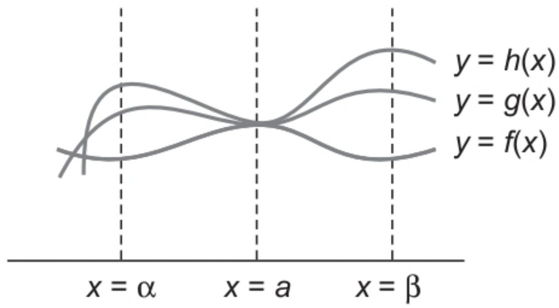
## Sandwich Theorem

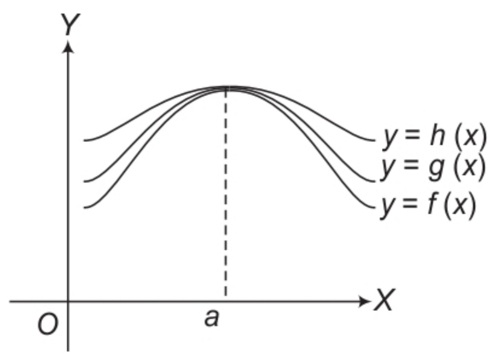
Let $f(x)$, $g(x)$ and $h(x)$ be real functions such that

$$f(x) \le g(x) \le h(x), \ \forall \ x \in (\alpha, \beta) - \{a\}$$

If $$\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x),$$

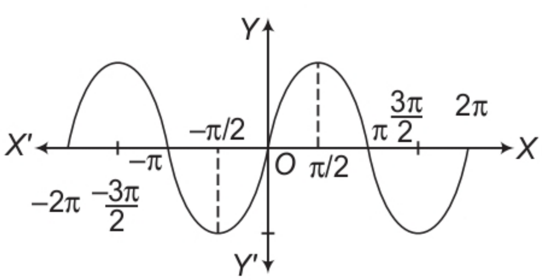
then $$\lim_{x \to a} g(x) = l$$



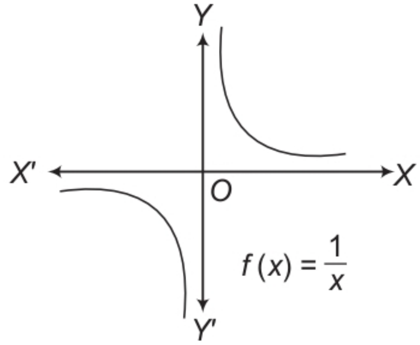




# Continuity

If the graph of a function has no break or gap, then it is continuous. A function which is not continuous is called a **discontinuous** function. e.g. $f(x) = \sin x$ is continuous, as its graph has no break or gap.




While $f(x) = \frac{1}{x}$ is discontinuous at $x = 0$.




## Continuity of a Function at a Point

Let $f$ be a real function and $a$ be a point in the domain of $f$. We say $f$ is continuous at $a$, if $\lim_{x \to a} f(x) = f(a)$.

i.e. $$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

Thus, $f(x)$ is continuous at $x = a$, if $\lim_{x \to a} f(x)$ exists and equals to $f(a)$.

**Note** If a function is not continuous at $x = a$, then it is said to be discontinuous at $x = a$.

### Continuity of a Function in an Interval

(i) A function $f(x)$ is said to be continuous in an open interval $(a, b)$, if $f(x)$ is continuous at every point of the interval.

(ii) A function $f(x)$ is said to be continuous in a closed interval $[a, b]$, if $f(x)$ is continuous in $(a, b)$. In addition, $f(x)$ is continuous at $x = a$ from right and $f(x)$ is continuous at $x = b$ from left.

**Note** A real function $f$ is said to be continuous in its domain, if it is continuous at every point of its domain.

## Discontinuity of a Function

A function $f(x)$ can be discontinuous at a point $x = a$ in any one of the following ways.

(i) $f(a)$ is not defined.

(ii) LHL and RHL both exist but unequal i.e.

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$

(iii) Either $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ or both non-existing or infinite.

(iv) LHL and RHL both exist and equal but not equal to $f(a)$,
i.e. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$

# Types of Discontinuity

1. **Removable Discontinuity**

If $\lim_{x \to a} f(x)$ exists and either it is not equal to $f(a)$ or $f(a)$ is not defined, then the function $f(x)$ is said to have a removable discontinuity (**missing point discontinuity**) of $x = a$.

This discontinuity can be removed by suitably defining the function at $x = a$.

2. **Non-removable discontinuity**

Non-removable discontinuity is of following two types

(i) **Discontinuity of first kind**

If $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist but are not equal, then the function $f(x)$ is said to have a non-removable discontinuity of first kind at $x = a$.

**Note** In this case, we also say that $f(x)$ has jump discontinuity at $x = a$ and we defind $\left| \lim_{x \to a^-} f(x) - \lim_{x \to a^+} f(x) \right|$ = jump of the function at $x = a$.

(ii) **Discontinuity of second kind**

If at least one of the limits $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ does not exist or at least one of these is $\infty$ or $-\infty$, then the function $f(x)$ is said to have a non-removable discontinuity of second kind at $x = a$.

**Important Points to be Remembered**

(i) If $f(x)$ is continuous and $g(x)$ is discontinuous at $x = a$, then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$.

(ii) If $f(x)$ and $g(x)$ both are discontinuous at $x = a$, then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$.

(iii) There are some functions which are continuous only at one point.

e.g. $f(x) = \begin{cases} +x, \text{if } x \in Q \\ -x, \text{if } x \notin Q \end{cases}$ and $g(x) = \begin{cases} x, \text{if } x \in Q \\ 0, \text{if } x \notin Q \end{cases}$ are both continuous only at $x = 0$.

# Fundamental Theorems of Continuity

(i) If $f$ and $g$ are continuous functions, then

(a) $f \pm g$ and $fg$ are continuous.

(b) $cf$ is continuous, where $c$ is a constant.

(c) $\frac{f}{g}$ is continuous at those points, where $g(x) \neq 0$.

(ii) If $g$ is continuous at a point $a$ and $f$ is continuous at $g(a)$, then $fog$ is continuous at $a$.

(iii) If $f$ is continuous in $[a, b]$, then it is bounded in $[a, b]$ i.e. there exist $m$ and $M$ such that

$$m \leq f(x) \leq M, \forall\, x \in [a, b],$$

where $m$ and $M$ are called minimum and maximum values of $f(x)$ respectively in the interval $[a, b]$.

(iv) If $f$ is continuous in its domain, then $|f|$ is also continuous in its domain.

(v) If $f$ is continuous at $a$ and $f(a) \neq 0$, then there exists an open interval $(a - \delta, a + \delta)$ such that for all $x \in (a - \delta, a + \delta)$, $f(x)$ has the same sign as $f(a)$.

(vi) If $f$ is a continuous function defined on $[a, b]$ such that $f(a)$ and $f(b)$ are of opposite sign, then there exists atleast one solution of the equation $f(x) = 0$ in the open interval $(a, b)$.

(vii) If $f$ is continuous on $[a, b]$ and maps $[a, b]$ into $[a, b]$, then for some $x \in [a, b]$, we have $f(x) = x$.

(viii) If $f$ is continuous in domain $D$, then $\frac{1}{f}$ is also continuous in $D - \{x : f(x) = 0\}$.

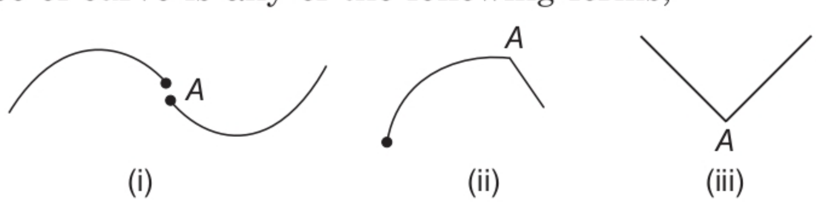
## Differentiability

If the curve has no break point and no sharp edge, then it is differentiable.

## Differentiability (or Derivability) of a Function at a Point

The function $f(x)$ is differentiable at a point $P$ iff there exists a unique tangent at point $P$.

In other words, $f(x)$ is differentiable at a point $P$ iff the curve does not have $P$ as a corner point i.e. the function is not differentiable at those points on which function has holes or sharp edges.
If the shape of curve is any of the following forms,




then the function is not differentiable at point $A$.

***Mathematically*** A function $f(x)$ is said to be differentiable at a point $a$ in its domain, if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exist finitely

or if $\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$

i.e. Left Hand Derivative (LHD) = Right Hand Derivative (RHD)

or $$Lf'(a) = Rf'(a)$$

## Differentiability of a Function in an Interval

(i) A function $f(x)$ is said to be differentiable in an interval $(a, b)$, if $f(x)$ is differentiable at every point of this interval $(a, b)$.

(ii) A function $f(x)$ is said to be differentiable in a closed interval $[a, b]$, if $f(x)$ is differentiable in $(a, b)$, in addition $f(x)$ is differentiable at $x = a$ from right and at $x = b$ from left.

**Note** A real function $f$ is said to be differentiable if it is differentiable at every point of its domain.

# Fundamental Theorems of Differentiability

(i) The sum, difference, product and quotient of two differentiable function, provided it is defined, is differential.

(ii) The composition of differential function is a differential function.

(iii) If $f(x)$ and $g(x)$ both are not differential function, then the sum function $f(x) + g(x)$ and the product function $f(x) \cdot g(x)$ can be differential function.

## Relation between Continuity and Differentiability

(i) If a function $f(x)$ is differentiable at $x = a$, then $f(x)$ is necessarily continuous at $x = a$ but the converse is not necessary true, i.e. if a function is continuous at $x = a$, then it is not necessary that $f$ is differentiable at $x = a$

(ii) If $f$ is not continuous at $x = a$, then $f$ is not differential at $x = a$.

# Continuity and Differentiability of Different Functions

| Function | Curve | Domain and Range | Continuity and Differentiability |
|---|---|---|---|
| Identity | $f(x) = x$ | Domain $= R$,<br>Range $= ]-\infty, \infty[ = R$ | Continuous and Differentiable everywhere |
| Constant | $f(x) = c$ | Domain $= R$,<br>Range $= \{c\}$, where $c \to$ constant | |
| Polynomial | $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where $a_0, a_1, \ldots, a_n$ are real numbers and $n \in N$. | Domain $= R$ | |
| Square Root | $f(x) = \sqrt{x}$ | Domain $= [0, \infty)$,<br>Range $= [0, \infty)$ | Continuous and differentiable in $(0, \infty)$ |
| Greatest integer | $f(x) = [x]$ | Domain $= R$,<br>Range $= I$ | Other than integral values it is continuous and differentiable |
| Least integer | $f(x) = (x)$ | Domain $= R$,<br>Range $= I$ | |
| Fractional part | $f(x) = \{x\} = x - [x]$ | Domain $= R$,<br>Range $= [0, 1)$ | |
| Signum | $f(x) = \frac{\lvert x \rvert}{x} = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$ | Domain $= R$,<br>Range $= \{-1, 0, 1\}$ | Continuous and differentiable everywhere except at $x = 0$ |
| Exponential | $f(x) = a^x, a > 0, a \neq 1$ | Domain $= R$,<br>Range $= ]0, \infty[$ | Continuous and differentiable in their domain |
| Logarithmic | $f(x) = \log_a x; x, a > 0$ and $a \neq 1$ | Domain $= (0, \infty)$,<br>Range $= R$ | |

| Functions | Curve | Domain and Range | Continuity and Differentiability |
|---|---|---|---|
| sine | $y = \sin x$ | Domain $= R$, Range $= [-1, 1]$ | Continuous and differentiable in their domain |
| cosine | $y = \cos x$ | Domain $= R$, Range $= [-1, 1]$ | |
| tangent | $y = \tan x$ | Domain $= R - \left\{(2n+1)\frac{\pi}{2} \mid n \in Z\right\}$, Range $= R$ | |
| cosecant | $y = \text{cosec } x$ | Domain $= R - \{n\pi \mid n \in Z\}$ Range $= \{-\infty, -1] \cup [1, \infty)$ | |
| secant | $y = \sec x$ | Domain $= R - \left\{(2n+1)\frac{\pi}{2} \mid n \in Z\right\}$, Range $= (-\infty, -1] \cup [1, \infty)$ | |
| cotangent | $y = \cot x$ | Domain $= R - \{n\pi \mid n \in Z\}$, Range $= R$ | |
| Arc sine | $y = \sin^{-1} x$ | Domain $= [-1, 1]$, Range $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ | Continuous and differentiable in their domain |
| Arc cosine | $y = \cos^{-1} x$ | Domain $= [-1, 1]$, Range $= [0, \pi]$ | |
| Arc tangent | $y = \tan^{-1} x$ | Domain $= R$, Range $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ | |
| Arc cosecant | $y = \text{cosec}^{-1} x$ | Domain $= (-\infty, 1] \cup [1, \infty)$, Range $= \left(\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$ | |
| Arc secant | $y = \sec^{-1} x$ | Domain $= (-\infty, -1] \cup [1, \infty)$, Range $= [0, \pi] - \left\{\frac{\pi}{2}\right\}$ | |
| Arc cotangent | $y = \cot^{-1} x$ | Domain $= R$, Range $= (0, \pi)$ | |