Limits and Derivatives

1. Limit

If x approches a i.e. $x \to a$, then $f(x)$ approaches l i.e. $f(x) \to l$, where l is a real number, then l is called limit of the function $f(x)$. In symbolic form, it can be written as $\lim_{x \to a} f(x) = l$.

2. Left Hand and Right Hand Limits

If values of the function at the point which are very near to a on the left tends to a definite unique number as x tends to a, then the unique number so obtained is called the Left Hand Limit (LHL) of $f(x)$ at $x = a$, we write it as

$$f(a-0) = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$$

Similarly, Right Hand Limit (RHL) is

$$f(a+0) = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

3. Existence of Limit

If the right hand limit and left hand limit coincide (i.e. same), then we say that limit exists and their common value is called the limit of $f(x)$ at $x = a$ and denoted it by $\lim_{x \to a} f(x)$.

4. Algebra of Limits

Let f and g be two functions such that both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

(i) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

(ii) $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$

(iii) $\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$

(iv) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, where $\lim_{x \to a} g(x) \neq 0$

5. **Limits of a Polynomial Function**

A function f is said to be a polynomial function if $f(x)$ is zero function or if $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, where a_i's are real number such that $a_n \neq 0$.

Then, limit of polynomial functions is

$$f(x) = \lim_{x \to a} f(x) = \lim_{x \to a} [a_0 + a_1x + a_2x^2 + \dots + a_nx^n]$$
$$= a_0 + a_1a + a_2a^2 + \dots + a_na^n = f(a)$$

6. **Limits of Rational Functions**

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)}$$

(i) If $g(a) = 0$ and $f(a) \neq 0$, then limit does not exist.

(ii) If $g(a) = 0$ and $f(a) = 0$, then we can find limit by using suitable method as direct substitution, factorisation, rationalisation method, etc.

7. **Some Standard Limits**

(i) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ (ii) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(iii) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$ (iv) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

(v) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$ (vi) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(vii) $\lim_{x \to 0} \frac{\log(1 + x)}{x} = 1$ (viii) $\lim_{x \to 0} \frac{\log(1 - x)}{-x} = 1$

8. **Derivative at a Point**

Suppose f is a real valued function, then

$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ is called the derivative of f at x,

iff $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ exists finitely.

9. **First Principle of Derivative**

If f is a real valued function, then

$$\frac{d}{dx} f(x) = f'(x) = \lim_{x \to 0} \frac{f(x + h) - f(x)}{h}$$

provided this limit exists.

10. **Algebra of Derivative of Functions**

Let f and g be two functions such that their derivatives are defined in a common domain, then

(i) $\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} [f(x)] + \frac{d}{dx} [g(x)]$

(ii) $\frac{d}{dx} [(f(x) - g(x)] = \frac{d}{dx} [f(x)] - \frac{d}{dx} [g(x)]$

(iii) $\frac{d}{dx} [f(x) \cdot g(x)] = \left[\frac{d}{dx} f(x)\right] \cdot g(x) + f(x) \cdot \left[\frac{d}{dx} g(x)\right]$

(iv) $\frac{d}{dx} \left[\frac{f(x)}{g(x)}\right] = \frac{\left[\frac{d}{dx} f(x)\right] \cdot g(x) - f(x) \cdot \left[\frac{d}{dx} g(x)\right]}{[g(x)]^2}$

11. **Some Standard Derivatives**

(i) $\frac{d}{dx}(\text{Constant}) = 0$ (ii) $\frac{d}{dx}(x^n) = nx^{n-1}$

(iii) $\frac{d}{dx}(\sin x) = \cos x$ (iv) $\frac{d}{dx}(\cos x) = -\sin x$

(v) $\frac{d}{dx}(\tan x) = \sec^2 x$ (vi) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(vii) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(viii) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

(ix) $\frac{d}{dx}(a^x) = a^x \log_e a$ (x) $\frac{d}{dx}(e^x) = e^x$

(xi) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$