Rotational Motion

Centre of Mass

Centre of mass of a system is the point that behaves as whole mass of the system is concentrated on it and all external forces are acting on it. For rigid bodies, centre of mass is independent of the state of the body, *i.e.* whether it is in rest or in accelerated motion centre of mass will remain same.

Centre of Mass of System of n Particles

If a system consists of n particles of masses $m_1, m_2, m_3, \ldots, m_n$ having position vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, \mathbf{r}_n$, then position vector of centre of mass of the system, $\mathbf{r}_{CM} = \dfrac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 + \ldots + m_n\mathbf{r}_n}{m_1 + m_2 + m_3 + \ldots + m_n} = \dfrac{\sum_{i=1}^{n} m_i\mathbf{r}_i}{\sum m_i}$

In terms of coordinates,

$$\mathbf{x}_{CM} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2 + \ldots + m_n\mathbf{x}_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{n} m_i\mathbf{x}_i}{\sum m_i}$$

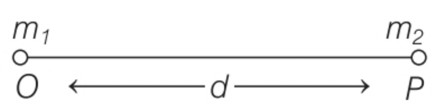
$$\mathbf{y}_{CM} = \frac{m_1\mathbf{y}_1 + m_2\mathbf{y}_2 + \ldots + m_n\mathbf{y}_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{n} m_i\mathbf{y}_i}{\sum m_i}$$

$$\mathbf{z}_{CM} = \frac{m_1\mathbf{z}_1 + m_2\mathbf{z}_2 + \ldots + m_n\mathbf{z}_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{n} m_i\mathbf{z}_i}{\sum m_i}$$

Centre of Mass of Two Particles System

Choosing O as origin of the coordinate axis.

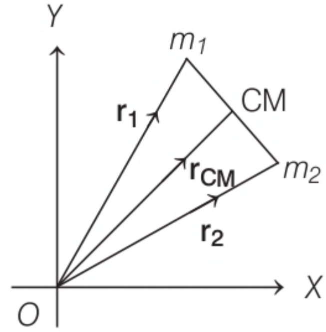
(i) Then, position of centre of mass from $m_1 = \dfrac{m_2 d}{m_1 + m_2}$



(ii) Position of centre of mass from $m_2 = \dfrac{m_1 d}{m_1 + m_2}$

(iii) If position vectors of particles of masses m_1 and m_2 are $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, then

$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$



(iv) If in a two particles system, particles of masses m_1 and m_2 are moving with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively, then velocity of the centre of mass,

$$\mathbf{v}_{CM} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}$$

(v) If accelerations of the particles are $\mathbf{a}_1$ and $\mathbf{a}_2$ respectively, then acceleration of the centre of mass,

$$\mathbf{a}_{CM} = \frac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2}{m_1 + m_2}$$

(vi) Centre of mass of an isolated system has a constant velocity. It means, isolated system will remain at rest if it is initially at rest or will move with a same velocity, if it is in motion initially.

(vii) The position of centre of mass depends upon the shape, size and distribution of the mass of the body.

(viii) The centre of mass of an object need not to lie with in the object.

(ix) In symmetrical bodies having homogeneous distribution of mass the centre of mass coincides with the geometrical centre of the body.

(x) The position of centre of mass of an object changes in translatory motion but remains unchanged in rotatory motion.

Linear Momentum of a System of Particles

For a system of n particles, the total momentum of a system of particles is equal to the product of the total mass and velocity of its centre of mass.

$$\mathbf{p} = M\mathbf{v}_{CM}$$

According to Newton's second law for system of particles

Net external force, $F_{ext} = \frac{dp}{dt}$.

Rigid Body

A body is said to be a rigid body, when it has perfectly definite shape and size. The distance between all points of particles of such a body do not change, while applying any force on it. General motion of a rigid body consists of both the translational motion and the rotational motion.

Translational Motion

A rigid body performs a pure translational motion, if each particle of the body undergoes the same displacement in the same direction in a given interval of time.

Rotational Motion

A rigid body performs a pure rotational motion, if each particle of the body moves in a circle, and the centre of all the circles lie on a straight line called the axes of rotation.

Equations of Rotational Motion

(i) $\omega = \omega_0 + \alpha t$ (ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ (iii) $\omega^2 = \omega_0^2 + 2\alpha\theta$

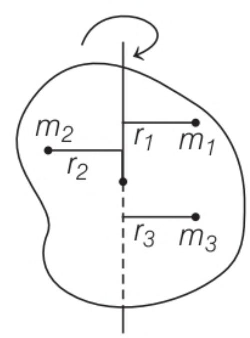
where, θ is displacement in rotational motion, ω_0 is initial velocity, ω is final velocity and α is acceleration.

Moment of Inertia

The inertia of rotational motion is called **moment of inertia**. It is denoted by I.

Moment of inertia is the property of an object by virtue of which it opposes any change in its state of rotation about an axis.

The moment of inertia of a body about a given axis is equal to the sum of the products of the masses of its constituent particles and the square of their respective distances from the axis of rotation.



Moment of inertia of a body,

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots = \sum_{i=1}^{n} m_i r_i^2$$

Its SI unit is kg-m^2 and its dimensional formula is $[ML^2]$.

The moment of inertia of a body depends upon

(a) position of the axis of rotation.
(b) orientation of the axis of rotation.
(c) shape and size of the body.
(d) distribution of mass of the body about the axis of rotation.

The physical significance of the moment of inertia is same in rotational motion as the mass in linear motion.

The Radius of Gyration

The root mean square distance of its constituent particles from the axis of rotation is called the radius of gyration of a body.

It is denoted by K.

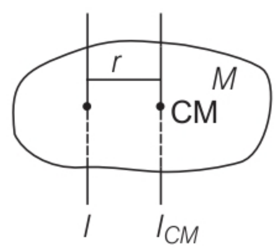
Radius of gyration, $K = \sqrt{\dfrac{r_1^2 + r_2^2 + \ldots + r_n^2}{n}}$

The product of the mass of the body (M) and square of its radius of gyration (K) gives the same moment of inertia of the body about the rotational axis.

Therefore, moment of inertia, $I = MK^2 \Rightarrow K = \sqrt{\dfrac{I}{M}}$

Parallel Axes Theorem

The moment of inertia of any object about any arbitrary axis is equal to the sum of moment of inertia about a parallel axis passing through the centre of mass and the product of mass of the body and the square of the perpendicular distance between the two axes.

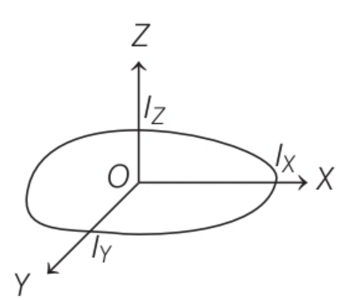


Mathematically, $I = I_{\text{CM}} + Mr^2$

where, I is the moment of inertia about the arbitrary axis, I_{CM} is the moment of inertia about the parallel axis through the centre of mass, M is the total mass of the object and r is the perpendicular distance between the axis.

Perpendicular Axes Theorem

The moment of inertia of any two dimensional body about an axis perpendicular to its plane (I_Z) is equal to the sum of moments of inertia of the body about two mutually perpendicular axes lying in its own plane and intersecting each other at a point, where the perpendicular axis passes through it.



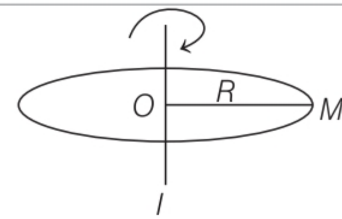
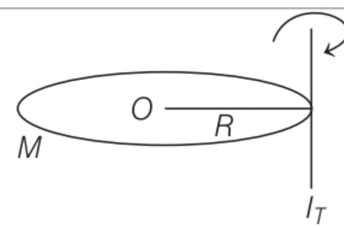
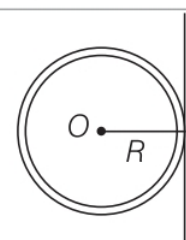
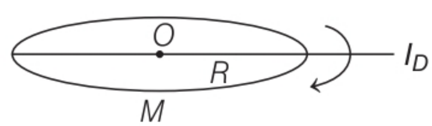
Mathematically, $I_Z = I_X + I_Y$

where, I_X and I_Y are the moments of inertia of plane lamina about the perpendicular axes X and Y, respectively which lie in the plane of lamina and intersect each other.

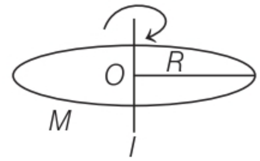
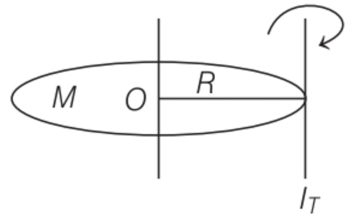
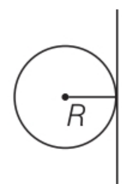
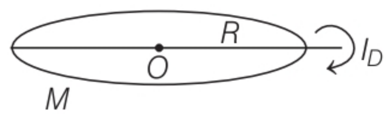
> Theorem of parallel axes is applicable for any type of rigid body whether it is a two dimensional or three dimensional, while the theorem of perpendicular is applicable for laminar type or two dimensional bodies only.

Moment of Inertia of Homogeneous Rigid Bodies

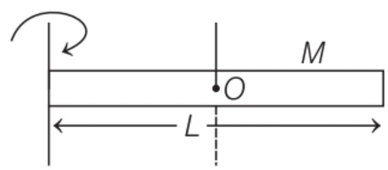
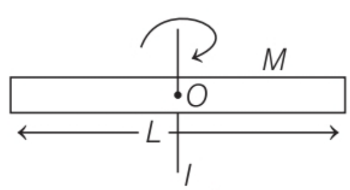
For a Thin Circular Ring

S. No.	Axis of Rotation	Moment of Inertia
(a)	About an axis passing through its centre and perpendicular to its plane	 $I = MR^2$
(b)	About a tangent perpendicular to its plane	 $I_T = 2MR^2$
(c)	About a tangent in the plane of ring	 $I_{T'} = \frac{3}{2} MR^2$
(d)	About a diameter	 $I_D = \frac{1}{2} MR^2$

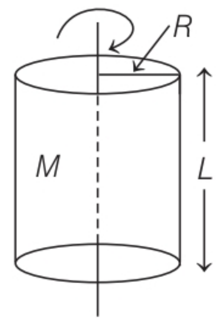
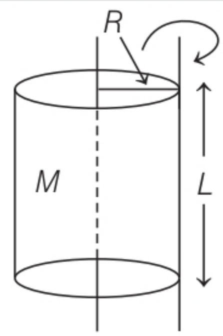
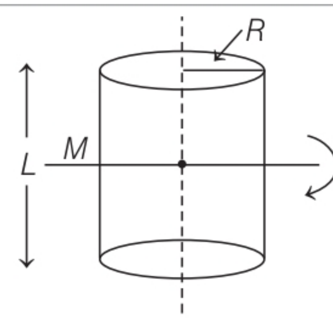
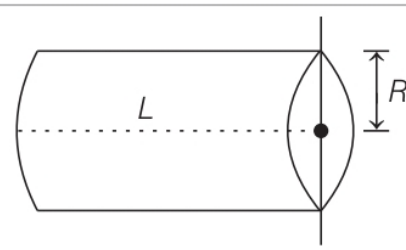
For a Circular Disc

S. No.	Axis of Rotation	Moment of Inertia
(a)	About an axis passing through its centre and perpendicular to its plane	 $I = \frac{1}{2} MR^2$
(b)	About a tangent perpendicular to its plane	 $I_T = \frac{3}{2} MR^2$
(c)	About a tangent in its plane	 $I_{T'} = \frac{5}{4} MR^2$
(d)	About a diameter	 $I_D = \frac{1}{4} MR^2$

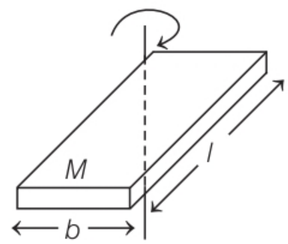
For a Thin Rod

S. No.	Axis of Rotation	Moment of Inertia
(a)	About an axis passing through its centre and perpendicular to its length	 $I = \frac{ML^2}{12}$
(b)	About an axis passing through its one end and perpendicular to its length	 $I = \frac{ML^2}{3}$

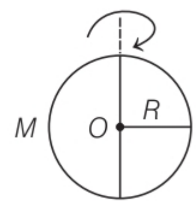
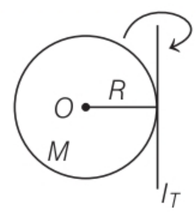
For a Solid Cylinder

S. No.	Axis of Rotation	Moment of Inertia
(a)	About its geometrical axis	 $I = \frac{MR^2}{2}$
(b)	About an axis passing through its outer face along its length	 $I = \frac{3}{2} MR^2$
(c)	About an axis passing through its centre and perpendicular to its length	 $I = \left(\frac{ML^2}{12} + \frac{MR^2}{4}\right)$
(d)	About an axis passing through its diameter of circular surface	 $I = \left(\frac{ML^2}{3} + \frac{MR^2}{4}\right)$

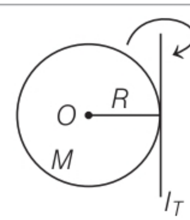
For a Rectangular Plate

Axis of Rotation	Moment of Inertia
About an axis passing through its centre and perpendicular to its plane $I = \frac{M(l^2 + b^2)}{12}$	

For a Thin Spherical Shell

S. No.	Axis of Rotation	Moment of Inertia
(a)	About its any diameter $I_D = \frac{2}{3} MR^2$	
(b)	About its any tangent $I_T = \frac{5}{3} MR^2$	

For a Solid Sphere

S. No.	Axis of Rotation	Moment of Inertia
(a)	About its any diameter $I_D = \frac{2}{5} MR^2$	
(b)	About its any tangent $I_T = \frac{7}{5} MR^2$	

Torque

Torque or moment of a force about the axis of rotation

$$\tau = \mathbf{r} \times \mathbf{F} = rF \sin\theta \, \hat{\mathbf{n}}$$

It is a vector quantity. It is also known as moment of force or couple.

If the nature of the force is to rotate the object clockwise, then torque is called **negative torque** and if rotate the object anti-clockwise, then it is called **positive torque.**

Its SI unit is N-m and its dimensional formula is $[ML^2T^{-2}]$.

In rotational motion, torque, $\tau = I\alpha$

where, α is angular acceleration and I is moment of inertia.

Angular Momentum

The moment of linear momentum is called angular momentum.

It is denoted by L.

Angular momentum, $L = I\omega = mvr$

In vector form, $L = I\omega = \mathbf{r} \times m\mathbf{v}$

Its SI unit is J-s and its dimensional formula is $[ML^2T^{-1}]$.

Torque, $$\tau = \frac{d\mathbf{L}}{dt}$$

Principle of Moment

When an object is in rotational equilibrium, then algebraic sum of all torques acting on it is zero. Clockwise torques are taken negative and anti-clockwise torques are taken positive.

Conservation of Angular Momentum

If the external torque acting on a system is zero, then its angular momentum remains conserved.

If $\tau_{ext} = 0$, then $L = I\omega = \text{constant} \Rightarrow I_1\omega_1 = I_2\omega_2$

Torque and Angular Momentum for a System of Particles

The rate of change of the total angular momentum of a system of particles about a point is equal to the sum of the external torques acting on the system taken about the same point.

$$\frac{d\mathbf{L}}{dt} = \tau_{ext}$$

Equilibrium of Rigid Body

A rigid body is said to be in equilibrium, if both of its linear momentum and angular momentum are not changing with time. Thus, for equilibrium body does not possess linear acceleration or angular acceleration.

Couple

A pair of equal and opposite forces with parallel lines of action is called couple. It produces rotation without translation.

Centre of Gravity

If a body is supported on a point such that total gravitational torque about this point is zero, then this point is called centre of gravity of the body.

Centre of gravity coincides with centre of mass, if g is constant. But for large objects g will vary, hence centre of gravity does not coincide with centre of mass.

Angular Impulse

Total effect of a torque applied on a rotating body in a given time is called angular impulse. Angular impulse is equal to total change in angular momentum of the system in given time. Thus, angular impulse

$$J = \int_0^{\Delta L} \tau \, dt = L_f - L_i$$

Rotational Kinetic Energy

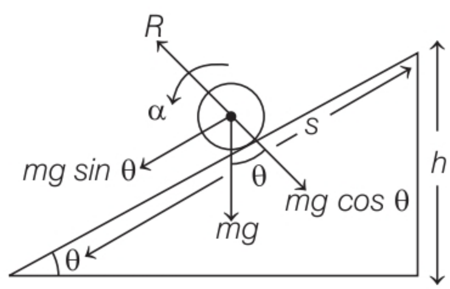
Rotational kinetic energy of a body is equal to the sum of kinetic energies of its constituent particles.

Rotational kinetic energy, $K = \frac{1}{2} I\omega^2$

Motion of a Body Rolling Down Without Slipping on an Inclined Plane

Acceleration of the body,

$$a = \frac{mg \sin \theta}{m + \frac{I}{r^2}} = \frac{g \sin \theta}{1 + \frac{K^2}{r^2}}$$



where, K = radius of gyration, m = mass of the body, r = radius of the body and θ = inclination of the plane. Velocity attained at the bottom,

$$v = \sqrt{\frac{2gh}{1 + \frac{K^2}{r^2}}} = \sqrt{\frac{2gs \sin \theta}{1 + \frac{K^2}{r^2}}} \qquad \left(\because s = \frac{h}{\sin \theta}\right)$$

where, h = height of slope and s = length of slope.

$$\text{Time} = \sqrt{\frac{2s\left(1 + \frac{k^2}{r^2}\right)}{g \sin \theta}}$$

where, s = length of slope and r = radius of rolling body.

If a cylinder, ring, disc and sphere rolls on inclined plane, then the sphere will reach the bottom first with greater velocity while ring will reach the bottom with least velocity.

If a solid and hollow body of same shape are allowed to roll down an inclined plane, then solid body will reach the bottom first with greater velocity.

For rolling without slipping, the minimum value of coefficient of friction

$$\mu = \frac{F}{R} = \frac{(Ia / r^2)}{mg \cos \theta}$$

Total kinetic energy of a rolling object

$= $ Kinetic energy of translation + Kinetic energy of rotation.

$$= \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2$$

Power delivered by torque $P = \tau \cdot \omega$

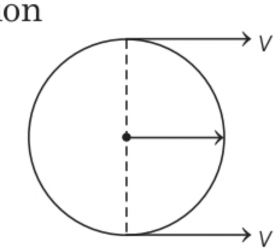
Work done by torque $W = \int_{\theta_1}^{\theta_2} \tau \, d\theta$

If τ is constant, then $W = \tau (\theta_2 - \theta_1)$

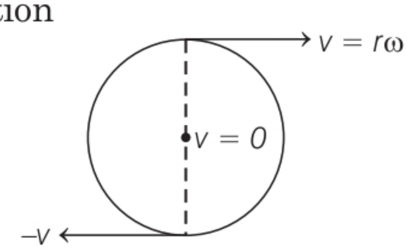
or $W = \tau \times$ Angle moved by the particle.

In rolling motion, all points of the body have same angular speed but different linear speeds.

In pure translational motion



In pure rotational motion



In combined motion, *i.e.* translational as well as rotational motion.