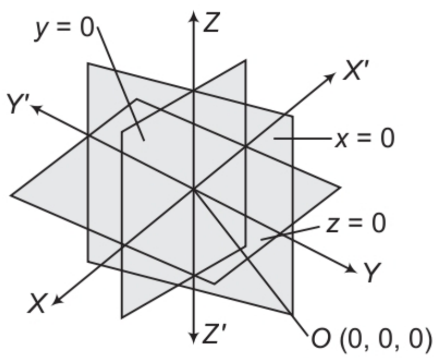
Three Dimensional Geometry

Coordinate System

The three mutually perpendicular lines in a space which divides the space into eight parts and if these perpendicular lines are the coordinate axes, then it is said to be a coordinate system.



Note The coordinates of any point on the X, Y and Z-axes will be the form $(x, 0, 0)$, $(0, y, 0)$ and $(0, 0, z)$ respectively.

Sign Convention

Octant Coordinate	x	y	z
$OXYZ$	+	+	+
$OX'YZ$	−	+	+
$OXY'Z$	+	−	+
$OXYZ'$	+	+	−
$OX'Y'Z$	−	−	+
$OX'YZ'$	−	+	−
$OXY'Z'$	+	−	−
$OX'Y'Z'$	−	−	−

Distance between Two Points

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points. Then, distance between these points is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The distance of a point $P(x, y, z)$ from origin O is

$$OP = \sqrt{x^2 + y^2 + z^2}$$

Section Formulae

(i) The coordinates of any point, which divides the join of points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m : n$ internally are

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right)$$

(ii) The coordinates of any point, which divides the join of points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m : n$ externally are

$$\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}, \frac{mz_2 - mz_1}{m - n}\right)$$

(iii) The coordinates of mid-point of P and Q are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

(iv) Coordinates of the centroid of a triangle formed with vertices $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

(v) **Centroid of a Tetrahedron**

If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) are the vertices of a tetrahedron, then its centroid G is given by

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right).$$

Area of Triangle

If the vertices of a triangle be $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, then

Area of $\Delta ABC = \sqrt{\Delta_{xy}^2 + \Delta_{yz}^2 + \Delta_{zx}^2}$

where, $\Delta_{yz} = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}$, $\Delta_{xz} = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}$ and $\Delta_{xy} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

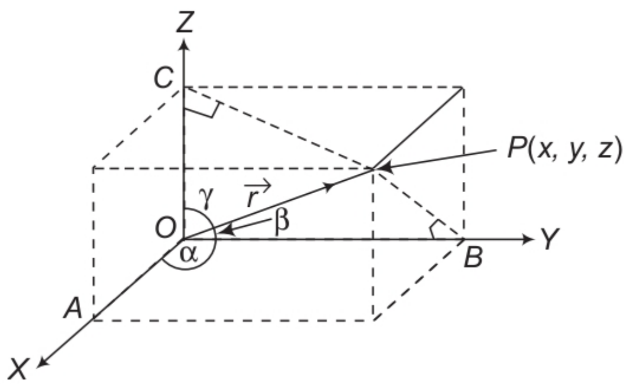
Direction Cosines

If a directed line segment OP makes angle α, β and γ with OX, OY and OZ respectively, then $\cos\alpha, \cos\beta$ and $\cos\gamma$ are called direction cosines of OP and it is represented by l, m, n.

i.e. $\quad l = \cos\alpha$

$m = \cos\beta$

and $\quad n = \cos\gamma$



If $OP = r$, then coordinates of OP are (lr, mr, nr)

(i) If l, m, n are direction cosines of a vector $\mathbf{r}$, then

(a) $\mathbf{r} = |\mathbf{r}|(l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}) \quad \Rightarrow \quad \hat{\mathbf{r}} = l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}$

(b) $l^2 + m^2 + n^2 = 1$

(c) Projections of $\mathbf{r}$ on the coordinate axes are

$$l|\mathbf{r}|, m|\mathbf{r}|, n|\mathbf{r}|$$

(d) $|\mathbf{r}| = \sqrt{\text{sum of the squares of projections of } \mathbf{r} \text{ on the coordinate axes}}$

(ii) If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points, such that the direction cosines of $\mathbf{PQ}$ are l, m, n. Then,

$$x_2 - x_1 = l|\mathbf{PQ}|, \; y_2 - y_1 = m|\mathbf{PQ}|, \; z_2 - z_1 = n|\mathbf{PQ}|$$

These are projections of $\mathbf{PQ}$ on X, Y and Z-axes, respectively.

(iii) If l, m, n are direction cosines of a vector $\mathbf{r}$ and a, b, c are three numbers, such that $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$. Then, we say that a, b and c are the direction ratios of $\mathbf{r}$ which are proportional to l, m, n.

Also, we have $l = \pm \dfrac{a}{\sqrt{a^2 + b^2 + c^2}}$, $m = \pm \dfrac{b}{\sqrt{a^2 + b^2 + c^2}}$,

$$n = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

(iv) If θ is the angle between two lines having direction cosines l_1, m_1, n_1 and l_2, m_2, n_2, then $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

(a) Lines are parallel, if $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$.

(b) Lines are perpendicular, if $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$.

(v) If θ is the angle between two lines whose direction ratios are proportional to a_1, b_1, c_1 and a_2, b_2, c_2 respectively, then the angle θ between them is given by

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Lines are parallel, if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

Lines are perpendicular, if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$.

(vi) The projection of the line segment joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ to the line having direction cosines l, m, n is

$$|(x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)n|.$$

(vii) The direction ratio of the line passing through points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$. Then, direction cosines of **PQ** are

$$\frac{x_2 - x_1}{|\mathbf{PQ}|}, \frac{y_2 - y_1}{|\mathbf{PQ}|}, \frac{z_2 - z_1}{|\mathbf{PQ}|}$$

Angle between Two Intersecting Lines

If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two given lines, then the angle θ between them is given by

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

(i) The angle between any two diagonals of a cube is $\cos^{-1}\left(\dfrac{1}{3}\right)$.

(ii) The angle between a diagonal of a cube and the diagonal of a face of the cube is $\cos^{-1}\left(\sqrt{\dfrac{2}{3}}\right)$.

Line in Space

A line (or straight line) is a curve such that all the points on the line segment joining any two points of it lies on it.

A line can be determined uniquely, if

(i) its direction and the coordinates of a point on it are known.

(ii) it passes through two given points.

1. **Equation of a Line Passing through a given Point and Parallel to a given Vector**

Vector Equation Equation of a line passing through a point with position vector **a** and parallel to vector **b** is $\mathbf{r} = \mathbf{a} + \lambda\,\mathbf{b}$, where λ is a parameter.

Cartesian Equation Equation of a line passing through a fixed point $A(x_1, y_1, z_1)$ and having direction ratios a, b, c is given by $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$, it is also called the **symmetrically** form of a line.

2. **Equation of Line Passing through Two given Points**

Vector Equation A line passing through two given points having position vectors **a** and **b** is $\mathbf{r} = \mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a})$, where λ is a parameter.

Cartesian Equation Equation of a straight line joining two fixed points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

3. **Perpendicular Distance of a Point from a Line**

Vector form The length of the perpendicular from a point $P(\vec{\alpha})$ on the line

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} \text{ is given by } \sqrt{|\vec{\alpha} - \mathbf{a}|^2 - \left\{\frac{(\vec{\alpha} - \mathbf{a})\cdot\mathbf{b}}{|\mathbf{b}|}\right\}^2}$$

Cartesian Form The length of the perpendicular from a point $P(x_1, y_1, z_1)$ on the line

$$\frac{x - a}{l} = \frac{y - b}{m} = \frac{z - c}{n} \text{ is given by}$$

$$\sqrt{\{(a-x_1)^2+(b-y_1)^2+(c-z_1)^2\}-\{(a-x_1)\,l + (b-y_1)\,m+(c-z_1)\,n\}^2}$$

where, l, m, n are direction cosines of the line.

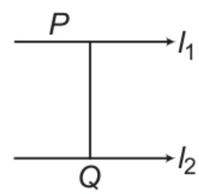
Skew Lines

Two straight lines in space are said to be skew lines, if they are neither parallel nor intersecting. Thus, skew-lines are such pair of lines which are non-coplanar.

Shortest Distance

If l_1 and l_2 are two skew lines, then a line perpendicular to each of lines l_1 and l_2 is known as the line of shortest distance.

If the line of shortest distance intersects the lines l_1 and l_2 at P and Q respectively, then the distance PQ between points P and Q is known as the shortest distance between l_1 and l_2.



Vector Form

(i) The shortest distance between lines $\mathbf{r}=\mathbf{a}_1+\lambda\mathbf{b}_1$ and $\mathbf{r}=\mathbf{a}_2+\mu\mathbf{b}_2$ is given by

$$d=\left|\frac{(\mathbf{b}_1\times\mathbf{b}_2)\cdot(\mathbf{a}_2-\mathbf{a}_1)}{|\mathbf{b}_1\times\mathbf{b}_2|}\right|$$

(ii) The shortest distance between parallel lines $\mathbf{r}=\mathbf{a}_1+\lambda\mathbf{b}$ and $\mathbf{r}=\mathbf{a}_2+\mu\mathbf{b}$ is given by

$$d=\left|\frac{(\mathbf{a}_2-\mathbf{a}_1)\times\mathbf{b}}{|\mathbf{b}|}\right|$$

(iii) Two lines $\mathbf{r}=\mathbf{a}_1+\lambda\mathbf{b}_1$ and $\mathbf{r}=\mathbf{a}_2+\mu\mathbf{b}_2$ are intersecting, when $(\mathbf{b}_1\times\mathbf{b}_2)\cdot(\mathbf{a}_2-\mathbf{a}_1)=0$.

Cartesian Form

(i) The shortest distance between the lines

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$$

and $$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$ is given by

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}}$$

(ii) Two lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_1}{b_2} = \frac{z - z_1}{c_2}$ are intersecting, when

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Important Points to be Remembered

Since, X, Y and Z-axes pass through the origin and have direction cosines (1, 0, 0), (0, 1, 0) and (0, 0, 1), respectively. Therefore, their equations are

$$X\text{-axis}: \frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0} \text{ or } y = 0, z = 0$$

$$Y\text{-axis}: \frac{x-0}{0} = \frac{y-0}{1} = \frac{z-0}{0} \text{ or } x = 0, z = 0$$

$$Z\text{-axis}: \frac{x-0}{0} = \frac{y-0}{0} = \frac{z-0}{1} \text{ or } x = 0, y = 0$$

Plane

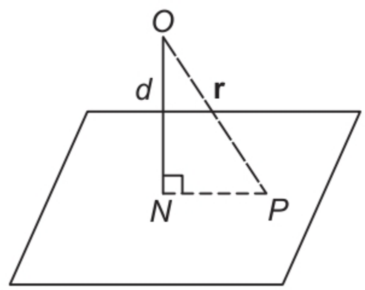
A plane is a surface such that, if two points are taken on it, a straight line joining them lies wholly on the surface. A straight line, which is perpendicular to every line lying on a plane is called a normal to the plane.

General Equation of the Plane

The general equation of the first degree in x, y, z always represents a plane. Hence, the general equation of the plane is $ax + by + cz + d = 0$. The coefficient of x, y and z in the cartesian equation of a plane are the direction ratios of normal to the plane.

Equation of Plane in Normal Form

Vector Form



The equation of plane having normal unit vector $\hat{\mathbf{n}}$ to the plane is $\mathbf{r} \cdot \hat{\mathbf{n}} = \mathrm{d}$, where d is the perpendicular distance of the plane from origin and $\mathbf{r}$ in the position vector of any point P on the plane and $\hat{\mathbf{n}}$ is the unit normal vector.

Cartesian Form

The equation of a plane, which is at a distance p from origin and the direction cosines of the normal from the origin to the plane are l, m, n is given by $lx + my + nz = p$.

Note The coordinates of foot of perpendicular N from the origin on the plane are (lp, mp, np).

Equation of the Plane Passing Through a Fixed Point

Vector Form

The vector equation of a plane passing through a given point A with position vector $\mathbf{a}$ and perpendicular to a given vector $\mathbf{n}$ is $(\mathbf{r} - \mathbf{a})\,\mathbf{n} = 0$.

Cartesian Form

The equation of a plane passing through a given point (x_1, y_1, z_1) is given by $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

where, a, b, c are direction ratios of normal to the plane.

Intercept Form

The intercept form of equation of plane represented in the form of

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where, a, b and c are intercepts on X, Y and Z-axes, respectively.

Note There is no vector form of plane in intercept form.

For x intercept Put $y = 0$, $z = 0$ in the equation of the plane and obtain the value of x. Similarly, we can determine for other intercepts.

Equation of Plane Passing Through Three Non-collinear Points

Vector Form

The equation of plane passing through three non-collinear points A, B and C with position vectors **a, b** and **c** is

$$(\mathbf{r}-\mathbf{a})\,[(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})]=0$$

where, $\vec{r}$ is the position vector of any point P on the plane.

Cartesian Form

The cartesian equation of a plane passing through three non-collinear points $A\,(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C\,(x_3, y_3, z_3)$ is

$$\begin{bmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{bmatrix}=0.$$

where, $P(x, y, z)$ be any point on the plane.

Equation of Plane Passing Through the Intersection of Two given Planes

Vector Form

The equation of plane passing through the intersection of the planes $\mathbf{r}\cdot\mathbf{n}_1 = d_1$ and $\mathbf{r}\cdot\mathbf{n}_2 = d_2$ is $\mathbf{r}\cdot(\mathbf{n}_1+\lambda\,\mathbf{n}_2)=d_1+\lambda\,d_2$, where λ is a scalar.

Cartesian Form

The carteian equation of plane passing through the intersection of two planes $a_1x + b_1\,y + c_1\,z - d_1$ and $a_2\,x + b_2\,y + c_2z - d_2 = 0$
is $(a_1\,x + b_1\,y + c_1\,z - d_1) + \lambda\,(a_2\,x + b_2\,y + c_2\,z - d_2) = 0$
or $x(a_1 + \lambda\,a_2) + y(b_1 + \lambda\,b_2) + z(c_1 + \lambda\,c_2) = d_1 + \lambda\,d_2$, where $\lambda \in R$.

Equation of a Plane Parallel to a Given Plane

Vector Form

The vector equation of a plane parallel to the given plane $\mathbf{r}.\,\mathbf{n} = d_1$ is $\mathbf{r}\cdot\mathbf{n} = d_2$.

Cartesian Form

The cartesian equation of a plane parallel to the given plane $ax + by + cz + d_1 = 0$ is $ax + by + cz + d_2 = 0$.

Important Results

(i) Equation of a plane passing through the point $A(x_1, y_1, z_1)$ and parallel to two given lines with direction ratios

a_1, b_1, c_1 and a_2, b_2, c_2 is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$

(ii) Equation of a plane passing through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ and parallel to a line with direction ratios a, b, c is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a & b & c \end{vmatrix} = 0.$$

(iii) Four points $A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ are coplanar if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0.$$

Condition for Coplanarity of Two Lines

Vector Form

Two lines $\vec{\mathbf{r}} = \mathbf{a}_1 + \lambda\, \mathbf{b}_1$ and $\vec{\mathbf{r}} = \mathbf{a}_2 + \mu\, \mathbf{b}_2$ are coplanar or intersecting if $(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2) = 0$ i.e $(\mathbf{a}_2 - \mathbf{a}_1)$ is perpendicular to $(\mathbf{b}_1 \times \mathbf{b}_2)$.

Cartesian Form

The lines $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and $\dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$

are coplanar if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$

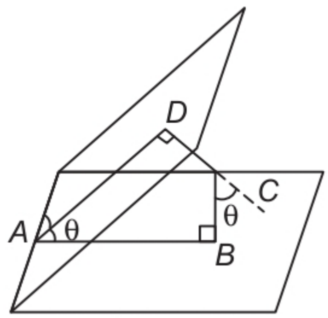
Angle between Two Planes

The angle between two planes is defined as the angle between their normals.

Vector Form

If $\mathbf{n}_1$ and $\mathbf{n}_2$ are normals to the planes, and θ be the angle between the planes $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\mathbf{r} . \mathbf{n}_2 = d_2$.

Then,
$$\cos\theta = \left| \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} \right|$$



Cartesian Form

The angle between the two planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

$$\cos\theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

Parallelism and Perpendicularity of Two Planes

Two planes are parallel or perpendicular according as their normals are parallel or perpendicular.

Vector Form

Two planes $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = d_2$ are parallel, if $\mathbf{n}_1 = \lambda\, \mathbf{n}_2$ for some scalar and perpendicular, if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$.

Cartesian Form

The planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ and perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Note The equation of plane parallel to a given plane $ax + by + cz + d = 0$ is given by $ax + by + cz + k = 0$, where k may be determined from given conditions.

Distance of a Point From a Plane

Vector Form

Let the equation of plane be $\mathbf{r} \cdot \mathbf{n} = d$. The perpendicular distance from a point P whose position vector is $\mathbf{a}$, to the plane is

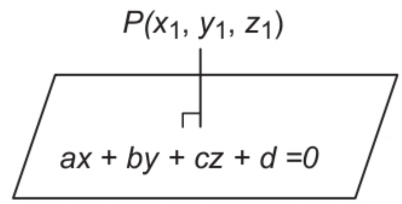
$$\frac{|\mathbf{a} \cdot \mathbf{n} - d|}{|\mathbf{n}|}$$

Note The length of perpendicular from origin to the plane $\mathbf{r} \cdot \mathbf{n} = d$ is $\frac{|d|}{|\mathbf{n}|}$.

Cartesian Form

The perpendicular distance of a point $P(x_1, y_1, z_1)$ from the plane

$ax + by + cz + d = 0$ is $\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$.



If the plane is given in normal form $lx + my + nz = p$. Then, the distance of the point $P(x_1, y_1, z_1)$ from the plane is $|lx_1 + my_1 + nz_1 - p|$.

Note The length of perpendicular from origin to the plane $ax + by + cz + d = 0$ is $\frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$.

Distance between Two Parallel Planes

If $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ be equation of two parallel planes. Then, the distance between them is

$$\left|\frac{d_2 - d_1}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Angle between a Line and a Plane

The angle between a line and plane is the complement of the angle between the line and normal to the plane.

Vector Form

If the equation of a line is $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ and plane is $\mathbf{r} \cdot \mathbf{n} = d$, then the angle between the line and normal is

$$\cos\theta = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}||\mathbf{b}|}$$

and the angle between the line and plane is

$$\sin\phi = \frac{\mathbf{n} \cdot \mathbf{b}}{|\mathbf{n}||\mathbf{b}|} \qquad [\because \phi = 90° - \theta]$$

Cartesian Form

The angle between a line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and normal to the plane $a_2x + b_2y + c_2z + d_2 = 0$ is

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

and the angle between a line and the plane is

$$\sin\phi = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \qquad [\because \phi = 90° - \theta]$$

Bisectors of Angles between Two Planes

The bisector planes of the angles between the planes

$$a_1x + b_1y + c_1z + d_1 = 0, a_2x + b_2y + c_2z + d_2 = 0 \text{ is}$$

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{\Sigma a_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{\Sigma a_2^2}}$$

One of these planes will bisect the acute angle and the other obtuse angle between the given plane.

(i) If $a_1a_2 + b_1b_2 + c_1c_2 < 0$, then origin lies is in acute angle and the acute angle bisector is obtained by taking positive sign in the above equation. The obtuse angle bisector is obtained by taking negative sign in the above equation.

(ii) If $a_1a_2 + b_1b_2 + c_1c_2 > 0$, then origin lies in obtuse angle and the obtuse angle bisector is obtained by taking positive sign in above equation. Acute angle bisector is obtained by taking negative sign.

Important Points to be Remembered

(i) The image or reflection (x, y, z) of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$ is given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

(ii) The foot (x, y, z) of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$ is given by

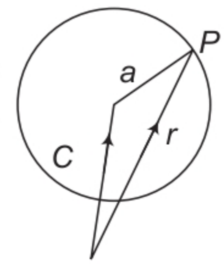
$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \frac{-(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

Sphere

A sphere is the locus of a point which moves in a space, such a way that its distance from a fixed point always remains constant.

General Equation of the Sphere

Vector Form



The vector equation of a sphere of radius a and centre having position vector $\mathbf{c}$ is $|\mathbf{r} - \mathbf{c}| = a$. The vector equation of sphere of radius a with centre at the origin, is $|\vec{r}| = a$.

Cartesian Form

The equation of the sphere with centre (a, b, c) and radius r is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2 \quad \text{...(i)}$$

the equation of a sphere with centre at origin and radius r is $x^2 + y^2 + z^2 = r^2$.

In generally, we can write as $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.

Here, its centre is $(-u, -v, -w)$ and radius $= \sqrt{u^2 + v^2 + w^2 - d}$

Important Points to be Remembered

(i) The general equation of second degree in x, y, z is

$$ax^2 + by^2 + cz^2 + 2hxy + 2kyz + 2lzx + 2ux + 2vy + 2wz + d = 0$$

represents a sphere, if

(a) $a = b = c \, (\neq 0)$ (b) $h = k = l = 0$

Then, the equation becomes

$$ax^2 + ay^2 + az^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{...(i)}$$

To find its centre and radius first we make the coefficients of x^2, y^2 and z^2 each unity by dividing throughout by a.

Thus, we have $x^2 + y^2 + z^2 + \frac{2u}{a}x + \frac{2v}{a}y + \frac{2w}{a}z + \frac{d}{a} = 0 \quad \text{...(ii)}$

$\therefore$ Centre is $\left(\frac{-u}{a}, \frac{-v}{a}, \frac{-w}{a}\right)$

and radius $= \sqrt{\frac{u^2}{a^2} + \frac{v^2}{a^2} + \frac{w^2}{a^2} - \frac{d}{a}} = \frac{\sqrt{u^2 + v^2 + w^2 - ad}}{|a|}$.

(ii) Any sphere concentric with the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

is $$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + k = 0$$

Contd. ...

(iii) Since, $r^2 = u^2 + v^2 + w^2 - d$, therefore, the Eq. (ii) represents a real sphere, if $u^2 + v^2 + w^2 - d > 0$.

(iv) The equation of a sphere on the line joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2) as a diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0.$$

(v) The equation of a sphere passing through four non-coplanar points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) is

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

Condition for Tangent Plane to a Sphere

We know that plane touch the sphere, if the perpendicular distance from centre to the sphere is equal to the radius.

Vector Form

The plane $\mathbf{r} \cdot \mathbf{n} = d$ touches the sphere the $|\mathbf{r} - \mathbf{c}| = a$, if $\dfrac{|\mathbf{c}\,\mathbf{n} - d|}{|\mathbf{n}|} = a$.

Cartesian Form

The plane $lx + my + nz = p$ will touch the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, if

$$\frac{|lu + mv + nw + p|}{\sqrt{l^2 + m^2 + n^2}} = \sqrt{u^2 + v^2 + w^2 - d}$$

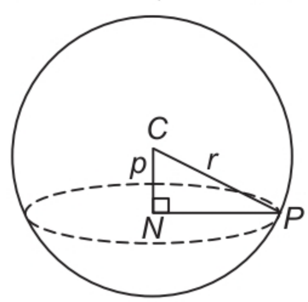
or $$(lu + mv + nw + p)^2 = (u^2 + v^2 + w^2 - d)(l^2 + m^2 + n^2)$$

Plane Section of a Sphere

Consider a sphere intersected by a plane. The set of points common to both sphere and plane is called a plane section of a sphere.

In ΔCNP, $NP^2 = CP^2 - CN^2 = r^2 - p^2$ $\quad [\therefore NP = \sqrt{r^2 - p^2}]$

Hence, the locus of P is a circle whose centre is at the point N, the foot of the perpendicular from the centre of the sphere to the plane.



The section of sphere by a plane through its centre is called a great circle. The centre and radius of a great circle are the same as those of the sphere.