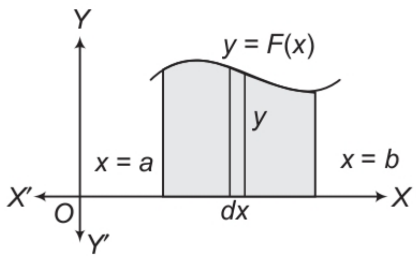
Applications of Integrals

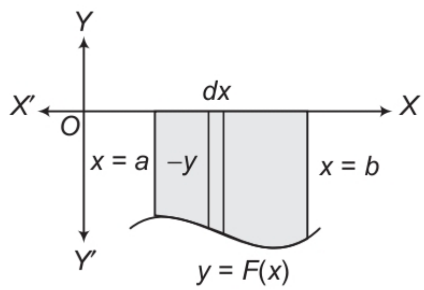
The space occupied by the curve along with the axis, under the given condition is called **area of bounded region**.

(i) The area bounded by the curve $y = F(x)$ above the X-axis and between the lines $x = a$, $x = b$ is given by



$$\int_a^b y\, dx = \int_a^b F(x)\, dx$$

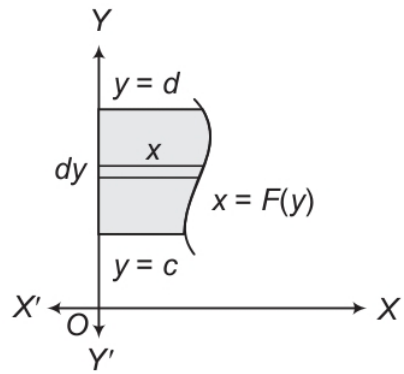
(ii) If the curve between the lines $x = a$, $x = b$ lies below the X-axis, then the required area is given by



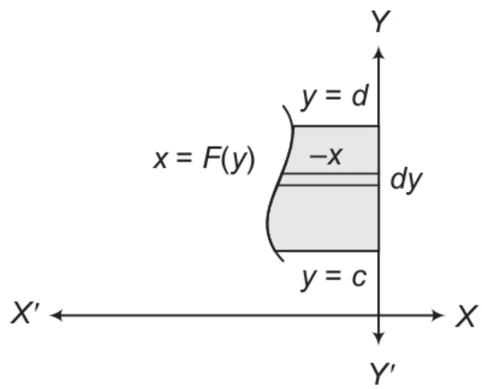
$$\left|\int_a^b (-y)\, dx\right| = \left|-\int_a^b y\, dx\right| = \left|-\int_a^b F(x)\, dx\right|$$

(iii) The area bounded by the curve $x = F(y)$ right to the Y-axis and between the lines $y = c$, $y = d$ is given by

$$\int_c^d x\,dy = \int_c^d F(y)\,dy$$

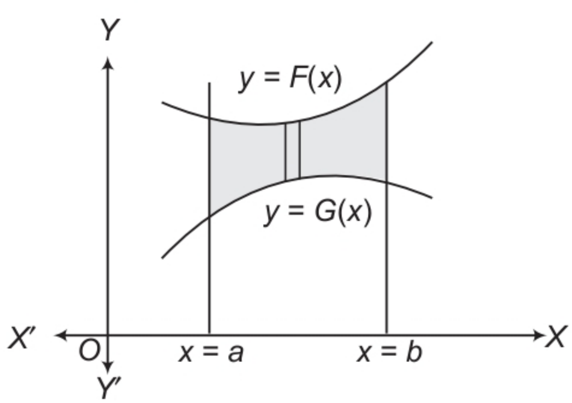


(iv) If the curve between the lines $y = c$, $y = d$ left to the Y-axis, then the area is given by



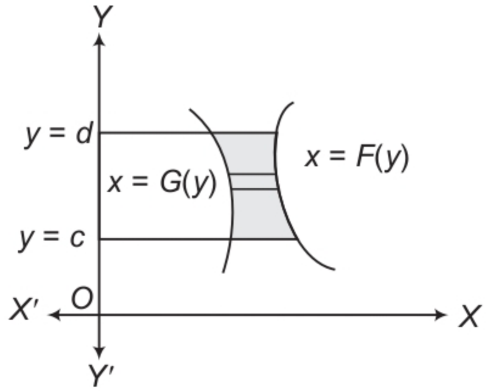
$$\left|\int_c^d (-x)\,dy\right| = \left|-\int_c^d x\,dy\right| = \left|-\int_c^d F(y)\,dy\right|$$

(v) Area bounded by two curves $y = F(x)$ and $y = G(x)$ between $x = a$ and $x = b$ is given by



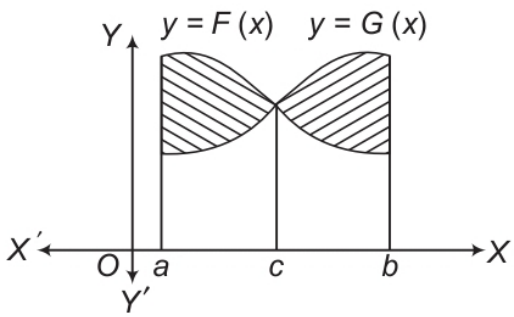
$$\int_a^b \{F(x) - G(x)\}\,dx$$

(vi) Area bounded by two curves $x = F(y)$ and $x = G(y)$ between $y = c$ and $y = d$ is given by



$$\int_c^d [F(y) - G(y)]\, dy$$

(vii) If $F(x) \geq G(x)$ in $[a, c]$ and $F(x) \leq G(x)$ in $[c, d]$, where $a < c < b$, then area of the region bounded by the curves is given as



$$\text{Area} = \int_a^c \{F(x) - G(x)\}\, dx + \int_c^b \{G(x) - F(x)\} dx$$

Area of Curves Given by Polar Equations

Let $f(\theta)$ be a continuous function, $\theta \in (\alpha, \beta)$, then the area bounded by the curve $r = f(\theta)$ and radius α, β $(\alpha < \beta)$ is

$$A = \frac{1}{2}\int_\alpha^\beta r^2 d\theta$$

Area of Curves Given by Parametric Curves

Let $x = \phi(t)$ and $y = \psi(t)$ be two parametric curves, then area bounded by the curve, X-axis and ordinates $x = \phi(t_1), x = \psi(t_2)$ is

$$A = \left| \int_{t=t_1}^{t=t_2} y \times \left(\frac{dx}{dt}\right) dt \right|$$

Curve Sketching

1. Symmetry

(i) If powers of y in an equation of curve are all even, then curve is symmetrical about X-axis.

(ii) If powers of x in an equation of curve are all even, then curve is symmetrical about Y-axis.

(iii) When x is replaced by $-x$ and y is replaced by $-y$, then curve is symmetrical in opposite quadrant.

(iv) If x and y are interchanged and equation of curve remains unchanged, then curve is symmetrical about line $y = x$.

2. Nature of Origin

(i) If point $(0, 0)$ satisfies the equation, then curve passes through origin.

(ii) If curve passes through origin, then equate lowest degree term to zero and get equation of tangent. If there are two tangents, then origin is a double point.

3. Point of Intersection with Axes

(i) Put $y = 0$ and get intersection with X-axis, put $x = 0$ and get intersection with Y-axis.

(ii) Now, find equation of tangent at this point i.e. shift origin to the point of intersection and equate the lowest degree term to zero.

(iii) Find regions where curve does not exists i.e. curve will not exit for those values of variable when makes the other imaginary or not defined.

4. Asymptotes

(i) Equate coefficient of highest power of x to get asymptote parallel to X-axis.

(ii) Similarly equate coefficient of highest power of y to get asymptote parallel to Y-axis.

5. The Sign of $\frac{dy}{dx}$

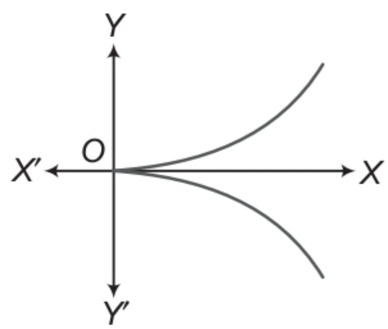
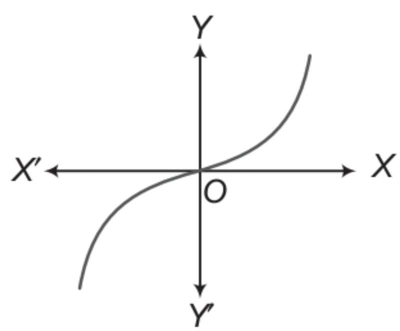
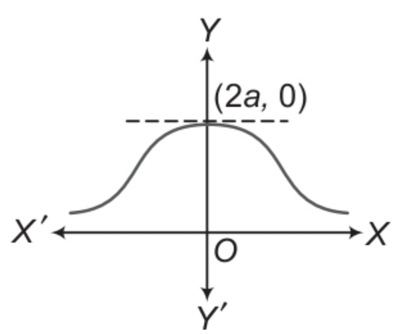
Find points at which $\frac{dy}{dx}$ vanishes or becomes infinite. It gives us the points where tangent is parallel or perpendicular to the X-axis.

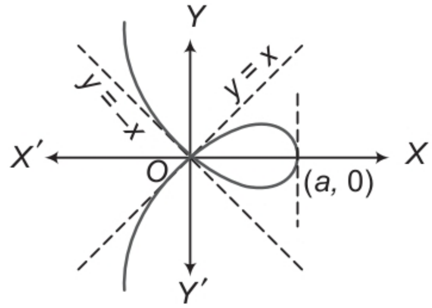
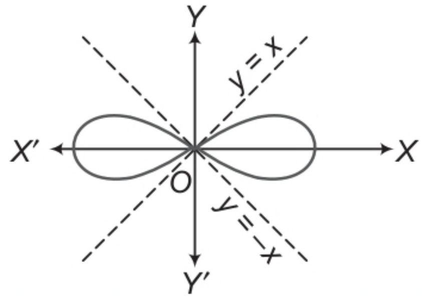
6. Points of Inflexion

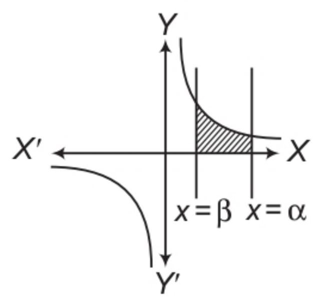
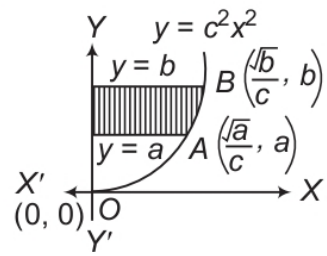
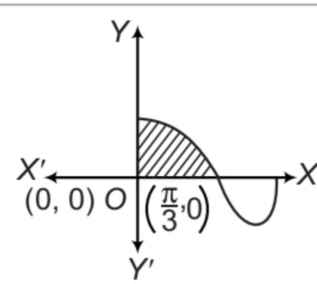
Put $\frac{d^2y}{dx^2} = 0$ or $\frac{d^2x}{dy^2} = 0$ and solve the resulting equation. If some point of inflexion is there, then locate it exactly.

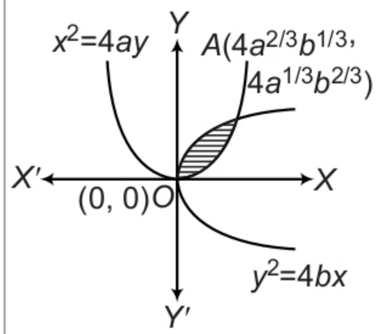
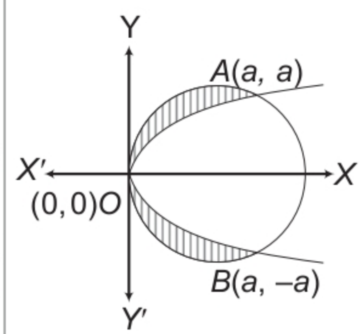
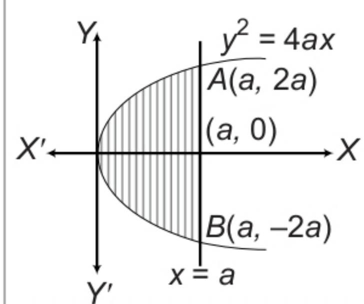
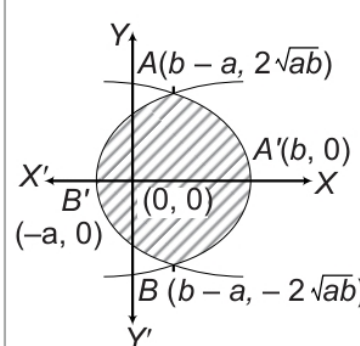
Taking in consideration of all above information, we draw an approximate shape of the curve.

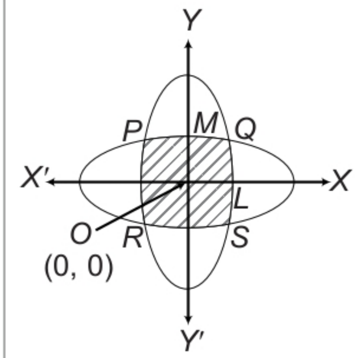
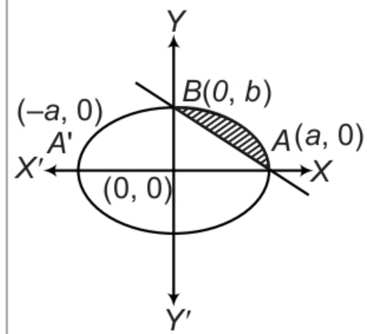
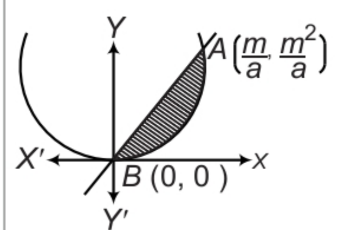
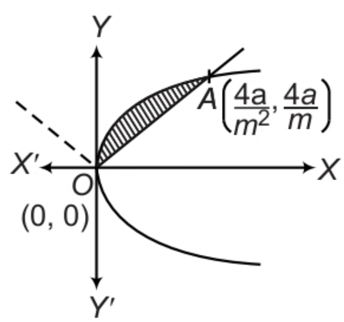
Shapes of Some Curves

S.No.	Equation	Curve
(i)	$ay^2 = x^3$ (Semi-cubical parabola)	
(ii)	$y = x^3$ (Cubical parabola)	
(iii)	$(x^2 + 4a^2)y = 8a^3$	

S.No.	Equation	Curve
(iv)	$ay^2 = x^2(a - x)$	
(v)	$a^2y^2 = x^2(a^2 - x^2)$	

S.No.	Equation	Intersection points	Area of shaded region	Graph
(i)	If $\alpha, \beta > 0, \alpha > \beta$, then area bounded by the curve $xy = p^2$, X-axis and ordinate $x = \alpha, x = \beta$	—	$p^2 \log\left(\frac{\alpha}{\beta}\right)$ sq units	
(ii)	Area between the curve $y = c^2x^2$, Y-axis and line $y = a$, $y = b$	O (0,0), $A\left(\frac{\sqrt{a}}{c}, a\right)$, $B\left(\frac{\sqrt{b}}{c}, b\right)$	$\frac{2(b^{3/2} - a^{3/2})}{3c}$ sq units	
(iii)	$y = k\cos 3x$, $\forall\, 0 \le x \le \frac{\pi}{6}$,	when $0 \le x \le \frac{\pi}{6}$, then $0 \le 3x \le \frac{\pi}{2}$	$\frac{k}{3}$ sq units	

S.No.	Equation	Intersection points	Area of shaded region	Graph
(iv)	$f(x, y); x^2 = 4ay$, $y^2 = 4bx$	$O(0,0)$ $A(4a^{2/3}b^{1/3}, 4a^{1/3}b^{2/3})$	$\frac{16}{3}(ab)$ sq units	
(v)	$f(x, y)$; $x^2 + y^2 \le 2ax$ and $y^2 \ge ax$	$O(0,0), A(a, a), B(a, -a)$	(i) For $x \ge 0, y \ge 0$ Area $= a^2\left(\frac{\pi}{4} - \frac{2}{3}\right)$ sq units (ii) For $x \ge 0$, Area $= 2a^2\left(\frac{\pi}{4} - \frac{2}{3}\right)$ sq units	
(vi)	Area bounded by the parabola $y^2 = 4ax$ and its latus rectum $x = a$	$A(a, 2a)$, $B(a, -2a)$	$\frac{8}{3}a^2$ sq units	
(vii)	Area bounded by the curves $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$	$A(b-a, 2\sqrt{ab})$, $B(b-a, -2\sqrt{ab})$	$\frac{8}{3}\sqrt{ab}\,(a + b)$ sq units	

S.No.	Equation	Intersection points	Area of shaded region	Graph
(viii)	Common area between $\frac{x^2}{b^2}+\frac{y^2}{a^2}=\frac{1}{a^2b^2}$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{a^2b^2}$	$\left(\pm\frac{1}{\sqrt{a^2+b^2}}, \pm\frac{1}{\sqrt{a^2+b^2}}\right)$	Area of region $PQRS$ $=4\times$ Area of $OLQM$ $\frac{4}{ab}\tan^{-1}\left(\frac{a}{b}\right)$ sq units	
(ix)	$f(x, y)$; $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$, $\frac{x}{a}+\frac{y}{b}\ge 1$ or $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1 \le \frac{x}{a}+\frac{y}{b}$	$A(a, 0)$, $B(0, b)$	$ab\cdot\frac{(\pi-2)}{4}$ sq units	
(x)	$f(x, y)$; $ax^2 \le y \le mx$ $\therefore y = ax^2$, $y = mx$	$B(0, 0)$, $A\left(\frac{m}{a}, \frac{m^2}{a}\right)$	$\frac{1}{6}\cdot\frac{m^3}{a^2}$ sq units	
(xi)	$f(x, y)$; $y^2 = 4ax$ and $y = \lvert mx \rvert$	$O(0, 0)$, $A\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$	$\frac{8a^2}{3m^3}$ sq units	

Volume and Surface Area

If we revolve any plane curve along any line, then solid so generated is called solid of revolution.

1. Volume of Solid Revolution

(i) The volume of the solid generated by revolution of the area bounded by the curve $y = f(x)$, X-axis and the ordinates $x = a, x = b$ is $\int_a^b \pi y^2 dx$, it is being given that $f(x)$ is a continuous function in the interval (a, b).

(ii) The volume of the solid generated by revolution of the area bounded by the curve $x = g(y)$, the axis of Y and two abscissae $y = c$ and $y = d$ is $\int_c^d \pi x^2 dy$, it is being given that $g(y)$ is a continuous function in the interval (c, d).

2. Surface Area of Solid Revolution

(i) The surface area of the solid generated by revolution of the area bounded by the curve $y = f(x)$, X-axis and the ordinates $x = a, x = b$ is $2\pi \int_a^b y \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx$, it is being given that $f(x)$ is a continuous function in the interval (a, b).

(ii) The surface area of the solid generated by revolution of the area bounded by the curve $x = f(y)$, Y-axis and $y = c, y = d$ is $2\pi \int_c^d x \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}\, dy$, it is being given that $f(y)$ is a continuous function in the interval (c, d).