Differential Equations

Differential Equation

An equation that involves an independent variable, dependent variable and differential coefficients of dependent variable with respect to the independent variable is called a **differential equation**.

e.g. (i) $x^2\left(\frac{d^2y}{dx^2}\right)+x^3\left(\frac{dy}{dx}\right)^3=7x^2y^2$

(ii) $(x^2+y^2)\,dx=(x^2-y^2)\,dy$

Order and Degree of a Differential Equation

The **order** of a differential equation is the order of the highest derivative occuring in the equation. The order of a differential equation is always a positive integer.

The **degree** of a differential equation is the exponent of the derivative of the highest order in the equation, when the equation is a polynomial in derivatives, i.e. in y', y'', y''' etc.

e.g. The order and degree of a differential equation

$\left(\frac{d^3y}{dx^3}\right)^2+2\left(\frac{d^2y}{dx^2}\right)^3+3y=0$ are 3 and 2 respectively.

Note If the differential equation is not a polynomial equation in derivatives, then its degree is not defined.

e.g. Degree of $\frac{dy}{dx}+\cos\left(\frac{dy}{dx}\right)=0$ is not defined,

as $\frac{dy}{dx}+\cos\left(\frac{dy}{dx}\right)=0$ is not a polynomial in derivatives.

Linear and Non-Linear Differential Equations

A differential equation is said to be linear, if the dependent variable and all of its derivatives occuring in the first power and there are no product of these.

A linear equation of nth order can be written in the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y = Q$$

where, $P_0, P_1, P_2, \dots, P_{n-1}$, P_n and Q must be either constants or functions of x only.

A linear differential equation is always of the first degree but every differential equation of the first degree need not be linear.

e.g. The equations $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + xy = 0$, $x \frac{d^2y}{dx^2} + y \frac{dy}{dx} + y = x^3$

and $\left(\frac{dy}{dx}\right) \frac{d^2y}{dx^2} + y = 0$ are not linear.

Solution of Differential Equations

A solution of a differential equation is a relation between the variables, of the equation not involving the differential coefficients, such that it satisfy the given differential equation (i.e., from which the given differential equation can be derived).

e.g. $y = A\cos x + B\sin x$ is a solution of $\frac{d^2y}{dx^2} + y = 0$, because it satisfy this equation.

1. General Solution

If the solution of the differential equation contains as many independent arbitrary constants as the order of the differential equation, then it is called the general solution or the complete integral of the differential equation.

e.g. The general solution of $\frac{d^2y}{dx^2} + y = 0$ is $y = A\cos x + B\sin x$ because it contains two arbitrary constants A and B, which is equal to the order of the equation.

2. **Particular Solution**

Solution obtained by giving particular values to the arbitrary constants in the general solution is called a particular solution. e.g. In the previous example, if $A = B = 1$, then $y = \cos x + \sin x$ is a particular solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Solution of a differential equation is also called its **primitive**.

Formation of Differential Equation

Suppose we have an equation $f(x, y, c_1, c_2, \dots, c_n) = 0$, where $c_1, c_2, \dots c_n$ are n arbitrary constants.

Then, to form a differential equation differentiate the equation successively n times to get n equations.

Eliminate the arbitrary constants from the $(n + 1)$ equations (the given equation and the n equations obtained above), which leads to the required differential equations.

Solutions of Differential Equations of the First Order and First Degree

A differential equation of first degree and first order can be solved if they belong to any of the following standard forms.

1. **Equation of the form**

$$f(f_1(x, y))\, d(f_1(x, y)) + \phi(f_2(x, y))d(f_2(x, y)) + \dots = 0$$

If the differential equation can be written as $f\,[f_1(x, y)]d\,\{f_1(x, y)\} + \phi\,[f_2(x, y)]\,d\,\{f_2(x, y)\} + \dots = 0$, then each term can be integrated separately.

For this, remember the following results

(i) $x\,dy + y\,dx = d\,(xy)$

(ii) $dx + dy = d(x + y)$

(iii) $\frac{x\,dy - y dx}{x^2} = d\left(\frac{y}{x}\right)$

(iv) $\frac{y dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$

(v) $\frac{2xy\,dx - x^2\,dy}{y^2} = d\left(\frac{x^2}{y}\right)$

(vi) $\frac{2xy\,dy - y^2\,dx}{x^2} = d\left(\frac{y^2}{x}\right)$

(vii) $\frac{2xy^2\,dx - 2x^2y\,dy}{y^4} = d\left(\frac{x^2}{y^2}\right)$

(viii) $\frac{2x^2y\,dy - 2xy^2\,dx}{x^4} = d\left(\frac{y^2}{x^2}\right)$

(ix) $\frac{x\,dy + y\,dx}{xy} = d\,(\log xy)$

(x) $\frac{y\,dx - x\,dy}{xy} = d\left(\log \frac{x}{y}\right)$

(xi) $\frac{x\,dy - y\,dx}{xy} = d\left(\log \frac{y}{x}\right)$

(xii) $\frac{dx + dy}{x + y} = d\,(\log\,(x + y))$

(xiii) $\dfrac{x\,dx + y\,dy}{x^2 + y^2} = d\left(\log\sqrt{x^2 + y^2}\right)$ (xiv) $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

(xv) $\dfrac{y\,dx - x\,dy}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$ (xvi) $\dfrac{xdy + ydx}{x^2 y^2} = d\left(\dfrac{-1}{xy}\right)$

(xvii) $\dfrac{ye^x dx - e^x dy}{y^2} = d\left(\dfrac{e^x}{y}\right)$ (xviii) $\dfrac{xe^y dy - e^y dx}{x^2} = d\left(\dfrac{e^y}{x}\right)$

(xix) $\dfrac{xdx + ydy}{\sqrt{x^2 + y^2}} = d(\sqrt{x^2 + y^2})$

(xx) $x^{m-1} \cdot y^{n-1}(mydx + nx\,dy) = d(x^m\ y^n)$

(xxi) $\dfrac{xdy - ydx}{x^2 - y^2} = d\left(\dfrac{1}{2}\log\dfrac{x + y}{x - y}\right)$

(xxii) $\dfrac{f'(x, y)}{[f(x, y)]^n} = \dfrac{d[f(x, y)]^{1-n}}{1 - n}$

(xxiii) $\dfrac{dx}{x^2} - \dfrac{dy}{y^2} = d\left(\dfrac{1}{y} - \dfrac{1}{x}\right)$

2. **Equations in which the Variables are Separable**

If the equation can be reduced into the form $f(x)\,dx = g(y)$, we say that the variables have been separated. On integrating this reduced form solution of given equation is obtained, which is $\int f(x)\,dx = \int g(y)\,dy + C$, where C is an arbitrary constant.

3. **Differential Equation Reducible to Variables Separable Form**

A differential equation of the form $\dfrac{dy}{dx} = f(ax + by + c)$

can be reduced to variables separable form by substituting

$$ax + by + c = z \Rightarrow a + b\frac{dy}{dx} = \frac{dz}{dx}$$

The given equation becomes

$$\frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z) \Rightarrow \frac{dz}{dx} = a + b\,f(z)$$

$$\Rightarrow \qquad \frac{dz}{a + bf(z)} = dx$$

Hence, the variables are separated in terms of z and x.

4. **Homogeneous Differential Equation**

A differential equation of the form

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$$

where, $f(x, y)$ and $g(x, y)$ are homogeneous function of same degree is called a homogeneous differential equation.

This equation can be reduced to the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ or $\frac{dx}{dy} = G\left(\frac{x}{y}\right)$.

To solve $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$, we put $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Then, the given equation reduces to

$$v + x\frac{dv}{dx} = F(v)$$

$$\Rightarrow \quad x\frac{dv}{dx} = F(v) - v$$

which is invariable separable form and so it can be solved in the usual manner.

Similarly, to solve $\frac{dx}{dy} = G\left(\frac{x}{y}\right)$, we put $x = vy$.

Note A function $f(x, y)$ is said to be homogeneous function of degree n, if

$$f(\lambda x, \lambda y) = \lambda^n f(x, y).$$

5. **Differential Equations Reducible to Homogeneous Form**

The differential equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}, \text{ where } a_1b_2 - a_2b_1 \neq 0, \text{i.e.} \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \qquad \text{...(i)}$$

can be reduced to homogeneous form by substituting

$$x = X + h \text{ and } y = Y + k$$

$$\therefore \quad \frac{dY}{dX} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)} \qquad \text{...(ii)}$$

For finding h and k, put $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$.

On solving, we get

$$\frac{h}{b_1c_2 - b_2c_1} = \frac{k}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\Rightarrow \qquad h = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \text{ and } k = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Then, Eq. (ii) reduces to $\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$, which is a homogeneous form and can be solved easily.

6. **Linear Differential Equation**

A linear differential equation of the first order can be either of the following forms

(i) $\frac{dy}{dx} + Py = Q$, where P and Q are the functions of x or constants.

(ii) $\frac{dx}{dy} + Rx = S$, where R and S are the functions of y or constants.

Consider the differential Eq. (i) i.e. $\frac{dy}{dx} + Py = Q$

For this now, integrating factor $\text{(IF)} = e^{\int P\,dx}$

and solution is $ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + C$

i.e. $$y\,\text{(IF)} = \int Q\,\text{(IF)}\,dx + C$$

Similarly, for the second differential equation $\frac{dx}{dy} + Rx = S$, the integrating factor, $\text{IF} = e^{\int R\,dy}$ and the general solution is

$$x \cdot e^{\int R\,dy} = \int S \cdot e^{\int R\,dy}\,dy + C$$

i.e. $$x\,\text{(IF)} = \int S\,(\text{ IF })\,dy + C$$

7. **Differential Equation Reducible to the Linear Form**

Equation of the form $f'(y)\frac{dy}{dx} + f(y)P = Q$, where P and Q are functions of x only or constants, can be reduced to linear form by substituting

i.e. $$f(y) = u \Rightarrow f'(y) = \frac{dy}{dx} = \frac{du}{dx}$$

This will reduce the given equation to $\frac{du}{dx} + uP = Q$,

which is in linear differential equation form and can be solved in the usual manner.

8. Bernoulli's Equation

An equation of the form $\frac{dy}{dx} + Py = Qy^n$ $(n \neq 0,1)$, where P and Q are functions of x only or constants, is called **Bernoulli's equation**.

It is easy to reduce the above equation into linear form as below

Dividing both the sides by y^n, we get

$$y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q$$

Put $$y^{-n+1} = z$$

$\Rightarrow$ $$(-n+1)\, y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$$

Then, the equation reduces to

$$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q \Rightarrow \frac{dz}{dx} + (1-n)\,Pz = Q\,(1-n)$$

which is linear differential equation in z and can be solved in the usual manner.

Orthogonal Trajectory

Any curve, which cuts every member of a given family of curves at right angle, is called an orthogonal trajectory of the family.

Procedure for Finding the Orthogonal Trajectory

(i) Let $f(x, y, c) = 0$ be the equation of the given family of curves, where 'c' is a parameter.

(ii) Differentiate $f = 0$, with respect to 'x' and eliminate c to form a differential equation.

(iii) Substitute $\left(\frac{-dx}{dy}\right)$ in place of $\left(\frac{dy}{dx}\right)$ in the above differential equation. This will give the differential equation of the orthogonal trajectories.

(iv) By solving this differential equation, we get the required orthogonal trajectories.