DCAM classes

Dynamic Classes for Academic Mastery

Inverse Trigonometric Functions

Inverse Function

If $y = f(x)$ and $x = g(y)$ are two functions such that $f(g(y)) = y$ and $g(f(y)) = x$, then f and y are said to be inverse of each other, i.e. $g = f^{-1}$. If $y = f(x)$, then $x = f^{-1}(y)$.

Inverse Trigonometric Functions

As we know that trigonometric functions are not one-one and onto in their natural domain and range, so their inverse do not exist but if we restrict their domain and range, then their inverse may exists.

Domain and Range of Inverse Trigonometric Functions

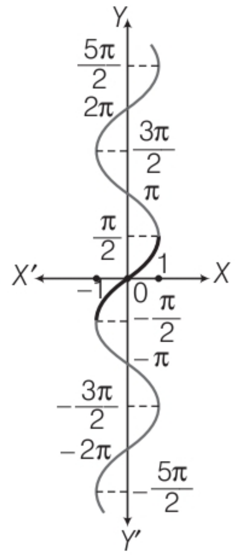
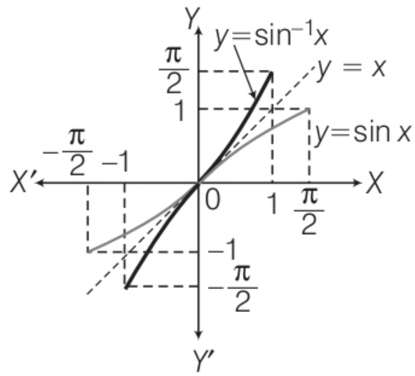
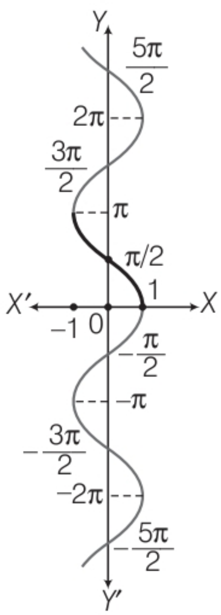
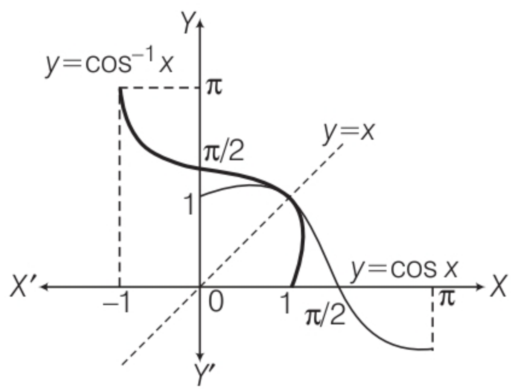
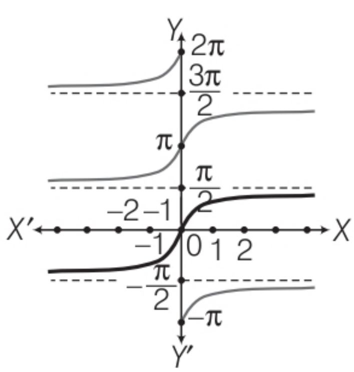
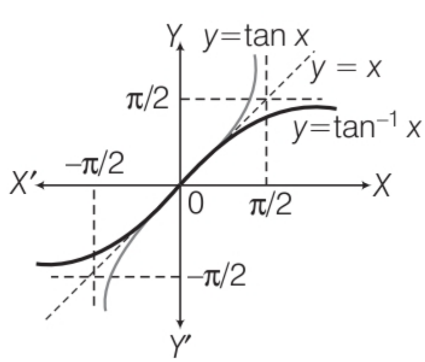
The range of trigonometric functions becomes the domain of inverse trigonometric functions and restricted domain of trigonometric functions becomes range or principal value branch of inverse trigonometric functions.

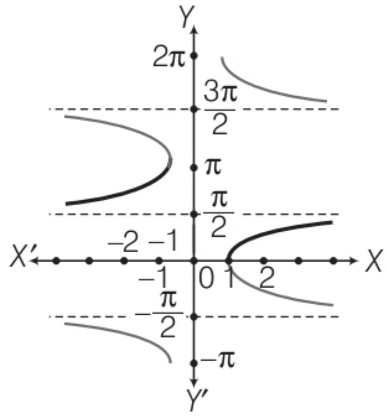
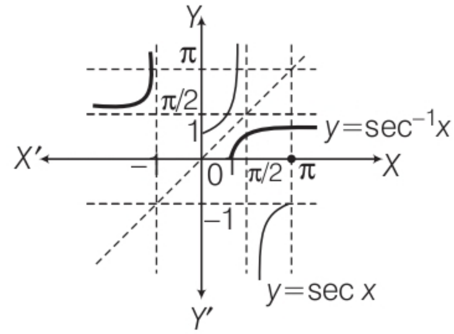
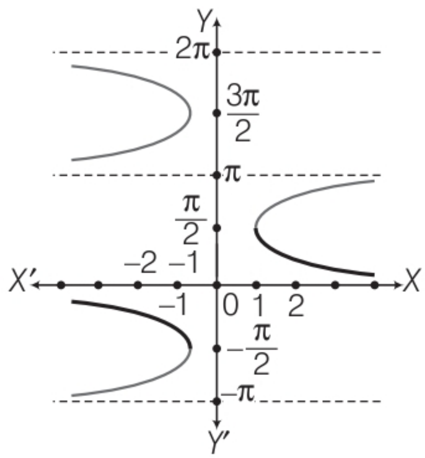
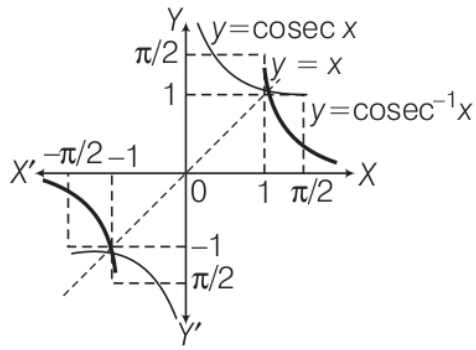
Table for Domain, Range and Other Possible Range of Inverse Trigonometric Functions

Function	Domain	Principal value branch (Range)	Other possible range
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right], \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ etc.
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	$[-\pi, 0], [\pi, 2\pi]$ etc.
$y = \tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ etc.
$y = \sec^{-1} x$	$R - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$	$[-\pi, 0] - \left\{-\frac{\pi}{2}\right\}, [\pi, 2\pi] - \left\{\frac{3\pi}{2}\right\}$ etc.
$y = \operatorname{cosec}^{-1} x$	$R - (-1, 1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$	$\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] - \{-\pi\}, \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$
$y = \cot^{-1} x$	R	$(0, \pi)$	$(-\pi, 0), (\pi, 2\pi)$ etc.

Graphs of Inverse Trigonometric Functions

The graphs of inverse trigonometric functions with respect to line $y = x$ are given in the following table

Function	Graph (By interchanging axes)	Graph (By mirror image)
$y = \sin^{-1} x$		
$y = \cos^{-1} x$		
$y = \tan^{-1} x$		

<table>
<tr><th>Function</th><th>Graph
(By interchanging axes)</th><th>Graph
(By mirror image)</th></tr>
<tr><td>$y = \sec^{-1} x$</td><td>
</td><td>
</td></tr>
<tr><td>$y = \text{cosec}^{-1} x$</td><td>
</td><td>
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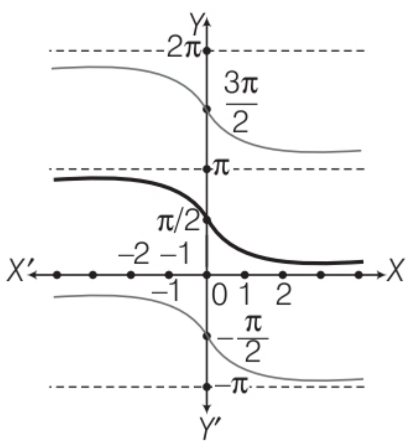
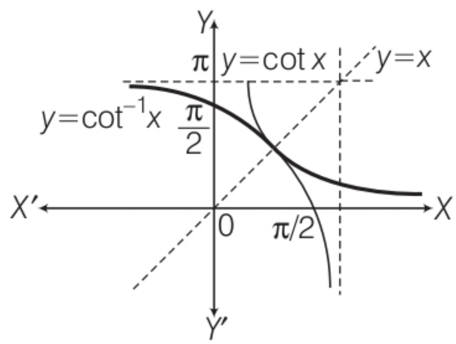
<tr><td>$y = \cot^{-1} x$</td><td>
</td><td>
</td></tr>
</table>

Elementary Properties of Inverse Trigonometric Functions

Property I

(i) $\sin^{-1}(\sin\theta) = \theta;\ \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos\theta) = \theta; \theta \in [0, \pi]$

(iii) $\tan^{-1}(\tan\theta) = \theta; \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta; \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \theta \neq 0$

(v) $\sec^{-1}(\sec\theta) = \theta; \theta \in [0, \pi], \theta \neq \frac{\pi}{2}$

(vi) $\cot^{-1}(\cot\theta) = \theta; \theta \in (0, \pi)$

Property II

(i) $\sin(\sin^{-1} x) = x; x \in [-1, 1]$

(ii) $\cos(\cos^{-1} x) = x; x \in [-1, 1]$

(iii) $\tan(\tan^{-1} x) = x; x \in R$

(iv) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x; x \in (-\infty, -1| \cup |1, \infty)$

(v) $\sec(\sec^{-1} x) = x; x \in (-\infty, -1| \cup |1, \infty)$

(vi) $\cot(\cot^{-1} x) = x; x \in R$

Property III

(i) $\sin^{-1}(-x) = -\sin^{-1} x; x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x; x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x; x \in R$

(iv) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x; x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x; x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1} x; x \in R$

Property IV

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x; x \in (-\infty, -1] \cup [1, \infty)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x; x \in (-\infty, -1] \cup [1, \infty)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x; & \text{if } x > 0 \\ -\pi + \cot^{-1} x; & \text{if } x < 0 \end{cases}$

Property V

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}; x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}; x \in R$

(iii) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}; x \in (-\infty, -1] \cup [1, \infty)$

Property VI

(i) $$\sin^{-1} x + \sin^{-1} y = \begin{cases} \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}; \\ \quad \text{if } -1 \le x, y \le 1 \text{ and } x^2 + y^2 \le 1 \text{ or} \\ \quad \text{if } xy < 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}; \\ \quad \text{if } 0 < x, y \le 1 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}; \\ \quad \text{if } -1 \le x, y < 0 \text{ and } x^2 + y^2 > 1 \end{cases}$$

(ii) $$\sin^{-1} x - \sin^{-1} y = \begin{cases} \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}; \\ \quad \text{if } -1 \le x, y \le 1 \text{ and } x^2 + y^2 \le 1 \\ \quad \text{or if } xy > 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}; \\ \quad \text{if } 0 < x \le 1, -1 \le y \le 0 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}; \\ \quad \text{if } -1 \le x < 0, 0 < y \le 1 \text{ and } x^2 + y^2 > 1 \end{cases}$$

Property VII

(i) $\cos^{-1} x + \cos^{-1} y$

$$= \begin{cases} \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\}; & \text{if } -1 \le x, y \le 1 \text{ and } x + y \ge 0 \\ 2\pi - \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\}; & \text{if } -1 \le x, y \le 1 \text{ and } x + y \le 0 \end{cases}$$

(ii) $\cos^{-1} x - \cos^{-1} y$

$$= \begin{cases} \cos^{-1}\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\}; \text{ if } -1 \le x, y \le 1 \text{ and } x \le y \\ -\cos^{-1}\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\}; \text{ if } -1 \le y \le 0, 0 < x \le 1 \text{ and } x \ge y \end{cases}$$

Property VIII

(i) $\tan^{-1} x + \tan^{-1} y$

$$= \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

(ii) $\tan^{-1} x - \tan^{-1} y$

$$= \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right); & \text{if } xy > -1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right); & \text{if } x > 0, y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right); & \text{if } x < 0, y > 0 \text{ and } xy < -1 \end{cases}$$

Property IX

(i) $$\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right), x \in (0, 1)$$

(ii) $$\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$= \cot^{-1}\frac{x}{\sqrt{1-x^2}} = \sec^{-1}\left(\frac{1}{x}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right), x \in (0, 1)$$

(iii) $$\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\left(\frac{1}{x}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$

$$= \sec^{-1}(\sqrt{1+x^2}), x \in (0, \infty)$$

Property X

(i) $$2\sin^{-1}x = \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}); & \text{if } -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}); & \text{if } \dfrac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}); & \text{if } -1 \le x \le -\dfrac{1}{\sqrt{2}} \end{cases}$$

(ii) $$2\cos^{-1}x = \begin{cases} \cos^{-1}(2x^2-1); & \text{if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2-1); & \text{if } -1 \le x \le 0 \end{cases}$$

(iii) $$2\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{2x}{1-x^2}\right); & \text{if } -1 < x < 1 \\ \pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right); & \text{if } x > 1 \\ -\pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right); & \text{if } x < -1 \end{cases}$$

Property XI

(i) $$3\sin^{-1}x = \begin{cases} \sin^{-1}(3x-4x^3); & \text{if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - \sin^{-1}(3x-4x^3); & \text{if } \dfrac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}(3x-4x^3); & \text{if } -1 \le x < -\dfrac{1}{2} \end{cases}$$

(ii) $$3\cos^{-1}x = \begin{cases} \cos^{-1}(4x^3-3x); & \text{if } \dfrac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3-3x); & \text{if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 2\pi + \cos^{-1}(4x^3-3x); & \text{if } -1 \le x \le -\dfrac{1}{2} \end{cases}$$

(iii) $$3\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right); & \text{if } -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right); & \text{if } x > \dfrac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right); & \text{if } x < -\dfrac{1}{\sqrt{3}} \end{cases}$$

Property XII

(i) $$2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & \text{if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & \text{if } x < -1 \end{cases}$$

(ii) $$2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); & \text{if } 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); & \text{if } -\infty < x < 0 \end{cases}$$

Some Important Results

(i) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right)$,

if $x > 0, y > 0, z > 0$ and $(xy + yz + zx) < 1$

(ii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \dfrac{\pi}{2}$, then $xy + yz + zx = 1$

(iii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then $x + y + z = xyz$

(iv) If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \dfrac{\pi}{2}$, then $x^2 + y^2 + z^2 + 2xyz = 1$

(v) If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, then

$$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$$

(vi) If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$, then $xy + yz + zx = 3$

(vii) If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, then $x^2 + y^2 + z^2 + 2xyz = 1$

(viii) If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \dfrac{3\pi}{2}$, then $xy + yz + zx = 3$

(ix) If $\sin^{-1}x + \sin^{-1}y = \theta$, then $\cos^{-1}x + \cos^{-1}y = \pi - \theta$

(x) If $\cos^{-1}x + \cos^{-1}y = \theta$, then $\sin^{-1}x + \sin^{-1}y = \pi - \theta$

(xi) If $\tan^{-1}x + \tan^{-1}y = \dfrac{\pi}{2}$, then $xy = 1$

(xii) If $\cot^{-1}x + \cot^{-1}y = \dfrac{\pi}{2}$, then $xy = 1$

(xiii) If $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\theta$, then $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\theta+\frac{y^2}{b^2}=\sin^2\theta$

(xiv) $\tan^{-1}x_1+\tan^{-1}x_2+\ldots+\tan^{-1}x_n=\tan^{-1}\left(\frac{S_1-S_3+S_5-\ldots}{1-S_2+S_4-S_6+\ldots}\right)$

where, S_k denotes the sum of the products of $x_1, x_2, \ldots, x_n$ takes k at a time.

Inverse Hyperbolic Functions

If $\sinh y = x$, then y is called the inverse hyperbolic sine of x and it is written as $y=\sinh^{-1}x$.

Similarly, $\cosh^{-1}x$, $\tan\mathrm{h}^{-1}x$ etc., can be defined.

Domain and Range of Inverse Hyperbolic Functions

Function	Domain	Range
$\sinh^{-1}x$	R	R
$\cosh^{-1}x$	$[1, \infty]$	R
$\tanh^{-1}x$	$(-1, 1)$	R
$\operatorname{cosech}^{-1}x$	$R-\{0\}$	$R-\{0\}$
$\operatorname{sech}^{-1}x$	$(0, 1]$	R
$\coth^{-1}x$	$R-[-1, 1]$	$R-\{0\}$

Relation between Inverse Circular Functions and Inverse Hyperbolic Functions

(i) $\sinh^{-1}x=-i\sin^{-1}(ix)$

(ii) $\cosh^{-1}x=-i\cos^{-1}x$

(iii) $\tanh^{-1}x=-i\tan^{-1}(ix)$

Some Important Results

(i) $\sinh^{-1}x=\log_e(x+\sqrt{x^2+1})$

(ii) $\cosh^{-1}x=\log_e(x+\sqrt{x^2-1})$

(iii) $\tanh^{-1}x=\frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$

(iv) $\coth^{-1}x=\frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right), |x|>1$

(v) $\operatorname{sech}^{-1}x=\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right), x\in(0,1]$

(vi) $\operatorname{cosech}^{-1}x=\begin{cases}\log_e\left(\frac{1+\sqrt{1+x^2}}{x}\right), & x>0\\ \log_e\left(\frac{1-\sqrt{1+x^2}}{x}\right), & x<0\end{cases}$