DCAM classes
Dynamic Classes for Academic Mastery

Sets and Relations

Set

Set is a collection of well defined objects which are distinct from each other. Sets are usually denoted by capital letters $A, B, C, \ldots$ and elements are usually denoted by small letters $a, b, c, \ldots$.

If a is an element of a set A, then we write $a \in A$ and say a belongs to A or a is in A or a is a member of A. If a does not belongs to A, we write $a \notin A$.

Standard Notations

N : A set of all natural numbers.
W : A set of all whole numbers.
Z : A set of all integers.
Z^+ / Z^- : A set of all positive/negative integers.
Q : A set of all rational numbers.
Q^+ / Q^- : A set of all positive/negative rational numbers.
R : A set of all real numbers.
R^+ / R^- : A set of all positive/negative real numbers.
C : A set of all complex numbers.

Methods for Describing a Set

(i) **Roster Form / Listing Method / Tabular Form** In this method, a set is described by listing the elements, separated by commas and enclosed within braces.

e.g. If A is the set of vowels in English alphabet, then

$$A = \{a, e, i, o, u\}$$

(ii) **Set Builder Form / Rule Method** In this method, we write down a property or rule which gives us all the elements of the set.

e.g. $A = \{x : x \text{ is a vowel in English alphabet}\}$

Types of Sets

(i) **Empty/Null/Void Set** A set containing no element, it is denoted by ϕ or $\{\}$.

(ii) **Singleton Set** A set containing a single element.

(iii) **Finite Set** A set containing finite number of elements or no element.

Note : **Cardinal Number (or Order) of a Finite Set** The number of elements in a given finite set is called its cardinal number. If A is a finite set, then its cardinal number is denoted by $n(A)$.

(iv) **Infinite Set** A set containing infinite number of elements.

(v) **Equivalent Sets** Two sets are said to be equivalent, if they have same number of elements.

If $n(A) = n(B)$, then A and B are equivalent sets.

(vi) **Equal Sets** Two sets A and B are said to be equal, if every element of A is a member of B and every element of B is a member of A and we write it as $A = B$.

Subset and Superset

Let A and B be two sets. If every element of A is an element of B, then A is called subset of B and B is called superset of A and written as $A \subseteq B$ or $B \supseteq A$.

Power Set

The set formed by all the subsets of a given set A, is called power set of A, denoted by $P(A)$.

Universal Set (U)

A set consisting of all possible elements which occurs under consideration is called a universal set.

Proper Subset

If A is a subset of B and $A \neq B$, then A is called proper subset of B and we write it as $A \subset B$.

Comparable Sets

Two sets A and B are comparable, if $A \subseteq B$ or $B \subseteq A$.

Non-comparable Sets

For two sets A and B, if neither $A \subseteq B$ nor $B \subseteq A$, then A and B are called non-comparable sets.

Disjoint Sets

Two sets A and B are called disjoint, if $A \cap B = \phi$. i.e. they do not have any common element.

Intervals as Subsets of *R*

(i) The set of real numbers x, such that $a \le x \le b$ is called a **closed interval** and denoted by $[a, b]$ i.e. $[a, b] = \{x : x \in R, a \le x \le b\}$.

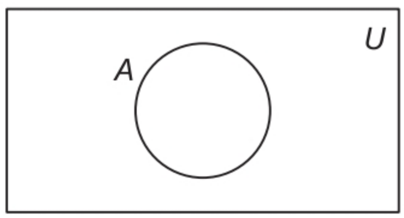
(ii) The set of real number x, such that $a < x < b$ is called an **open interval** and is denoted by (a, b)

i.e. $(a, b) = \{x : x \in R, a < x < b\}$

(iii) The sets $[a, b) = \{x : x \in R, a \le x < b\}$ and $(a, b] = \{x : x \in R, a < x \le b\}$ are called **semi-open** or **semi-closed** intervals.

Venn Diagram

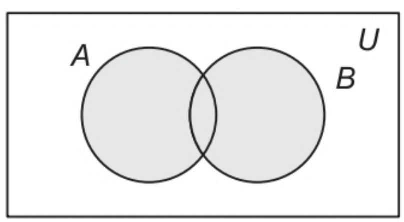
In a Venn diagram, the universal set is represented by a rectangular region and its subset is represented by circle or a closed geometrical figure inside the rectangular region.



Operations on Sets

1. Union of Sets

The union of two sets A and B, denoted by $A \cup B$, is the set of all those elements which are either in A or in B or both in A and B.



Laws of Union of Sets

For any three sets A, B and C, we have

(i) $A \cup \phi = A$ (Identity law)

(ii) $U \cup A = U$ (Universal law)

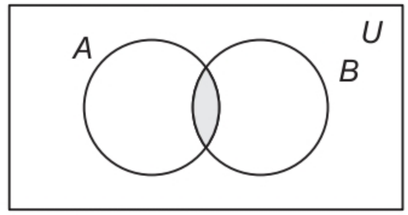
(iii) $A \cup A = A$ (Idempotent law)

(iv) $A \cup B = B \cup A$ (Commutative law)

(v) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law)

2. **Intersection of Sets**

The intersection of two sets A and B, denoted by $A \cap B$, is the set of all those elements which are common to both A and B.



If $A_1, A_2, \ldots, A_n$ is a finite family of sets, then their intersection is denoted by

$$\bigcap_{i=1}^{n} A_i \text{ or } A_1 \cap A_2 \cap \ldots \cap A_n .$$

Laws of Intersection

For any three sets, A, B and C, we have

(i) $A \cap \phi = \phi$ (Identity law)

(ii) $U \cap A = A$ (Universal law)

(iii) $A \cap A = A$ (Idempotent law)

(iv) $A \cap B = B \cap A$ (Commutative law)

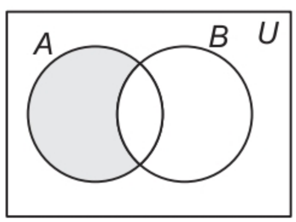
(v) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law)

(vi) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(intersection distributes over union)

(vii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(union distributes over intersection)

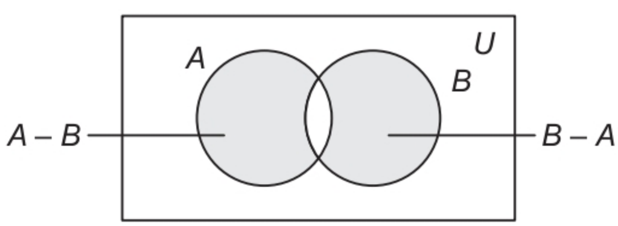
3. **Difference of Sets**

For two sets A and B, the difference $A - B$ is the set of all those elements of A which do not belong to B.



Symmetric Difference

For two sets A and B, symmetric difference is the set $(A - B) \cup (B - A)$ denoted by $A \Delta B$.



Laws of Difference of Sets

(a) For any two sets A and B, we have

(i) $A - B = A \cap B'$ (ii) $B - A = B \cap A'$

(iii) $A - B \subseteq A$ (iv) $B - A \subseteq B$

(v) $A - B = A \Leftrightarrow A \cap B = \phi$

(vi) $(A - B) \cup B = A \cup B$

(vii) $(A - B) \cap B = \phi$

(viii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

(b) If A, B and C are any three sets, then

(i) $A - (B \cap C) = (A - B) \cup (A - C)$

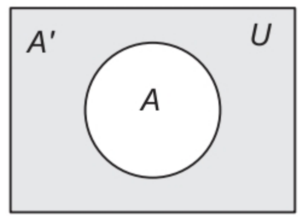
(ii) $A - (B \cup C) = (A - B) \cap (A - C)$

(iii) $A \cap (B - C) = (A \cap B) - (A \cap C)$

(iv) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

4. Complement of a Set

If A is a set with U as universal set, then complement of a set A, denoted by A' or A^c is the set $U - A$.



Properties of Complement of Sets are

(i) $(A')' = A = U - A'$ (law of double complementation)

(ii) (a) $A \cup A' = U$

(b) $A \cap A' = \phi$ (complement laws)

(iii) (a) $\phi' = U$

(b) $U' = \phi$ (laws of empty set and universal set)

(iv) $(A \cup B)' = U - (A \cup B)$

Important Points to be Remembered

(i) Every set is a subset of itself i.e. $A \subseteq A$, for any set A.

(ii) Empty set ϕ is a subset of every set i.e. $\phi \subset A$, for any set A.

(iii) For any set A and its universal set U, $A \subseteq U$

(iv) If $A = \phi$, then power set has only one element, i.e. $n(P(A)) = 1$.

(v) Power set of any set is always a non-empty set.

(vi) Suppose $A = \{1, 2\}$, then $P(A) = \{\{1\}, \{2\}, \{1, 2\}, \phi\}$.

(a) $A \in P(A)$ (b) $\{A\} \notin P(A)$

(vii) If a set A has n elements, then $P(A)$ has 2^n elements.

(viii) Equal sets are always equivalent but equivalent sets may not be equal.

(ix) The set $\{\phi\}$ is not a null set. It is a set containing one element ϕ.

Results on Number of Elements in Sets

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B)$, if A and B are disjoint sets.

(iii) $n(A - B) = n(A) - n(A \cap B)$

(iv) $n(B - A) = n(B) - n(A \cap B)$

(v) $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$

(vi) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$
$$- n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

(vii) n (number of elements in exactly two of the sets A, B, C)
$$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$$

(viii) n (number of elements in exactly one of the sets A, B, C)
$$= n(A) + n(B) + n(C) - 2n(A \cap B)$$
$$- 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$$

(ix) $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$

(x) $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

Ordered Pair

An ordered pair consists of two objects or elements grouped in a particular order.

Equality of Ordered Pairs

Two ordered pairs (a_1, b_1) and (a_2, b_2) are equal iff $a_1 = a_2$ and $b_1 = b_2$.

Cartesian (or Cross) Product of Sets

For two non-empty sets A and B, the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called Cartesian product $A \times B$, i.e.

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

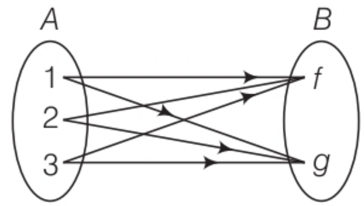
Ordered Triplet

If there are three sets A, B, C and $a \in A$, $b \in B$ and $c \in C$, then we form an ordered triplet (a, b, c). It is also called 3-triple. The set of all ordered triplets (a, b, c) is called the **cartesian product** of three sets A, B and C.

i.e. $$A \times B \times C = \{(a, b, c) : a \in A, b \in B \text{ and } c \in C\}$$

Diagramatic Representation of Cartesian Product of Two Sets

We first draw two circles representing sets A and B one opposite to the other as shown in the given figure and write the elements of sets in the corresponding circles.



Now, we draw line segments starting from each element of set A and terminating to each element of set B.

Properties of Cartesian Product

For three sets A, B and C,

(i) $n(A \times B) = n(A) \times n(B)$

(ii) $A \times B = \phi$, if either A or B is an empty set.

(iii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(iv) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(v) $A \times (B - C) = (A \times B) - (A \times C)$

(vi) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(vii) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$

(viii) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$

(ix) If $A \subseteq B$ and $C \subseteq D$, then $(A \times C) \subseteq (B \times D)$

(x) If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$

(xi) If $A \subseteq B$, then $A \times C \subseteq B \times C$ for any set C.

(xii) $A \times B = B \times A \Leftrightarrow A = B$

(xiii) If $A \neq B$, then $A \times B \neq B \times A$

(xiv) If either A or B is an infinite set, then $A \times B$ is an infinite set.

(xv) If A and B be any two non-empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.

Relation

If A and B are two non-empty sets, then a relation R from A to B is a subset of $A \times B$.

If $R \subseteq A \times B$ and $(a, b) \in R$, then we say that a is related to b by the relation R, written as aRb.

If $R \subseteq A \times A$, then we simply say R is a relation on A.

Representation of a Relation

(i) **Roster form** In this form, we represent the relation by the set of all ordered pairs belongs to R.

e.g. Let R is a relation from set $A = \{-3, -2, -1, 1, 2, 3\}$ to set $B = \{1, 4, 9, 10\}$, defined by $aRb \Leftrightarrow a^2 = b$,

Then, $(-3)^2 = 9, (-2)^2 = 4, (-1)^2 = 1, (2)^2 = 4, (3)^2 = 9$.

Then, in roster form, R can be written as

$$R = \{(-1, 1), (-2, 4), (1, 1), (2, 4), (-3, 9), (3, 9)\}$$

(ii) **Set-builder form** In this form, we represent the relation R from set A to set B as

$R = \{(a, b):\ a \in A, b \in B$ and the rule which relate the elements of A and $B\}$

e.g. Let R is a relation from set $A = \{1, 2, 4, 5\}$ to set $B = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}\right\}$ such that

$$R = \left\{(1, 1), \left(2, \frac{1}{2}\right)\left(4, \frac{1}{4}\right)\left(5, \frac{1}{5}\right)\right\}$$

Then, in set-builder form, R can be written as

$$R = \left\{(a, b) : a \in A, b \in B \text{ and } b = \frac{1}{a}\right\}$$

Note We cannot write every relation from set A to set B in set-builder form.

Domain, Codomain and Range of a Relation

Let R be a relation from a non-empty set A to a non-empty set B. Then, set of all first components or coordinates of the ordered pairs belonging to R is called the **domain** of R, while the set of all second components or coordinates of the ordered pairs belonging to R is called the **range** of R Also, the set B is called the **codomain** of relation R.

Thus, domain of $R = \{a : (a, b) \in R\}$ and range of $R = \{b : (a, b) \in R\}$

Types of Relations

(i) **Empty or Void Relation** As $\phi \subset A \times A$, for any set A, so ϕ is a relation on A, called the empty or void relation.

(ii) **Universal Relation** Since, $A \times A \subseteq A \times A$, so $A \times A$ is a relation on A, called the universal relation.

(iii) **Identity Relation** The relation $I_A = \{(a, a) : a \in A\}$ is called the identity relation on A.

(iv) **Reflexive Relation** A relation R on a set A is said to be reflexive relation, if every element of A is related to itself.

Thus, $(a, a) \in R, \forall\ a \in A \Rightarrow R$ is reflexive.

(v) **Symmetric Relation** A relation R on a set A is said to be symmetric relation iff $(a, b) \in R \Rightarrow (b, a) \in R, \forall\ a, b \in A$

i.e. $$a\ R\ b \Rightarrow bRa, \forall\ a, b \in A$$

(vi) **Transitive Relation** A relation R on a set A is said to be transitive relation, iff $(a, b) \in R$ and $(b, c) \in R$

$$\Rightarrow \quad (a, c) \in R, \forall\ a, b, c \in A$$

Equivalence Relation

A relation R on a set A is said to be an equivalence relation, if it is simultaneously reflexive, symmetric and transitive on A.

Equivalence Classes

Let R be an equivalence relation on A $(\neq \phi)$. Let $a \in A$.

Then, the equivalence class of a denoted by $[a]$ or (a) is defined as the set of all those points of A which are related to a under the relation R.

Inverse Relation

If A and B are two non-empty sets and R be a relation from A to B, then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$.

Composition of Relation

Let R and S be two relations from sets A to B and B to C respectively, then we can define relation SoR from A to C such that $(a, c) \in SoR \Leftrightarrow \exists\ b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

This relation SoR is called the composition of R and S.

(i) $RoS \neq SoR$ (ii) $(SoR)^{-1} = R^{-1}oS^{-1}$ known as **reversal rule**.

Important Results on Relation

(i) If R and S are two equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.

(ii) The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.

(iii) If R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.

(vi) Let A and B be two non-empty finite sets consisting of m and n elements, respectively. Then, $A \times B$ consists of mn ordered pairs. So, the total number of relations from A to B is 2^{nm}.

(v) If a set A has n elements, then number of reflexive relations from A to A is $2^{n^2 - n}$.

Functions and Binary Operations

Function

Let A and B be two non-empty sets, then a function f from set A to set B is a rule which associates each element of A to a unique element of B.

It is represented as $f: A \to B$ or $A \xrightarrow{f} B$ and function is also called the **mapping**.

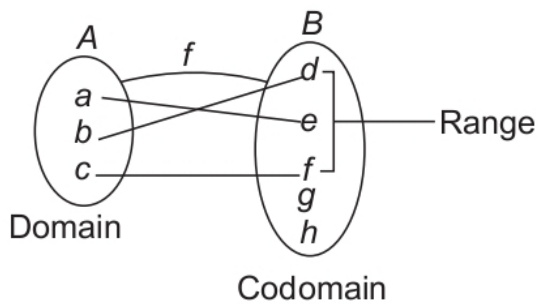
Domain, Codomain and Range of a Function

If $f: A \to B$ is a function from A to B, then

(i) the set A is called the **domain** of $f(x)$.

(ii) the set B is called the **codomain** of $f(x)$.

(iii) the subset of B containing only the images of elements of A is called the **range** of $f(x)$.



Characteristics of a Function $f: A \to B$

(i) For each element $x \in A$, there is unique element $y \in B$.

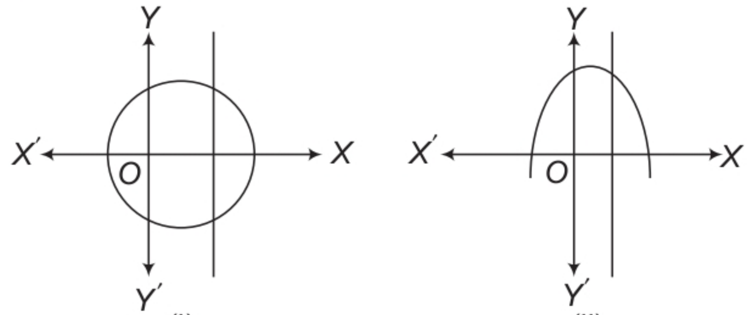
(ii) The element $y \in B$ is called the **image** of x under the function f. Also, y is called the **value of function** f at x i.e. $f(x) = y$.

(iii) $f: A \to B$ is not a function, if there is an element in A which has more than one image in B. But more than one element of A may be associated to the same element of B.

(iv) $f: A \to B$ is not a function, if an element in A does not have an image in B.

Identification of a Function from its Graph

Let us draw a vertical line parallel to Y-axis, such that it intersects the graph of the given expression. If it intersects the graph at more than one point, then the expression is a relation else, if it intersects at only one point, then the **expression is a function**.



(i) (ii)

In figure (i), the vertical parallel line intersects the curve at two points, thus the expression is a relation whereas in figure (ii), the vertical parallel line intersects the curve at one point. So, the expression is a function.

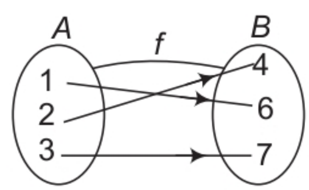
Types of Functions

1. One-One (or Injective) Function

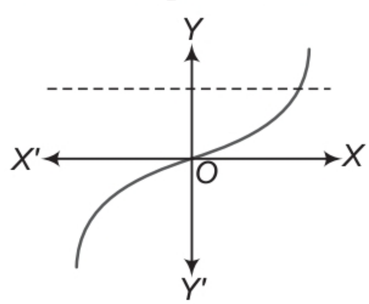
A mapping $f : A \to B$ is a called one-one (or injective) function, if different elements in A have different images in B, such a mapping is known as **one-one** or **injective function**.

Methods to Test One-One

(i) **Analytically** If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ or equivalently $x_1 \neq x_2$ $\Rightarrow f(x_1) \neq f(x_2), \forall\, x_1, x_2 \in A$, then the function is one-one.



(ii) **Graphically** If every line parallel to X-axis cuts the graph of the function atmost at one point, then the function is one-one.



(iii) **Monotonically** If the function is increasing or decreasing in whole domain, then the function is one-one.

Number of One-One Functions

Let A and B are finite sets having m and n elements respectively, then the number of one-one functions from A to B is $\begin{cases} {}^nP_m, n \geq m \\ 0, n < m \end{cases}$

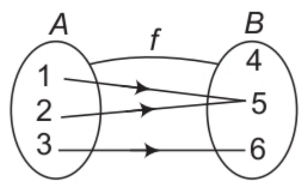
$$= \begin{cases} n(n-1)(n-2)\ldots(n-(m-1)), & n \geq m \\ 0, & n < m \end{cases}$$

2. Many-One Function

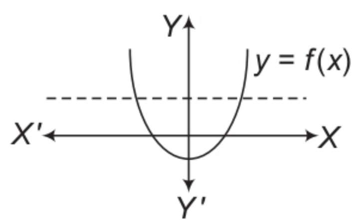
A function $f : A \longrightarrow B$ is called many-one function, if two or more than two different elements in A have the same image in B.

Method to Test Many-One

(i) **Analytically** If $x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$ for some $x_1, x_2 \in A$, then the function is many-one.



(ii) **Graphically** If any line parallel to X-axis cuts the graph of the function atleast two points, then the function is many-one.



(iii) **Monotonically** If the function is neither strictly increasing nor strictly decreasing, then the function is many-one.

Number of Many-One Function

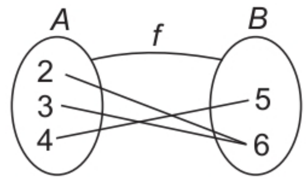
Let A and B are finite sets having m and n elements respectively, then the number of many-one function from A to B is

= Total number of functions – Number of one-one functions

$$= \begin{cases} n^m - {}^nP_m, \text{ if } n \geq m \\ n^m, \text{ if } n < m \end{cases}$$

3. **Onto** (or Surjective) **Function**

If the function $f : A \longrightarrow B$ is such that each element in B (codomain) is the image of atleast one element of A, then we say that f is a function of A onto B. Thus, $f : A \longrightarrow B$ is onto iff $f(A) = B$.



i.e. Range = Codomain

Note Every polynomial function $f : R \longrightarrow R$ of odd degree is onto.

Number of Onto (or Surjective) **Functions**

Let A and B are finite sets having m and n elements respectively, then number of onto (or surjective) functions from A to B is

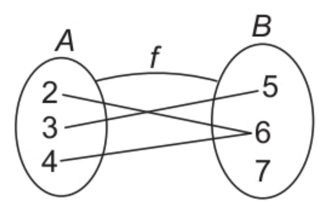
$$= \begin{cases} n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - {}^nC_3(n-3)^m + \dots, & n < m \\ n!, & n = m \\ 0, & n > m \end{cases}$$

4. **Into Function**

If $f : A \longrightarrow B$ is such that there exists atleast one element in codomain which is not the image of any element in domain, then f is into.

Thus, $f : A \longrightarrow B$, is into iff $f(A) \subset B$

i.e. Range $\subset$ Codomain



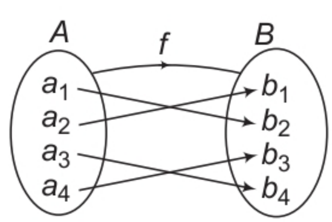
Number of Into Function

Let A and B be finite sets having m and n elements respectively, then number of into functions from A to B is

$$= \begin{cases} {}^nC_1(n-1)^m - {}^nC_2(n-2)^m + {}^nC_3(n-3)^m \dots, & n \le m \\ n^m, & n > m \end{cases}$$

5. **One-One and Onto Function** (or Bijective)

A function $f : A \to B$ is said to be one-one and onto (or bijective), if f is both one-one and onto.



Number of Bijective Functions

Let A and B are finite sets having m and n elements respectively, then number of onto functions from A to B is $\begin{cases} n!, \text{if } n = m \\ 0, \text{if } n > m \text{ or } n < m \end{cases}$.

Equal Functions

Two functions f and g are said to be equal iff

(i) domain of f = domain of g.

(ii) codomain of f = codomain of g.

(iii) $f(x) = g(x)$ for every x belonging to their common domain and then we write $f = g$.

Real Valued and Real Functions

A function $f : A \rightarrow B$ is called a **real valued function**, if $B \leq R$ and it is called a **real function** if, $A \leq R$ and $B \leq R$.

1. Domain of Real Functions

The domain of the real function $f(x)$ is the set of all those real numbers for which the expression for $f(x)$ or the formula for $f(x)$ assumes real values only.

2. Range of Real Functions

The range of a real function of a real variable is the set of all real values taken by $f(x)$ at points of its domain.

Working Rule for Finding Range of Real Functions

Let $y = f(x)$ be a real function, then for finding the range we may use the following steps

***Step* I** Find the domain of the function $y = f(x)$.

***Step* I** Transform the equation $y = f(x)$ as $x = g(y)$.
i.e. convert x in terms of y.

***Step* III** Find the values of y from $x = g(y)$ such that the values of x are real and lying in the domain of f.

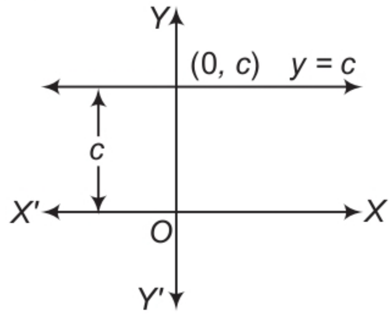
***Step* IV** The set of values of y obtained in step III be the range of function f.

Standard Real Functions and their Graphs

1. Constant Function

Let c be a fixed real number. The function which associates each real number x to this fixed number c, is called a constant function.

i.e. $y = f(x) = c$ for all $x \in R$.

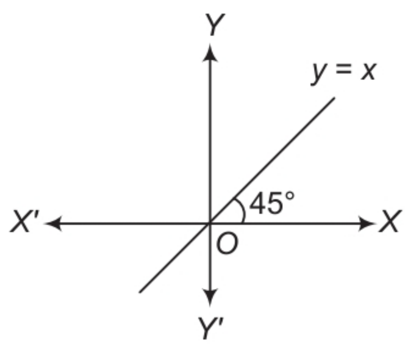


Domain of $f(x) = R$ and Range of $f(x) = \{c\}$.

2. Identity Function

The function which associates each real number x to the same number x, is called the identity function.

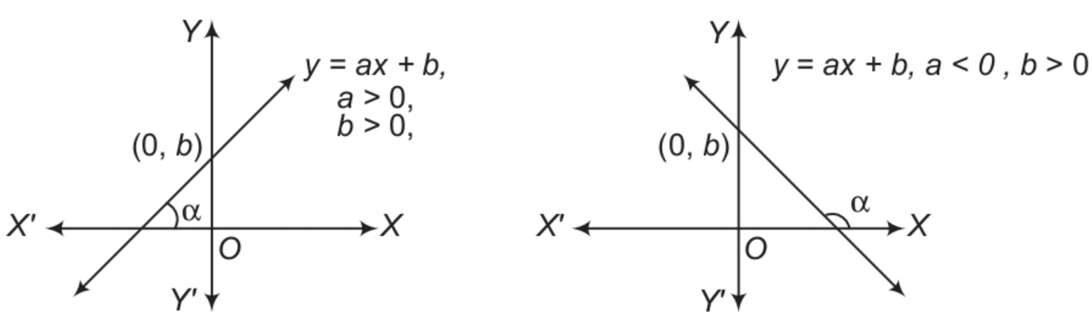
i.e. $y = f(x) = x, \forall x \in R$.



Domain of $f(x) = R$ and Range of $f(x) = R$

3. Linear Function

If a and b are fixed real numbers, then the linear function is defined as $y = f(x) = ax + b$. The graph of a linear function is given in the following diagram, which is a straight line with slope $\tan\alpha$.



Domain of $f(x) = R$ and Range of $f(x) = R$.

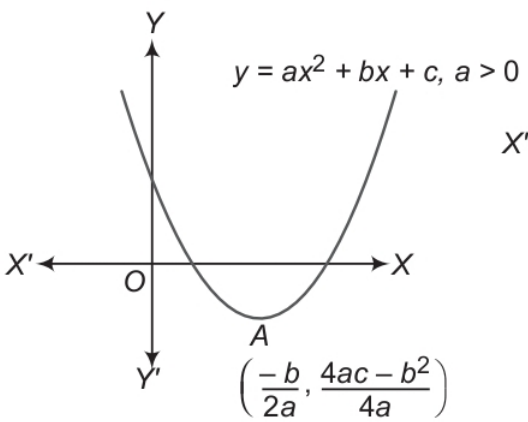
4. **Quadratic Function**

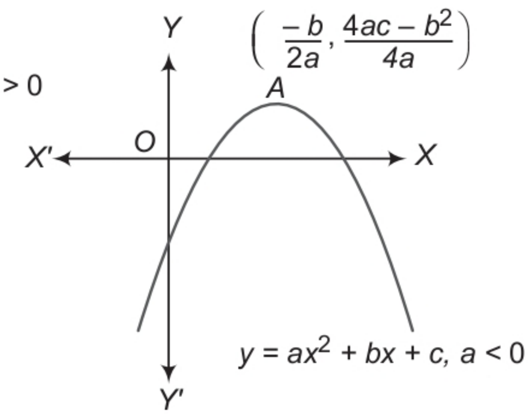
If a, b and c are fixed real numbers, then the quadratic function is expressed as

$$y = f(x) = ax^2 + bx + c, a \neq 0$$

$$\Rightarrow \qquad y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

which represents a downward parabola, if $a < 0$ and upward parabola, if $a > 0$ and vertex of this parabola is at $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$.





Domain of $f(x) = R$

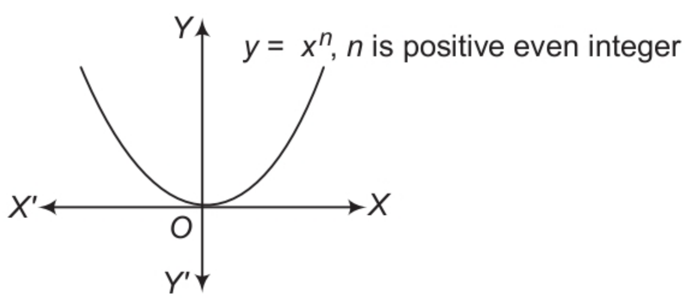
Range of $f(x)$ is $\left(-\infty, \frac{4ac - b^2}{4a}\right]$, if $a < 0$ and $\left[\frac{4ac - b^2}{4a}, \infty\right)$, if $a > 0$.

5. **Power Function**

The power function is given by $y = f(x) = x^n, n \in I, n \neq 1, 0$.

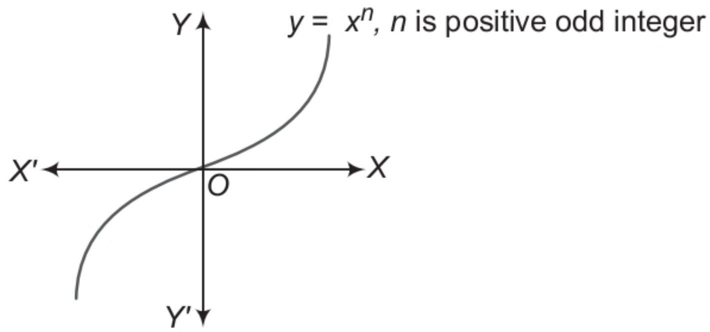
The domain and range of $y = f(x)$, is depend on n.

(a) If n is positive even integer, i.e. $f(x) = x^2, x^4, \ldots$



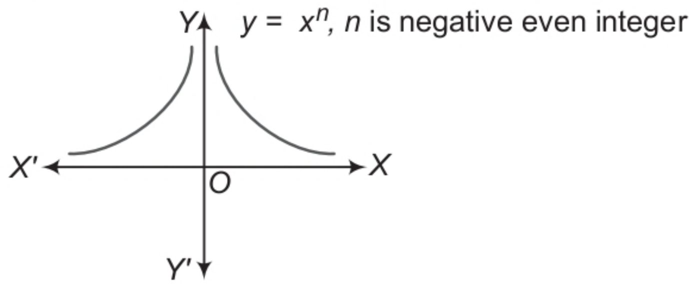
Domain of $f(x) = R$ and Range of $f(x) = [0, \infty)$

(b) If n is positive odd integer, i.e. $f(x) = x^3, x^5, \ldots$



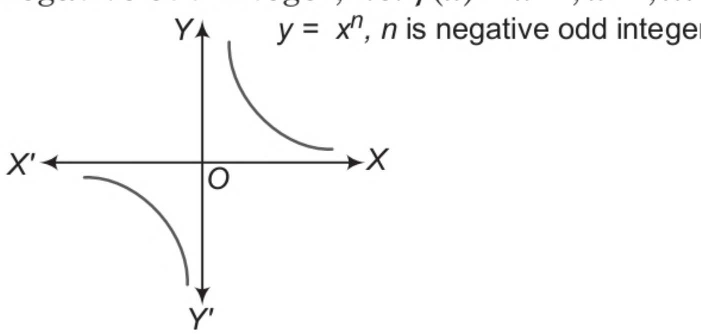
Domain of $f(x) = R$ and Range of $f(x) = R$

(c) If n is negative even integer, i.e. $f(x) = x^{-2}, x^{-4}, \ldots$



Domain of $f(x) = R - \{0\}$ and Range of $f(x) = (0, \infty)$

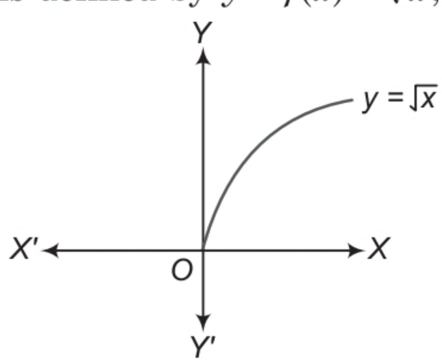
(d) If n is negative odd integer, i.e. $f(x) = x^{-1}, x^{-3}, \ldots$



Domain of $f(x) = R - \{0\}$ and Range of $f(x) = R - \{0\}$

6. **Square Root Function**

Square root function is defined by $y = f(x) = \sqrt{x}, x \geq 0$.

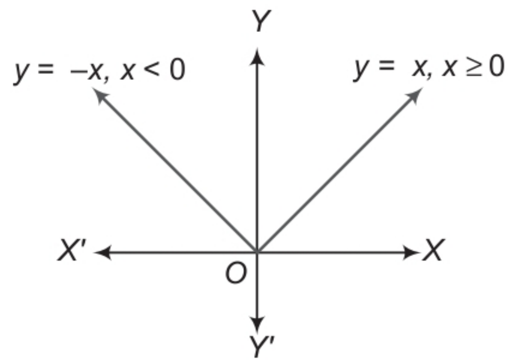


Domain of $f(x) = [0, \infty)$ and Range of $f(x) = [0, \infty)$

7. **Modulus** (or Absolute Value) **Function**

Modulus function is given by $y = f(x) = |x|$, where $|x|$ denotes the absolute value of x,

i.e. $$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}.$$

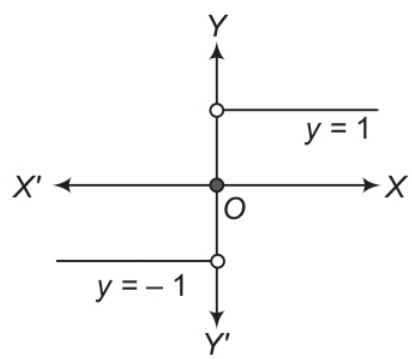


Domain of $f(x) = R$ and Range of $f(x) = [0, \infty)$.

8. **Signum Function**

Signum function is defined as follows

$$y = f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \text{ or } \begin{cases} \frac{x}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$



Symbolically, signum function is denoted by sgn (x).

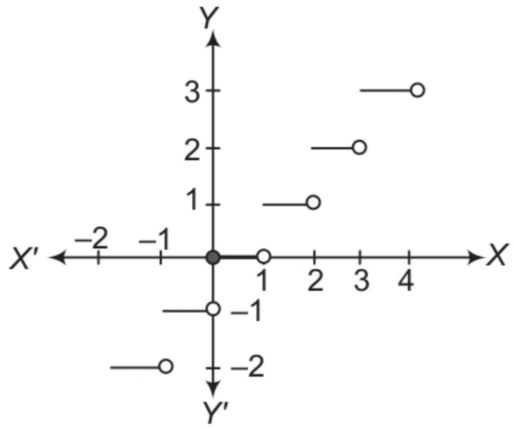
Thus, $y = f(x) = \text{sgn}(x)$

where, $$\text{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$$

Domain of sgn $(x) = R$ and Range of sgn $(x) = \{-1, 0, 1\}$

9. Greatest Integer Function/Step Function/ Floor Function

The greatest integer function is defined as $y = f(x) = [x]$



where, $[x]$ represents the greatest integer less than or equal to x. In general, if $n \le x < n + 1$ for any integer n, $[x] = n$.

Thus, $[2.304] = 2$, $[4] = 4$ and $[-8.05] = -9$

x	$[x]$
$0 \le x < 1$	0
$1 \le x < 2$	1
$-1 \le x < 0$	-1
$-2 \le x < -1$	-2
⋮	⋮

Domain of $f(x) = R$ and Range of $f(x) = I$, the set of integers.

Properties of Greatest Integer Function

(i) $[x + n] = n + [x], n \in I$

(ii) $[-x] = -[x], x \in I$

(iii) $[-x] = -[x] - 1, x \notin I$

(iv) $[x] \ge n \Rightarrow x \ge n, n \in I$

(v) $[x] > n \Rightarrow x \ge n + 1, n \in I$

(vi) $[x] \le n \Rightarrow x < n + 1, n \in I$

(vii) $[x] < n \Rightarrow x < n, n \in I$

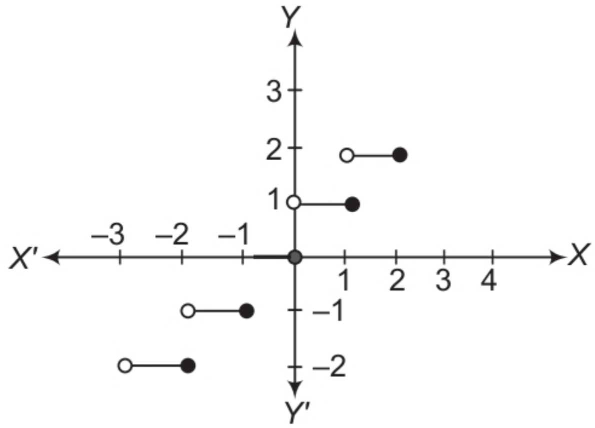
(viii) $[x + y] = [x] + [y + x - [x]]$ for all $x, y \in R$

(ix) $[x + y] \ge [x] + [y]$

(x) $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \ldots + \left[x + \frac{n-1}{n}\right] = [nx], n \in N$.

10. **Least Integer Function/Ceiling Function/Smallest Function**

The least integer function is defined as $y = f(x) = (x)$, where (x) represents the least integer greater than or equal to x.



Thus, $(3.578) = 4$, $(0.87) = 1$, $(4) = 4$, $(-8.239) = -8$, $(-0.7) = 0$

In general, if n is an integer and x is any real number such that $n < x \le n + 1$, then $(x) = n + 1$

$\therefore \qquad f(x) = (x) = [x] + 1$

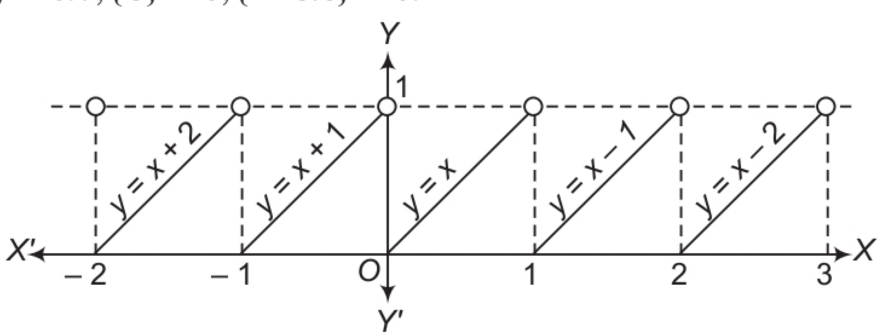
x	(x)
$-1 < x \le 0$	0
$0 < x \le 1$	1
$1 < x \le 2$	2
$2 < x \le 3$	3
$-2 < x \le -1$	-1
⋮	⋮

Domain of $f = R$ and Range of $f = I$

11. **Fractional Part Function**

It is defined as $f(x) = \{x\}$, where $\{x\}$ represents the fractional part of x, i.e., if $x = n + f$, where $n \in I$ and $0 \le f < 1$, then $\{x\} = f$

e.g. $\{0.7\} = 0.7, \{3\} = 0, \{-3.6\} = 0.4$



Properties of Fractional Part Function

(i) $\{x\} = x - [x]$

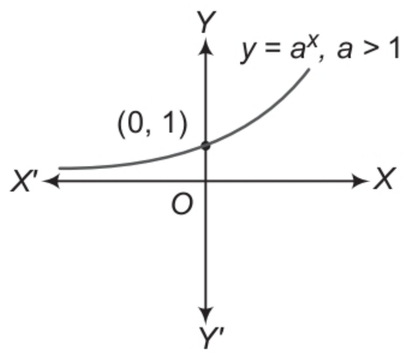
(ii) $\{x\} = x$, if $0 \le x < 1$

(iii) $\{x\} = 0$, if $x \in I$

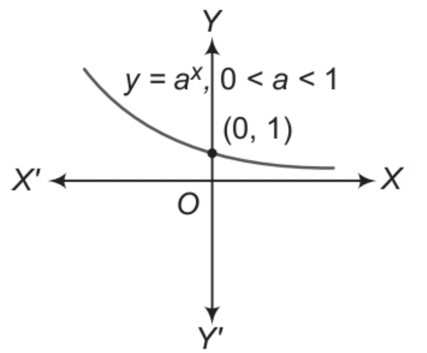
(iv) $\{-x\} = 1 - \{x\}$, if $x \notin I$

12. **Exponential Function**

Exponential function is given by $y = f(x) = a^x$, where $a > 0, a \neq 1$. The graph of the exponential function is as shown, which is increasing, if $a > 1$ and decreasing, if $0 < a < 1$.



(i)



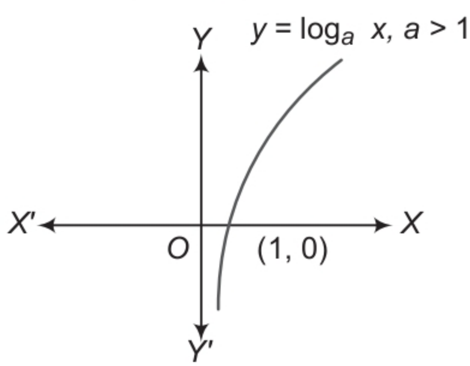
(ii)

Domain of $f(x) = R$ and Range of $f(x) = (0, \infty)$

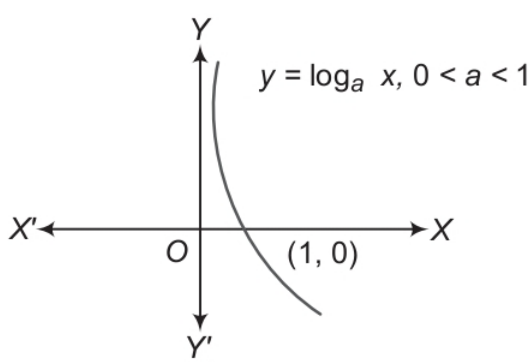
13. **Logarithmic Function**

A logarithmic function may be given by $y = f(x) = \log_a x$, where $a > 0, a \neq 1$ and $x > 0$.

The graph of the function is as shown below, which is increasing, if $a > 1$ and decreasing, if $0 < a < 1$.



(i)



(ii)

Domain of $f(x) = (0, \infty)$ and Range of $f(x) = R$

Operations on Real Functions

Let $f : A \to B$ and $g : A \longrightarrow B$ be two real functions, then

(i) **Addition** The addition $f + g$ is defined as

$f + g : A \longrightarrow B$ such that $(f + g)(x) = f(x) + g(x)$.

(ii) **Difference** The difference $f - g$ is defined as $f - g : A \to B$ such that $(f - g)(x) = f(x) - g(x)$.

(iii) **Product** The product fg is defined as

$$fg : A \longrightarrow B \text{ such that } (fg)(x) = f(x)\, g(x).$$

Clearly, $f \pm g$ and fg are defined only, if f and g have the same domain. In case, the domain of f and g are different, then

$$\text{domain of } f + g \text{ or } fg = \text{domain of } f \cap \text{domain of } g.$$

(iv) **Multiplication by a Number** (or a Scalar) The function cf, where c is a real number is defined as

$$cf : A \longrightarrow B, \text{ such that } (cf)(x) = cf(x).$$

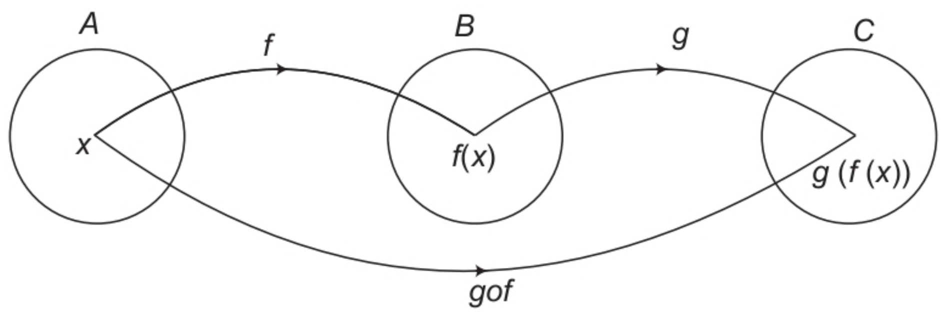
(v) **Quotient** The quotient $\frac{f}{g}$ is defined as

$\frac{f}{g} : A \to B$ such that $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$.

Composition of Two Functions

Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two functions. Then, we define $gof : A \longrightarrow C$, such that

$$gof(x) = g(f(x)), \forall\, x \in A$$



Important Points to be Remembered

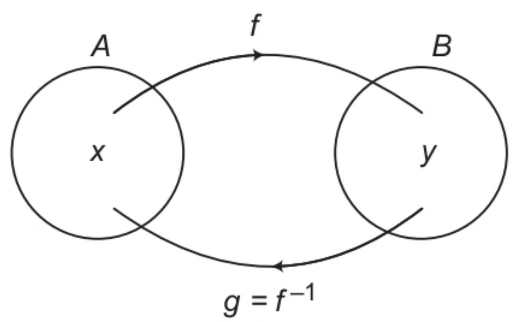
(i) If f and g are injective, then fog and gof are injective.

(ii) If f and g are surjective, then fog and gof are surjective.

(iii) If f and g are bijective, then fog and gof are bijective.

Inverse of a Function

Let $f : A \longrightarrow B$ is a bijective function, i.e. it is one-one and onto function. Then, we can define a function $g : B \longrightarrow A$, such that $f(x) = y \Rightarrow g(y) = x$, which is called inverse of f and *vice-versa*. Symbolically, we write $g = f^{-1}$



A function whose inverse exists, is called an **invertible function or inversible**.

(i) Domain $(f^{-1}) =$ Range (f)

(ii) Range $(f^{-1}) =$ Domain (f)

(iii) If $f(x) = y$, then $f^{-1}(y) = x$ and *vice-verse*.

Periodic Functions

A function $f(x)$ is said to be a periodic function of x, if there exists a real number $T > 0$, such that

$$f(T + x) = f(x), \forall\ x \in \text{Dom}(f).$$

The smallest positive real number T, satisfying the above condition is known as the period or the fundamental period of $f(x)$.

Testing the Periodicity of a Function

(i) Put $f(T + x) = f(x)$ and solve this equation to find the positive values of T independent of x.

(ii) If no positive value of T independent of x is obtained, then $f(x)$ is a non-periodic function.

(iii) If positive values of T which is independent of x are obtained, then $f(x)$ is a periodic function and the least positive value of T is the period of the function $f(x)$.

Important Points to be Remembered

(i) Constant function is periodic with no fundamental period.

(ii) If $f(x)$ is periodic with period T, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ are also periodic with same period T.

(iii) If $f(x)$ is periodic with period T_1 and $g(x)$ is periodic with period T_2, then $f(x)+g(x)$ is periodic with period equal to

(*a*) LCM of $\{T_1, T_2\}$, if there is no positive k, such that $f(k+x)=g(x)$ and $g(k+x)=f(x)$.

(*b*) $\frac{1}{2}$ LCM of $\{T_1, T_2\}$, if there exist a positive number k such that $f(k+x)=g(x)$ and $g(k+x)=f(x)$

(iv) If $f(x)$ is periodic with period T, then $kf(ax+b)$ is periodic with period $\frac{T}{|a|}$, where $a, b, k \in R$ and $a, k \neq 0$.

(v) If $f(x)$ is a periodic function with period T and $g(x)$ is any function, such that range of $f \subseteq$ domain of g, then gof is also periodic with period T.

Even and Odd Functions

Even Function A real function $f(x)$ is an even function, if $f(-x)=f(x)$.

Odd Function A real function $f(x)$ is an odd function, if $f(-x)=-f(x)$.

Properties of Even and Odd Functions

(i) Even function ± Even function = Even function.

(ii) Odd function ± Odd function = Odd function.

(iii) Even function × Odd function = Odd function.

(iv) Even function × Even function = Even function.

(v) Odd function × Odd function = Even function.

(vi) gof or fog is even, if both f and g are even or if f is odd and g is even or if f is even and g is odd.

(vii) gof or fog is odd, if both of f and g are odd.

(viii) If $f(x)$ is an even function, then $\frac{d}{dx} f(x)$ or $\int f(x)\,dx$ is an odd function and if $f(x)$ is an odd function, then $\frac{d}{dx} f(x)$ or $\int f(x)\,dx$ is an even function.

(ix) The graph of an even function is symmetrical about Y-axis.

(x) The graph of an odd function is symmetrical about origin or symmetrical in opposite quadrants.

(xi) An every function can never be one-one, however an odd function may or may not be one-one.

Binary Operations

Let S be a non-empty set. A function $*$ from $S \times S$ to S is called a binary operation on S i.e. $* : S \times S \rightarrow S$ is a binary operation on set S.

Note Generally binary operations are represented by the symbols $\star$, $\oplus$, ... etc., instead of letters figure etc.

Closure Property

An operation $*$ on a non-empty set S is said to satisfy the closure property, if

$$a \in S, b \in S \Rightarrow a * b \in S, \forall\ a,\ b \in S$$

Also, in this case we say that S is closed under $*$.

An operation $*$ on a non-empty set S, satisfying the closure property is known as a binary operation.

Some Particular Cases

(i) Addition is a binary operation on each one of the sets N, Z, Q, R and C, i.e. on the set of natural numbers, integers, rationals, real and complex numbers, respectively. While addition on the set S of all irrationals is not a binary operation.

(ii) Multiplication is a binary operation on each one of the sets N, Z, Q, R and C, i.e. on the set of natural numbers, integers, rationals, real and complex numbers, respectively. While multiplication on the set S of all irrationals is not a binary operation.

(iii) Subtraction is a binary operation on each one of the sets Z, Q, R and C, i.e. on the set of integers, rationals, real and complex numbers, respectively. While subtraction on the set of natural numbers is not a binary operation.

(iv) Let S be a non-empty set and $P(S)$ be its power set. Then, the union, intersection and difference of sets, on $P(S)$ is a binary operation.

(v) Division is not a binary operation on any of the sets N, Z, Q, R and C. However, it is a binary operation on the sets of all non-zero rational (real or complex) numbers.

(vi) Exponential operation $(a, b) \to a^b$ is a binary operation on set N of natural numbers while it is not a binary operation on set Z of integers.

Properties of Binary Operations

(i) **Commutative Property** A binary operation $*$ on a non-empty set S is said to be commutative or abelian, if

$$a * b = b * a, \forall\, a, b \in S.$$

Addition and multiplication are commutative binary operations on Z but subtraction is not a commutative binary operation, since $2-3 \neq 3-2$.

Union and intersection are commutative binary operations on the power set $P(S)$ of S. But difference of sets is not a commutative binary operation on $P(S)$.

(ii) **Associative Property** A binary operation $*$ on a non-empty set S is said to be associative, if $(a * b) * c = a * (b * c), \forall\, a, b, c \in S$.

Let R be the set of real numbers, then addition and multiplication on R satisfies the associative property.

(iii) **Distributive Property** Let $*$ and o be two binary operations on a non-empty sets. We say that $*$ is distributed over o, if

$a * (b \, o \, c) = (a * b) \, o \, (a * c), \forall\, a, b, c \in S$ also (called left distributive law) and $(b \, o \, c) * a = (b * a) \, o \, (c * a), \forall\, a, b, c \in S$ also (called right distributive law).

Let R be the set of all real numbers, then multiplication distributes over addition on R.

Since, $a \cdot (b + c) = a \cdot b + a \cdot c, \forall\, a, b, c \in R$.

Identity Element

Let $*$ be a binary operation on a non-empty set S. An element $e \in S$, if it exist, such that $a * e = e * a = a, \forall\, a \in S$, is called an identity elements of S, with respect to $*$.

For addition on R, zero is the identity element in R.

Since, $a + 0 = 0 + a = a, \forall\, a \in R$

For multiplication on R, 1 is the identity element in R.
Since, $a \times 1 = 1 \times a = a, \forall\ a \in R$

Let $P(S)$ be the power set of a non-empty set S. Then, ϕ is the identity element for union on $P(S)$, as $A \cup \phi = \phi \cup A = A, \forall\ A \in P(S)$

Also, S is the identity element for intersection on $P(S)$.

Since, $A \cap S = A \cap S = A, \forall\ A \in P(S)$.

For addition on N the identity element does not exist. But for multiplication on N the identity element is 1.

Inverse of an Element

Let $*$ be a binary operation on a non-empty set S and let e be the identity element.

Suppose $a \in S$, we say that a is invertible, if there exists an element $b \in S$ such that $a * b = b * a = e$

Also, in this case, b is called the inverse of a and we write, $a^{-1} = b$

Addition on N has no identity element and accordingly N has no invertible element.

Multiplication on N has 1 as the identity element and no element other than 1 is invertible.

Important Points to be Remembered

If S be a finite set containing n elements, then

(i) the total number of binary operations on S is n^{n^2}.

(ii) the total number of commutative binary operations' on S is $n^{n(n+1)/2}$.