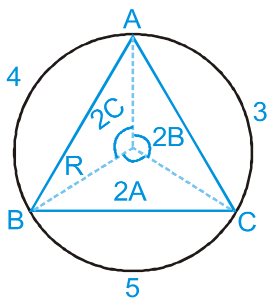
HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

2. $\text{angle} = \dfrac{\text{arc}}{\text{radius}}$ .....**(i)**



$4 + 5 + 3 = 2\pi R$

$\Rightarrow R = 6/\pi$

$\because \quad 2A = \dfrac{5}{R} = \dfrac{5\pi}{6},$

$2B = \dfrac{3}{R} = \dfrac{\pi}{2}$ and

$2C = \dfrac{4}{R} = \dfrac{2\pi}{3}$

$$\text{Area of } \Delta ABC = \frac{1}{2}R^2\left[\sin\frac{2\pi}{3} + \sin\frac{5\pi}{6} + \sin\frac{\pi}{2}\right] = \frac{R^2}{2}$$

$$\left[\frac{\sqrt{3}}{2} + \frac{1}{2} + 1\right] = \frac{R^2}{2}\left[\frac{\sqrt{3}+3}{2}\right]$$

$$= \frac{\sqrt{3}(\sqrt{3}+1)}{4} \times \frac{36}{\pi^2} = \frac{9\sqrt{3}(\sqrt{3}+1)}{\pi^2}$$

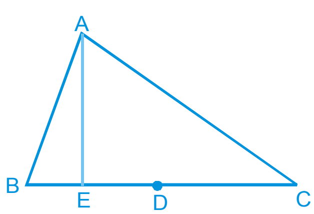
4. $\left(\dfrac{4R^2\sin^2 A}{\sin A} + \dfrac{4R^2\sin^2 B}{\sin B} + \dfrac{4R^2\sin^2 C}{\sin C}\right)\prod \sin\dfrac{A}{2}$

$= 4R^2(\sin A + \sin B + \sin C)\prod \sin\dfrac{A}{2}$

$= 16R^2\prod\cos\dfrac{A}{2}.\prod\sin\dfrac{A}{2} = 2R^2\sin A.\sin B.\sin C$

$= 2R^2\dfrac{abc}{8R^3} = \dfrac{abc}{4R} = \Delta$

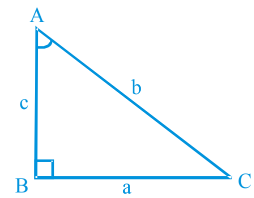
5. $ED = \dfrac{a}{2} - c\cos B$



$= \dfrac{a}{2} - c\left(\dfrac{a^2+c^2-b^2}{2ac}\right)$

$= \dfrac{a}{2} - \left(\dfrac{a^2+c^2-b^2}{2a}\right) = \dfrac{a^2-a^2-c^2+b^2}{2a} = \dfrac{b^2-c^2}{2a}$

6. $\tan^2\dfrac{A}{2} = \dfrac{1-\cos A}{1+\cos A}$



$= \dfrac{1-\frac{c}{b}}{1+\frac{c}{b}} = \dfrac{b-c}{b+c}$

7. $a + b + c = 2s \quad \Rightarrow \quad s = 2a$

Applying half angle formulae.

$\cot\dfrac{B}{2}.\cot\dfrac{C}{2} = \sqrt{\dfrac{(s)(s-b)}{(s-a)(s-c)}.\dfrac{(s)(s-c)}{(s-a)(s-b)}}$

$= \dfrac{s}{s-a} = 2$

8. $r_1 + r_2 = \dfrac{\Delta c}{(s-a)(s-b)}$

$\therefore \quad \Pi\,(r_1 + r_2) = \dfrac{\Delta^3 abc}{(s-a)^2(s-b)^2(s-c)^2} = \dfrac{\Delta^3(abc)s^2}{\Delta^4}$

$= \dfrac{(abc)\,s^2}{\Delta} = \dfrac{4R\Delta\,s^2}{\Delta} = 4Rs^2$

$\therefore \quad \dfrac{\Pi(r_1 + r_2)}{Rs^2} = 4$

9. $a = 1$

$\because \quad 2s = 6\left(\dfrac{\sin A + \sin B + \sin C}{3}\right)$

$\Rightarrow \quad 2s = 2\left(\dfrac{a+b+c}{2R}\right)$

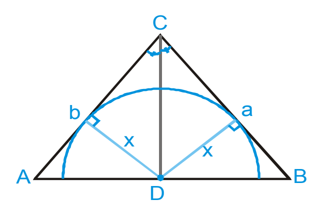
$R = 1$

$\because \quad \dfrac{a}{\sin A} = 2R \quad \Rightarrow \quad \sin A = \dfrac{1}{2}$

$A = \dfrac{\pi}{6}$

10. $\text{Area}(\Delta ADC) = \dfrac{1}{2}b.x$, $\text{Area}(\Delta BCD) = \dfrac{1}{2}x.a$

$\Rightarrow \quad \Delta = \dfrac{1}{2}x(b+a)$



$\Rightarrow \quad x = \dfrac{2\Delta}{a+b}$

12. Using $r_1 = \frac{\Delta}{(s-a)}$, $r_2 = \frac{\Delta}{(s-b)}$, $r_3 = \frac{\Delta}{(s-c)}$

we get $\frac{(2s-(a+b))(2s-(b+c))(2s-(c+a))}{\Delta^3}$

$\Rightarrow \frac{abc}{\Delta^3} = \frac{KR^3}{(abc)^2} \quad \Rightarrow \quad \frac{64R^3}{(abc)^2} = \frac{KR^3}{(abc)^2}$

hence $K = 64$

14. $\frac{\sum \frac{\Delta}{s-a}}{\sqrt{\sum \frac{\Delta^2}{(s-a)(s-b)}}} = \frac{\sum \frac{1}{s-a}}{\sqrt{\sum \frac{1}{(s-a)(s-b)}}}$

$= \frac{\sum \frac{1}{s-a}}{\sqrt{\sum \frac{s(s-c)}{\Delta^2}}} = \sum \frac{1}{(s-a)} \times \frac{\Delta}{s} = \sum \tan \frac{A}{2}$

$(\because \; r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2})$

15. $\cos C = \frac{(19)^2 + (17)^2 - c^2}{2 \cdot 19 \cdot 17} > 0$

$\therefore \; c^2 < (19)^2 + (17)^2$

$c^2 < (18+1)^2 + (18-1)^2$

$c^2 < 2(18^2 + 1)$

$c^2 < 650$ ....**(1)**

$c > 2$ and $c < 26$

$\therefore \; 2 < c < 26$ ....**(2)**

$4 < c^2 < 676$ ....**(3)**

$\therefore$ c is 3, 4, 5, ..., 25

$\therefore$ number of integral values of c is 23

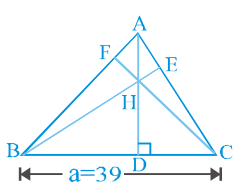
16. $AH = 2R\cos A$ (distance of orthocentre from the vertex)

given $\sin A = 3/5$; $\cos A = 4/5$

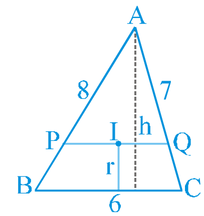
$a = 39$

$\therefore \; \frac{a}{\sin A} = 2R; \; \frac{39 \cdot 5}{3} = 2R \; \Rightarrow \; 2R = 65$

$\therefore \; AH = 65 \cdot \frac{4}{5} = 52$



17. $\Delta = \frac{6 \cdot h}{2} = 3h$; (where h is the altitude from A)



Also $\Delta = \frac{21 \cdot r}{2}$ (using $\Delta = r \cdot s$)

$\frac{21 \cdot r}{2} = \frac{6 \cdot h}{2} = 3h \Rightarrow \frac{r}{h} = \frac{2}{7}$

now APQ and ABC are similar

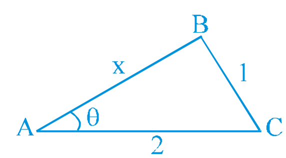
$\frac{h-r}{h} = \frac{PQ}{6} \Rightarrow 1 - \frac{r}{h} = \frac{PQ}{6} \Rightarrow 1 - \frac{2}{7} = \frac{PQ}{6}$

$\Rightarrow \frac{5}{7} = \frac{PQ}{6} \quad \Rightarrow PQ = \frac{30}{7}$

18. Using cosine rule

$\cos\theta = \frac{x^2 + 4 - 1}{4x} = \frac{x^2 + 3}{4x} = \frac{1}{4}\left[x + \frac{3}{x}\right]$

$= \frac{1}{4}\left[\left(\sqrt{x} - \sqrt{\frac{3}{x}}\right)^2 + 2\sqrt{3}\right]$



hence $\cos\theta$ is minimum if $x = \sqrt{3}$

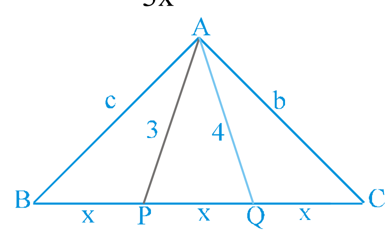
$\therefore$ minimum value of $\cos\theta = 2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

maximum value of $\theta = \frac{\pi}{6}$

19. In ΔABP

$9 = c^2 + x^2 - 2cx\cos B$; but $\cos B = \frac{c}{3x}$

$9 = c^2 + x^2 - 2cx\frac{c}{3x}$



$9 = x^2 + \frac{c^2}{3}$ ....**(1)** ; $\quad 16 = x^2 + \frac{b^2}{3}$ ....**(2)**

(1) + (2)

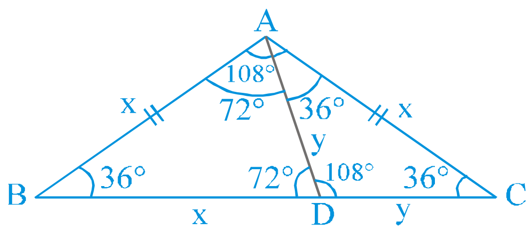
$25 = 2x^2 + \frac{1}{3}(b^2 + c^2) = 2x^2 + 3x^2$

$\Rightarrow 5x^2 = 25 \Rightarrow x = \sqrt{5}$; $\quad \therefore \; BC = 3\sqrt{5} = \sqrt{45}$

20. In $\Delta ABC \quad (\frac{x}{y} = ?)$

$$\frac{x+y}{\sin 108°} = \frac{x}{\sin 36°}$$

$$\frac{x+y}{x} = \frac{\sin 108°}{\sin 36°} \quad ....(1)$$



$$1 + \frac{y}{x} = \frac{\sin 72°}{\sin 36°} = 2\cos 36°$$

$$\frac{y}{x} = \frac{\sqrt{5}+1}{2} - 1 = \frac{\sqrt{5}-1}{2}$$

$$\frac{x}{y} = \frac{2(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)} = \frac{\sqrt{5}+1}{2}$$

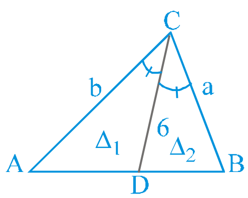
21. $\frac{\tan C}{\sin B} = \frac{\sin C}{\sin B} \cdot \frac{1}{\cos C}$.

nowusing sine law $\quad \frac{\sin C}{\sin B} = \frac{c}{b} = \frac{10}{9}$;

using cosine law $\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{5}{16}$

$$\therefore \quad \frac{\tan C}{\sin B} = \frac{\sin C}{\sin B} \cdot \frac{1}{\cos C} = \frac{10}{9} \cdot \frac{16}{5} = \frac{32}{9}$$

22. $\Delta = \Delta_1 + \Delta_2 = \frac{1}{2} ab\sin C = ab\sin\frac{C}{2}\cos\frac{C}{2}$



$$= \frac{1}{2} 6b\sin\frac{C}{2} + \frac{1}{2} 6a\sin\frac{C}{2} \quad \Rightarrow \quad \frac{1}{a} + \frac{1}{b} = \frac{1}{9}$$

23. $b^2x^2 + (b^2 + c^2 - a^2)x + c^2 = 0$

Note: $\quad b^2 + c^2 - a^2 = 2bc\cos A$ (from cosine rule)

let $\quad f(x) = b^2x^2 + (2bc\cos A)x + c^2 = 0 \quad$ also $A \in (0, \pi)$

in a triangle

$\therefore \quad \cos A \in (-1, 1)$

$2bc\cos A \in (-2bc, 2bc)$

$\Rightarrow \quad D = (2bc\cos A)^2 - 4b^2c^2 = 4b^2c^2 \underbrace{(\cos^2 A - 1)}_{-ve}$

$\Rightarrow \quad D < 0 \quad \Rightarrow$ **(A)** is correct

24. Put $a = 2R\sin A$ etc.

$E = R[\sin 2B + \sin 2C + \sin 2C + \sin 2A + \sin 2A + \sin 2B]$

$= 2R(\sin 2A + \sin 2B + \sin 2C]$

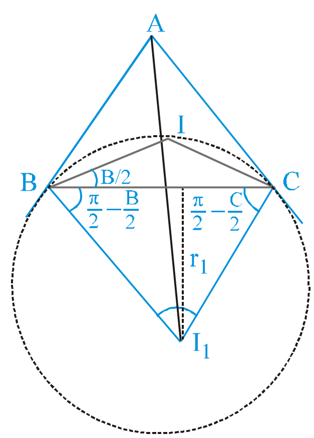
$$= 8R \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{abc}{R^2}$$

25. $BICI_1$ is a cyclic quadrilateral with II_1 as the diameter

also $\quad \angle BI_1C = \frac{\pi}{2} - \frac{A}{2}$

applying sine law in BCI_1

$$\frac{a}{\cos\frac{A}{2}} = II_1$$



$$\therefore \quad II_1 = \frac{2R \cdot 2\sin\frac{A}{2} \cdot \cos\frac{A}{2}}{\cos\frac{A}{2}} = 4R\sin\frac{A}{2}$$

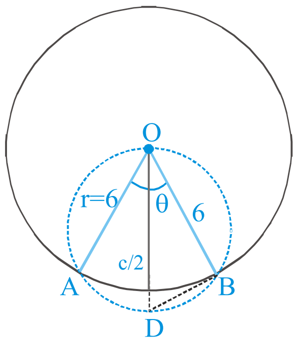
$$\therefore \quad \prod II_1 = 64R^3 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 16R^2 r$$

(using $4R\sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2} = r$)

26. $R = \frac{abc}{4\Delta}$; $\Delta = \frac{1}{2}\cdot 6\cdot 6\sin\theta = 18\sin\theta$

$a = b = 6$

$\sin\frac{\theta}{2} = \frac{c}{12} \Rightarrow c = 12\sin\frac{\theta}{2}$



$$R = \frac{36\cdot 12\sin(\theta/2)}{4\cdot 18\cdot\sin\theta} = 3\sec\frac{\theta}{2}$$

27. $a^2 + b^2 < c^2$ $\quad a^2 + b^2 < a^2 + b^2 - 2ab\cos C$

$\therefore \cos C < 0$

$\Rightarrow$ C is obtuse $\Rightarrow$ **(B)**

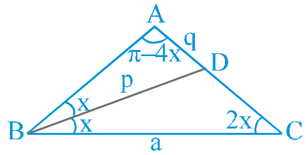
28. $BC = BD + AD$ $\quad a = p + q$

or $\frac{a}{p} = 1 + \frac{q}{p}$

using Sine law in Δ BDC and in Δ ABD

$$\frac{\sin 3x}{\sin 2x} = 1 + \frac{\sin x}{\sin 4x}$$

$$\Rightarrow \frac{\sin 3x\cdot\sin 4x - \sin x\cdot\sin 2x}{\sin 2x\cdot\sin 4x} = 1$$



$2\sin 3x\cdot\sin 4x - 2\sin x\cdot\sin 2x = 2\sin 2x\cdot\sin 4x$

$(\cos x - \cos 7x) - (\cos x - \cos 3x) = \cos 2x - \cos 6x$

$\cos 3x - \cos 7x = \cos 2x - \cos 6x$

$2\sin 5x\cdot\sin 2x = 2\sin 4x\cdot\sin 2x$

as $\sin 2x \neq 0$

hence $\sin 5x = \sin 4x$

$5x + 4x = 180°$

$\Rightarrow x = 20°$; hence $\angle A = 180° - 80° = 100°$

29. LHS $\frac{R[\sin 2A + \sin 2B + \sin 2C]}{2R[\sin A + \sin B + \sin C]}$

$$= \frac{4\sin A\sin B\sin C}{2\,.\,4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

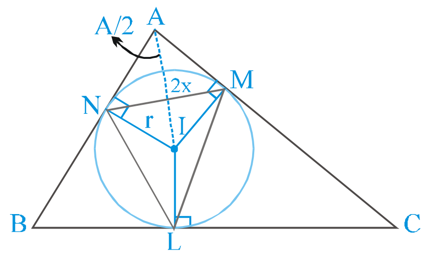
$$= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{R}$$

30. Numerator of RHS = $ks^2\,.\,4R$ and denominator of RHS = s^2 (using $r_1 = s\tan\frac{A}{2}$ etc.)

$$\Rightarrow \frac{\text{numerator}}{\text{denominator}} \text{ of RHS} = k\cdot 4R = \text{LHS} = R \Rightarrow k = \frac{1}{4}$$

31. note that ANIM is a cyclic quandrilateral.

$$\Rightarrow \operatorname{cosec}\frac{A}{2} = \frac{2x}{r} \Rightarrow 2x = r\operatorname{cosec}\frac{A}{2}$$

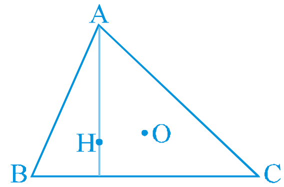


$$x = \frac{r}{2\sin\frac{A}{2}} \Rightarrow xyz = \frac{r^3}{2.4\prod\sin\frac{A}{2}} = \frac{r^3.R}{2.r} = \frac{r^2R}{2}$$

32. $OA = HA$

$R = 2R\cos A$ (distance of orthocentre from the vertex A is $2R\cos A$)

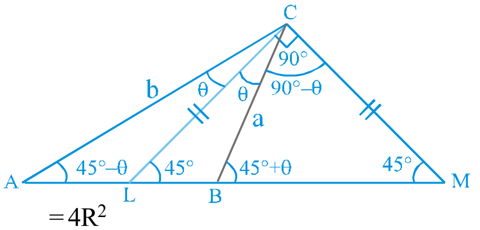
$$\Rightarrow \cos A = \frac{1}{2}$$



$\Rightarrow A = \frac{\pi}{3}$ $\Rightarrow$ **(C)**

33. $a^2+b^2=4R^2[\sin^2(45°-\theta)+\sin^2(135°-\theta)]$

$=4R^2[\sin^2(45°-\theta)+\cos^2(45°-\theta)]$



$=4R^2$

34. $\cot\frac{B}{2}\cdot\cot\frac{C}{2}=\frac{s(s-b)}{\Delta}\cdot\frac{s(s-c)}{\Delta}\cdot\frac{(s-a)}{s-a}$

$=\frac{s}{s-a}=\frac{2s}{2s-2a}$

but given that $a+b+c=4a \Rightarrow 2s=4a$

Hence $\cot\frac{B}{2}\cdot\cot\frac{C}{2}=\frac{4a}{2a}=2$

35. $\tan A=\frac{a}{2f} \Rightarrow \frac{1}{2}\Sigma\tan A=\frac{1}{2}\Pi\tan A$

$=\frac{1}{4}\left(\frac{a}{f}\cdot\frac{b}{g}\cdot\frac{c}{h}\right) \Rightarrow$ A

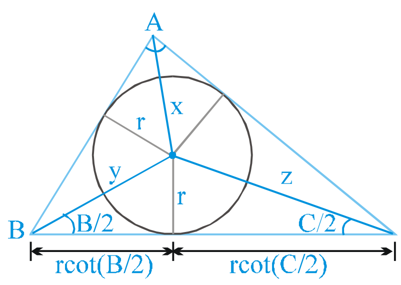
36. use $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$ to get $A-B=60°$ and

$A+B=150^0$ (given)

$\Rightarrow A=105°$

37. $x=r\,\text{cosec}\frac{A}{2}$

$a=r\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)$



$\frac{a}{x}=\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right).\sin\frac{A}{2}=\frac{\sin\frac{A}{2}.\cos\frac{A}{2}}{\sin\frac{B}{2}.\sin\frac{C}{2}}$

$\therefore \frac{abc}{xyz}=\frac{\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}.\sin\frac{C}{2}}=\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}$

In a triangle $\prod\cot\frac{A}{2}=\sum\cot\frac{A}{2}$

38. $\sin 2C=\frac{1-\cos A\cos B}{\sin A\sin B}\le 1$. Now proceed $A=B=\frac{3\pi}{8}$

and $C=\frac{\pi}{4}$

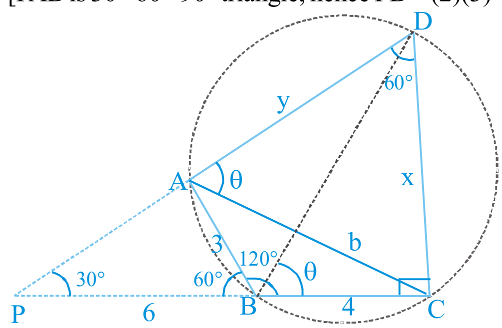
$1\le\cos(A-B) \Rightarrow \cos(A-B)=1 \Rightarrow A=B$

39. Extend CB and DA to meet at P.

note that Δ PCD is right angle as shown.

now $\tan 30°=\frac{x}{10}$

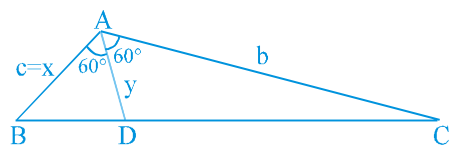
[PAB is 30°-60°-90° triangle, hence PB = (2)(3) = 6]



$\therefore x=\frac{10}{\sqrt{3}}=\frac{10\sqrt{3}}{3}$

40. $AD=y=\frac{2bc}{b+c}\cos\frac{A}{2}=\frac{bx}{b+x}$ (as $c=x$)

but $bx=1 \Rightarrow b=\frac{1}{x}$



$\therefore y=\frac{x}{1+x^2}=\frac{1}{x+\frac{1}{x}}$

$y_{max}=\frac{1}{2}$ since minimum value of the denominator is 2

if $x>0 \Rightarrow$ **(B)**

EXERCISE - 2

Part # I : Multiple Choice

1. **(A)** $\because \tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$ ....**(i)**

$\because \tan^2\left(\frac{A-B}{2}\right) = \frac{1-\cos(A-B)}{1+\cos(A-B)} = \frac{1-\frac{31}{32}}{1+\frac{31}{32}} = \frac{1}{63}$

$\therefore \tan\left(\frac{A-B}{2}\right) = \frac{1}{3\sqrt{7}}$ $\because$ $a = 5$ and $b = 4$

$\therefore$ from equation **(i)**, we get

$\frac{1}{3\sqrt{7}} = \left(\frac{5-4}{5+4}\right)\cot\frac{C}{2} \Rightarrow \frac{1}{3\sqrt{7}} = \frac{1}{9}\cot\frac{C}{2}$

$\Rightarrow \cot\frac{C}{2} = \frac{3}{\sqrt{7}}$

$\because \cos C = \frac{1-\tan^2 C/2}{1+\tan^2 C/2} = \frac{1-7/9}{1+7/9} = \frac{2}{16} = \frac{1}{8}$

$\because \cos C = \frac{b^2+a^2-c^2}{2ab}$

$\Rightarrow c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow c = 6$

(B), (C) $\because$ Area $= \frac{1}{2}ab\sin C$ $\because \cos C = \frac{1}{8}$

$\Rightarrow \sin C = \sqrt{1-\frac{1}{64}} = \frac{3\sqrt{7}}{8}$

Area $= \frac{1}{2} \times 5 \times 4 \times \frac{3\sqrt{7}}{8}$

Area $= \frac{15\sqrt{7}}{4}$ sq. unit.

$\because$ From Sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\Rightarrow \sin A = \frac{a\sin C}{c} = \frac{5\times 3\sqrt{7}}{6\times 8}$

$\therefore \sin A = \frac{5\sqrt{7}}{16}$

3. $\cos A + \cos C = 4\sin^2\frac{B}{2}$

$\Rightarrow \cos\left(\frac{A+C}{2}\right).\cos\left(\frac{A-C}{2}\right) = 2\sin^2\frac{B}{2}$

$= 2\cos^2\left(\frac{A+C}{2}\right)$

$\Rightarrow \frac{\cos\left(\frac{A-C}{2}\right)}{\cos\frac{A+C}{2}} = \frac{2}{1}$ $\Rightarrow \cot\frac{A}{2}.\cot\frac{C}{2} = \frac{2+1}{2-1}$

$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

$\Rightarrow \cot\frac{A}{2}+\cot\frac{C}{2} = 2\cot\frac{B}{2}$

4. $\beta_a = \frac{2bc}{b+c}\cos\frac{A}{2}$

(A) correct

(B) incorrect

(C) $\frac{abc\,\mathrm{cosec}\frac{A}{2}}{2R(b+c)} = \frac{abc\,\mathrm{cosec}\frac{A}{2}}{\frac{a}{\sin A}.(b+c)}$

$= \frac{bc\,2\sin\frac{A}{2}\cos\frac{A}{2}}{\sin\frac{A}{2}.(b+c)} = \frac{2bc}{(b+c)}\cos\frac{A}{2}$

(D) $\because \frac{2\Delta}{(b+c)}\mathrm{cosec}\frac{A}{2} = \frac{bc\sin A}{(b+c)}.\frac{1}{\sin\frac{A}{2}}$

$= \frac{2bc\sin\frac{A}{2}\cos\frac{A}{2}}{(b+c)}.\frac{1}{\sin\frac{A}{2}} = \frac{2bc}{b+c}\cos\frac{A}{2}$

6. $(r_1-r)(r_2-r)(r_3-r)$

$= \left(\frac{\Delta}{s-a}-\frac{\Delta}{s}\right)\left(\frac{\Delta}{s-b}-\frac{\Delta}{s}\right)\left(\frac{\Delta}{s-c}-\frac{\Delta}{s}\right)$

$= \frac{\Delta^3 abc}{s^2.\Delta^2}$ $\left(R = \frac{abc}{4\Delta}\right)$

$= \frac{\Delta}{s^2}\frac{abc}{4\Delta}\cdot 4\Delta = 4r^2R$

9. Product of distances of incenter from angular points

$$= \frac{r^3}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{r^3}{r/4R}$$

$= 4Rr^2$

$$= \frac{abc}{\Delta}r^2 = \frac{(abc)(r)}{\frac{\Delta}{r}} = \frac{(abc)(r)}{s}.$$

10. (A) $\dfrac{\Delta}{s-b} + \dfrac{\Delta}{s-c} = \dfrac{\Delta}{s-a} - \dfrac{\Delta}{s}$

$\Rightarrow \dfrac{1}{(s-b)(s-c)} = \dfrac{1}{s(s-a)}$

$\Rightarrow \tan^2\dfrac{A}{2} = 1 \quad \Rightarrow \quad A = 90°$

(B) $4R^2(\sin^2A + \sin^2B + \sin^2C) = 8R^2$

$1 - (\cos^2A - \sin^2B) + \sin^2C = 2$

$1 - \cos(A+B)\cos(A-B) + 1 - \cos^2C = 2.$

$\cos C\cos(A-B) - \cos^2C = 0$

$\cos C[\cos(A-B) - \cos C] = 0$

$\cos C[\cos(A-B) + \cos(A+B)] = 0$

$2\cos A\cos B\cos C = 0 \quad \Rightarrow \quad A = 90°$

or $B = 90°$ or $C = 90°$

(C) $r_1 = s.$

$s\tan A/2 = s \quad \Rightarrow \quad \tan A/2 = 1 \quad \Rightarrow \quad A = 90°$

(D) $\dfrac{a}{\sin A} = \dfrac{a\Delta}{s(s-a)} \quad \Rightarrow \quad \dfrac{1}{\sin A} = \tan A/2$

$\Rightarrow 2\sin^2A/2 = 1$

$\Rightarrow 1 - \cos A = 1 \quad \Rightarrow \cos A = 0 \quad \Rightarrow \quad A = 90°$

11. $\because \quad r_1 - r = \dfrac{\Delta}{s-a} - \dfrac{\Delta}{s} = \dfrac{\Delta a}{s(s-a)} = a\tan\dfrac{A}{2}$

$\therefore \quad \Pi(r_1 - r) = abc\tan\dfrac{A}{2}\tan\dfrac{B}{2}\tan\dfrac{C}{2}$

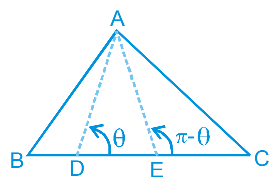
$= abc\Pi\tan\dfrac{A}{2}$

$$= abc\frac{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$= \frac{(abc)r}{4R.\frac{(\sin A + \sin B + \sin C)}{4}} = \frac{(abc)r}{R\left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}\right)}$$

$$= \frac{2(abc)r}{2s} = \frac{4R\Delta r}{s} = 4Rr^2$$

13.



if we apply m-n Rule in Δ ABE, we get

$(1+1)\cot\theta = 1.\cot B - 1.\cot\theta$

$2.\cot\theta = \cot B - \cot\theta$

$3.\cot\theta = \cot B$

$\tan\theta = 3\tan B$ ....(i)

Similarly, if we apply m-n Rule in Δ ACD, we get

$(1+1)\cot(\pi - \theta) = 1.\cot\theta - 1.\cot C.$

$\cot C = 3\cot\theta \quad \Rightarrow \quad \tan\theta = 3\tan C$ ....(ii)

form (1) and (2) we can say that $\tan B = \tan C$

$\Rightarrow B = C$

$\because \quad A + B + C = \pi$

$\therefore \quad A = \pi - (B + C) \quad = \pi - 2B$

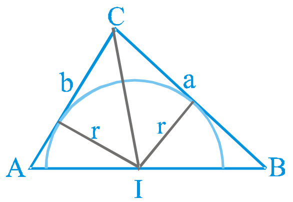
$\because \quad B = C$

$$\therefore \quad \tan A = -\tan 2B = -\left(\frac{2\tan B}{1-\tan^2 B}\right) = -\frac{\frac{2\tan\theta}{3}}{1-\frac{\tan^2\theta}{9}}$$

$$\Rightarrow \quad \tan A = \frac{6\tan\theta}{\tan^2\theta - 9}$$

14. $\frac{1}{2}ra + \frac{1}{2}rb = \frac{1}{2}ab\sin C$

$r(a+b) = 2\Delta$



$r = \frac{2\Delta}{a+b}$ ....**(1)**

$\therefore \quad r = \frac{2abc}{4R(2R\sin A + 2R\sin B)} = \frac{abc}{4R^2(\sin A + \sin B)}$

$\Rightarrow$ **(A)**

also $x = \frac{2ab}{a+b}\cos\frac{C}{2}$

from (1) $r = \frac{2\cdot\frac{1}{2}ab\sin C}{a+b} = \frac{2ab\sin\frac{C}{2}\cos\frac{C}{2}}{a+b}$

$= \frac{2ab\cos\frac{C}{2}}{a+b}\cdot\sin\frac{C}{2} = x\sin\frac{C}{2}$ $\Rightarrow$ **(C)**

15. $r_1 = 2r_2 = 3r_3$

$\Rightarrow \frac{\Delta}{s-a} = \frac{2\Delta}{s-b} = \frac{3\Delta}{s-c} \Rightarrow \frac{1}{s-a} = \frac{2}{s-b} = \frac{3}{s-c}$

(i) (ii) (iii)

From **(i)** and **(ii)** we get $a - b = c/3$ ...**(1)**

From **(i)** and **(iii)**, we get $2a - b = 2c$ ...**(2)**

From **(ii)** and **(iii)**, we get $a - 5b = -5c$ ...**(3)**

let $c = k$, then from (1) and (2), we get

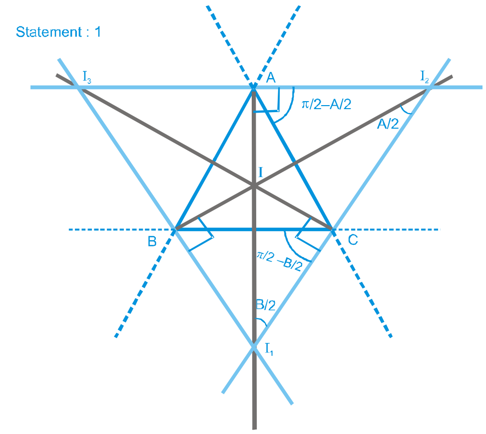
$a = \frac{5k}{3}$ and $b = \frac{4k}{3}$

$\therefore \frac{a}{b} = \frac{5}{4};\ \frac{a}{c} = \frac{5}{3}.$

Part # II : Assertion & Reason

1. $I_1I_2 = 4R\cos\frac{C}{2}$ if we apply Sine-Rule in $\Delta I_1I_2I_3$, then

$$2R_{ex} = \frac{I_1I_2}{\sin\left(\frac{A}{2}+\frac{B}{2}\right)} = \frac{4R\cos\frac{C}{2}}{\sin\left(\frac{A+B}{2}\right)} = \frac{4R\cos\frac{C}{2}}{\cos\frac{C}{2}}$$

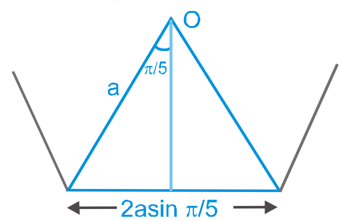


$2R_{ex} = 4R \quad R_{ex} = 2R$

$\therefore \quad \Delta ABC$ is pedal triangle of $\Delta I_1I_2I_3$

$\therefore$ statement - 1 and statement - 2 both are correct and statement -2 also explains Statement - 1

3. Perimeter $= 10a\sin\frac{\pi}{5}$



For n sided polygon, perimeter $= \left(2a\sin\frac{\pi}{n}\right)\times n$

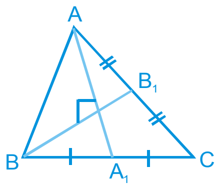
Hence statement II is false

4. Angles of ΔDEF are $\pi - 2A, \pi - 2B, \pi - 2C$

Incentre of ΔDEF is the orthocentre of ΔABC

EXERCISE - 3

Part # I : Matrix Match Type

2. 

(A) $\because$ AA_1 and BB_1 are perpendicular

$\therefore$ $a^2 + b^2 = 5c^2$

$\therefore$ $c^2 = \dfrac{a^2+b^2}{5} = 5 \quad \Rightarrow \quad c = \sqrt{5}$

$\because$ $\cos C = \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{16+9-5}{2\times4\times3} = \dfrac{5}{6}$

$\because$ $\sin C = \dfrac{\sqrt{11}}{6}$

$\therefore$ $\Delta = \dfrac{1}{2}ab\sin C = \sqrt{11}$

$\therefore$ $\Delta^2 = 11$

(B) G.M. $\geq$ H.M.

$$(r_1 r_2 r_3)^{1/3} \geq \frac{3}{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}}$$

$\Rightarrow$ $(r_1 r_2 r_3)^{1/3} \geq 3r$

$\Rightarrow$ $\dfrac{r_1 r_2 r_3}{r^3} \geq 27$

(C) $\because$ $\tan^2\dfrac{C}{2} = \dfrac{(s-a)(s-b)}{s(s-c)}$ $\quad \because a = 5, b = 4$

$2s = 9 + c$

$= \dfrac{(9+c-10)(9+c-8)}{(9+c)(9-c)} = \dfrac{c^2-1}{81-c^2}$

$\Rightarrow$ $\dfrac{7}{9} = \dfrac{c^2-1}{81-c^2} \quad \Rightarrow \quad c^2 = 36 \quad \Rightarrow \quad c = 6$

(D) $\because$ $2a^2 + 4b^2 + c^2 = 4ab + 2ac.$

$\Rightarrow$ $(a-2b)^2 + (a-c)^2 = 0 \quad \Rightarrow \quad a = 2b = c$

$\because$ $\cos B = \dfrac{a^2+c^2-b^2}{2ac} = \dfrac{7}{8}$

$\therefore$ $8\cos B = 7$

3. Use $p_1 = \dfrac{2\Delta}{a},\ p_2 = \dfrac{2\Delta}{b},\ p_3 = \dfrac{2\Delta}{c}$

(A) $\dfrac{3}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}} \leq \sqrt[3]{p_1p_2p_3}$ $\quad$ (HM $\leq$ GM)

(B) $\dfrac{\cos A}{p_1} + \dfrac{\cos B}{p_2} + \dfrac{\cos C}{p_3}$

$= \dfrac{a\cos A + b\cos B + c\cos C}{2\Delta}$

$= \dfrac{R(\sin 2A + \sin 2B + \sin 2C)}{2\Delta}$

($\therefore$ a = 2R sin A)

$= \dfrac{R.4.\sin A.\sin B.\sin C}{2\Delta} = \dfrac{4R}{2\Delta}.\dfrac{abc}{8R^3} = \dfrac{1}{R}$

(C) $\dfrac{b^2}{c}.\dfrac{2\Delta}{a} + \dfrac{c^2}{a}.\dfrac{2\Delta}{b} + \dfrac{a^2}{b}.\dfrac{2\Delta}{c}$

$= 2\Delta\left(\dfrac{a^3+b^3+c^3}{abc}\right)$

Now, $\dfrac{a^3+b^3+c^3}{3} \geq abc$ $\quad$ (AM $\geq$ GM)

$\dfrac{a^3+b^3+c^3}{abc} \geq 3$

$\Rightarrow$ $2\Delta.\left(\dfrac{a^3+b^3+c^3}{abc}\right) \geq 6\Delta.$

(D) $\Sigma p_1^{-2} = \dfrac{\Sigma a^2}{4\Delta^2}$

Part # II : Comprehension

Comprehension # 1

Angles BEC, ABD, ABE and BAC are in A.P.

let BEC = $\alpha - 3\beta$

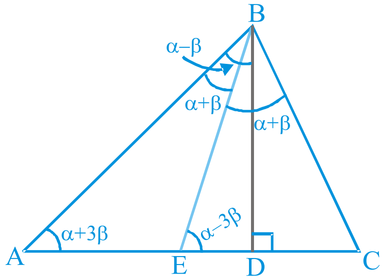
ABD = $\alpha - \beta$

ABE = $\alpha + \beta$

and BAC = $\alpha + 3\beta$

now, $\alpha - 3\beta = (\alpha + 3\beta) + (\alpha + \beta)$

[using exterior angle theorem]



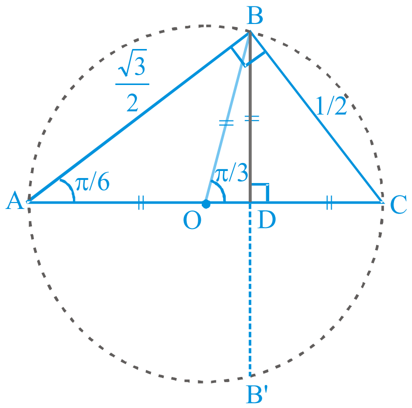
$\Rightarrow \quad \alpha = -7\beta$

$\therefore \quad \beta = -\dfrac{\pi}{24}, \alpha = \dfrac{7\pi}{24}$

and From ΔABD

$$\alpha - \beta + \alpha + 3\beta = \frac{\pi}{2}$$

$$2\beta + 2\beta = \frac{\pi}{2} \Rightarrow \alpha + \beta = \frac{\pi}{4}$$



$\therefore \quad \angle B = 2(\alpha + \beta) = \dfrac{\pi}{2}, \angle A = \dfrac{\pi}{6}, \angle C = \dfrac{\pi}{3}$

$\Rightarrow$ ABC is 30°–90°–60° triangle

(i) Area of circle circumscribing $\Delta ABC = \pi\left(\dfrac{1}{2}\right)^2 = \dfrac{\pi}{4}$

(ii) ΔBOC is equilateral $\Rightarrow r = \dfrac{\Delta}{s} = \dfrac{\dfrac{\sqrt{3}}{4}\left(\dfrac{1}{2}\right)^2}{\dfrac{1}{2}\left(\dfrac{3}{2}\right)} = \dfrac{1}{4\sqrt{3}}$

(iii) $BD = OB \sin\dfrac{\pi}{3} = \dfrac{1}{2}\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{4}$

$\therefore \quad BB' = 2BD = \dfrac{\sqrt{3}}{2}$

Comprehension # 2

1. Let $\angle I_3 I_1 I_2 = \theta$

Then angle of pedal triangle $= \pi - 2\theta = A$

$$\theta = \frac{\pi}{2} - \frac{A}{2}$$

2. Side of pedal triangle $= I_2 I_3 \cos\theta = BC$

$$I_2 I_3 = \frac{a}{\cos\left(\frac{\pi}{2} - \frac{A}{2}\right)}$$

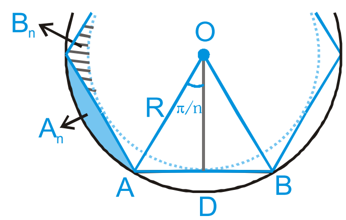
$$I_2 I_3 = 4R\cos\left(\frac{A}{2}\right)$$

3. $I I_1 = 4R\sin\dfrac{A}{2}$

$I_2 I_3 = 4R\cos\dfrac{A}{2}$

$\therefore \quad II_1^2 + I_2 I_3^2 = 16R^2$

Comprehension # 3



In Δ OAD

$OD = R\cos\dfrac{\pi}{n}, AD = R\sin\dfrac{\pi}{n}$

A_n = Area of circle (circumscribing polygon) – Area of polygon

$$A_n = \pi R^2 - \frac{n}{2}R^2 \sin\left(\frac{2\pi}{n}\right)$$

B_n = Area of polygon – Area of circle (Inscribed)

$$B_n = \frac{n}{2}R^2\sin\left(\frac{2\pi}{n}\right) - \pi R^2\cos^2\left(\frac{\pi}{n}\right)$$

1. If n = 6 then

$$A_n = \pi R^2 - \frac{3\sqrt{3}}{2}R^2$$

2. If n = 4 then value of

$$B_n = 2R^2 - \frac{\pi R^2}{2} = R^2\left(\frac{4-\pi}{2}\right)$$

3. $$\frac{A_n}{B_n} = \frac{\pi - \frac{n}{2}\sin\frac{2\pi}{n}}{\frac{n}{2}\sin\left(\frac{2\pi}{n}\right) - \pi.\cos^2\frac{\pi}{n}}$$

put $\pi = n\theta$

we get $$\frac{2\theta - \sin 2\theta}{\sin 2\theta - 2\theta\cos^2\theta}$$

$$= \frac{\theta - \sin\theta.\cos\theta}{\sin\theta.\cos\theta - \theta.\cos^2\theta} = \frac{\theta - \sin\theta.\cos\theta}{\cos\theta(\sin\theta - \theta\cos\theta)}$$

Comprehension # 5

b = AC = 3 and ratio is 1 : 2, Hence
AD = 1 and DC = 2

(i) Now $\sin\theta = \frac{\sqrt{15}}{4} \Rightarrow \theta = \sin^{-1}\frac{\sqrt{15}}{4}$ **Ans.**

(ii) Obviously $\angle FDC = \theta = \cos^{-1}\frac{1}{4} = \sin^{-1}\frac{\sqrt{15}}{4} = \angle FCD$

hence A, B, C are correct.

now area $\Delta ADE = 1 \cdot \frac{\sqrt{15}}{2}\cdot\frac{1}{2} = \frac{\sqrt{15}}{4}$

now $\tan\theta = \frac{BD}{2}$ (in ΔBDC)

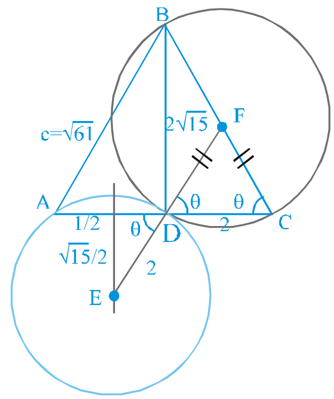
$\therefore \sqrt{15} = \frac{BD}{2} \Rightarrow BD = 2\sqrt{15}$

$\tan A = \frac{2\sqrt{15}}{1} = 2\sqrt{15}$

$\sin A = \frac{2\sqrt{15}}{\sqrt{61}}$ ....(1)

also $\sin A = \frac{2\sqrt{15}}{c} = \frac{2\sqrt{15}}{\sqrt{61}} \Rightarrow c = \sqrt{61}$

Area of $\Delta DBC = \frac{2\cdot 2\sqrt{15}}{2} = 2\sqrt{15}$



$\therefore \frac{\text{area. } \Delta ADE}{\text{area. } \Delta DBC} = \frac{\sqrt{15}/4}{2\sqrt{15}} = \frac{1}{8} \Rightarrow$ **(D) Ans.**

(iii) In ΔABC, $\frac{c}{\sin\theta} = 2R$;

$$R = \frac{c}{2\sin\theta} = \frac{\sqrt{61}\cdot 4}{2\sqrt{15}} = \frac{2\sqrt{61}}{\sqrt{15}}$$

Ans.

EXERCISE - 4

Subjective Type

1. LHS $= \frac{1}{2}(a^2 + (b + c - a) + b^2(c + a - b)$

$= \frac{1}{2}(a(b^2 + c^2 - a^2) + b(c^2 + a^2 - b^2)$

$+ c(a^2 + b^2 - c^2)$

$= \frac{1}{2}(2abc \cos A + 2abc \cos B + 2abc \cos C)$

$= abc\left(1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$

$= 4R\Delta\left(1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$

2. Since $a = 2R \sin A$, $b = 2R \sin B$, and $c = 2R \sin C$, we have

$(b^2 - c^2) \cot A = 4R^2(\sin^2 B - \sin^2 C) \cot A$

$= 4R^2 \sin(B + C) \sin(B - C) \cot A$

$= 4R^2 \sin A \sin(B - C) \frac{\cos A}{\sin A}$

$= -4R^2 \sin(B - C) \cos(B + C)$

$(\because \cos A = -\cos(B + C))$

$= -2R^2[2 \sin(B - C) \cos(B + C)]$

$= -2R^2[\sin 2B - \sin 2C]$ ....**(i)**

Similarly, $(c^2 - a^2) \cot B$

$= -2R^2[\sin 2C - \sin 2A]$ ....**(ii)**

and $(a^2 - b^2) \cot C$

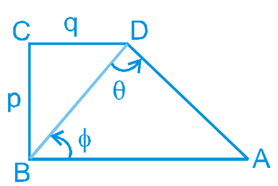
$= -2R^2[\sin 2A - \sin 2B]$ ....**(iii)**

Adding eq. **(i)**, **(ii)**, and **(iii)**, we get

$(b^2 - c^2) \cot A + (c^2 - a^2) \cot B$

$+ (a^2 - b^2) \cot C = 0$

3.



If we apply Sine-Rule in ΔABD, we get

$\frac{AB}{\sin\theta} = \frac{BD}{\sin(\pi - (\theta + \varphi))}$

$\Rightarrow AB = \frac{BD \sin\theta}{\sin(\theta + \varphi)} = \frac{BD \sin\theta}{\sin\theta\cos\varphi + \cos\theta\sin\varphi}$ ...**(i)**

$\sin\phi = \frac{p}{\sqrt{p^2 + q^2}}$ and $\cos\phi = \frac{q}{\sqrt{p^2 + q^2}}$

$\therefore$ from equation **(i)**, we get $AB = \frac{\left(\sqrt{p^2+q^2}\right)\sin\theta}{\frac{q\sin\theta}{\sqrt{p^2+q^2}} + \frac{p\cos\theta}{\sqrt{p^2+q^2}}}$

$\therefore AB = \frac{(p^2 + q^2)\sin\theta}{q\sin\theta + p\cos\theta}$

4. $\cos A(\sin B - \sin C) + (\sin 2B - \sin 2C) = 0$

$\Rightarrow \cos A.(\sin B - \sin C) + 2\cos(B + C)\sin(B - C) = 0$

$\because B + C = \pi - A$

$\Rightarrow \cos A.(\sin B - \sin C) - 2\cos A.\sin(B - C) = 0$

$\Rightarrow \cos A[(\sin B - \sin C) - 2(\sin B \cos C - \cos B \sin C)] = 0$

$\Rightarrow$ either $\cos A = 0 \Rightarrow A = 90° \Rightarrow$ right angled

or $(\sin B - \sin C) - 2(\sin B \cos C - \cos B \sin C) = 0$

$\Rightarrow (b - c) - 2\left(b.\frac{a^2 + b^2 - c^2}{2ab} - c.\frac{a^2 + c^2 - b^2}{2ac}\right) = 0$

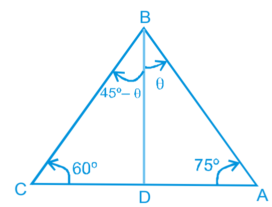
$\Rightarrow a(b - c) - 2(b^2 - c^2) = 0$

$(b - c)[a - 2(b + c)] = 0$

$\therefore b - c = 0 \Rightarrow b = c$

$\Rightarrow$ isosceles

5.



Area of $\Delta BAD = \sqrt{3} \times$ Area of ΔBCD

$\Rightarrow \frac{1}{2} BD \times BA \sin\theta = \sqrt{3} \times \frac{1}{2} BC \times BD \sin(45° - \theta)$

$\frac{BA}{BC} = \sqrt{3}\frac{\sin(45° - \theta)}{\sin\theta}$ .....**(i)**

$\because$ From Sine-Rule

$\frac{BC}{\sin 75°} = \frac{AB}{\sin 60°}$

$\therefore \frac{BA}{BC} = \frac{\sin 60°}{\sin 75°} = \frac{\sqrt{3}\sqrt{2}}{\sqrt{3}+1}$

$\therefore$ From equation **(i)**

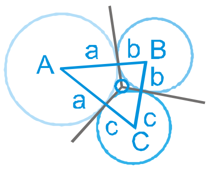
$$\frac{\sqrt{3}.\sqrt{2}}{(\sqrt{3}+1)} = \sqrt{3}\left[\frac{1}{\sqrt{2}}\cot\theta - \frac{1}{\sqrt{2}}\right]$$

$\Rightarrow \dfrac{2}{(\sqrt{3}+1)} = \cot\theta - 1 \quad \Rightarrow \dfrac{2(\sqrt{3}-1)}{2} = \cot\theta - 1$

$\Rightarrow \cot\theta = \sqrt{3} \quad \Rightarrow \theta = 30^\circ$

$\Rightarrow \angle ABD = 30^\circ$

6.



required distance = inradius of Δ ABC

$\because$ $2s = a + b + b + c + c + a$

$= 2(a + b + c)$

$s = a + b + c$

$\because$ $\Delta = \sqrt{s(s-(a+b))(s-(b+c))(s-(c+a))}$

$= \sqrt{(a+b+c)(abc)}$

$\therefore$ required distance $= \dfrac{\Delta}{s} = \dfrac{\sqrt{(a+b+c)(abc)}}{(a+b+c)}$

$= \sqrt{\dfrac{abc}{a+b+c}} = \left(\dfrac{abc}{a+b+c}\right)^{\frac{1}{2}}$

7. **(i)** L.H.S. $= (r_3+r_1)(r_3+r_2)\sin C$

$= \dfrac{\Delta b}{(s-a)(s-c)}\ \dfrac{\Delta a}{(s-c)(s-b)}\sin C$

$= \dfrac{ab\Delta^2}{(s-a)(s-b)(s-c)(s-c)}\sin C$

$= \dfrac{ab\,s(s-a)(s-b)(s-c)}{(s-a)(s-b)(s-c)(s-c)}\sin C = \dfrac{ab\,s\sin C}{(s-c)}$

$= \dfrac{2\Delta.s}{(s-c)} = 2\,sr_3$

R.H.S. $= 2r_3\sqrt{r_2r_3+r_3r_1+r_1r_2} = 2r_3\sqrt{s^2} = 2sr_3$

$\therefore$ L.H.S. = R.H.S.

(ii) L.H.S. $= -\dfrac{1}{\Delta}$

$$\left[\frac{(s-b)(s-c)}{(a-b)(c-a)} + \frac{(s-a)(s-c)}{(a-b)(b-c)} + \frac{(s-a)(s-b)}{(c-a)(b-c)}\right]$$

$= -\dfrac{1}{\Delta}$

$$\left[\frac{(s-b)(s-c)(b-c)+(s-a)(s-c)(c-a)+(s-a)(s-b)(a-b)}{(a-b)(b-c)(c-a)}\right]$$

$= \dfrac{1}{\Delta} =$ R.H.S.

(iii) First term $= (r+r_1)\tan\dfrac{B-C}{2}$

$= \left(\dfrac{\Delta}{s}+\dfrac{\Delta}{s-a}\right)\left(\dfrac{b-c}{b+c}\right)\cot\dfrac{A}{2} = \dfrac{\Delta(2s-a)}{s(s-a)}\ .$

$\left(\dfrac{b-c}{b+c}\right).\sqrt{\dfrac{s(s-a)}{(s-b)(s-c)}}$

$= b - c$

similarly second term = c – a

& third term = a – b

$\therefore$ L.H.S. = b – c + c – a + a – b = 0 = R.H.S.

(iv) $r_1+r_2+r_3-r = 4R$

$\therefore$ $(r_1+r_2+r_3-r)^2 = r_1^2+r_2^2+r_3^2+r^2-2r(r_1+r_2+r_3)+$ $2(r_1r_2+r_2r_3+r_3r_1)$ ....**(i)**

$\because$ $r(r_1+r_2+r_3) = ab+bc+ca-s^2$

and $r_1r_2+r_2r_3+r_3r_1 = s^2$

$\therefore$ from equation **(i)**

$16R^2 = r^2+r_1^2+r_2^2+r_3^2-2(ab+bc+ca-s^2)+2s^2$

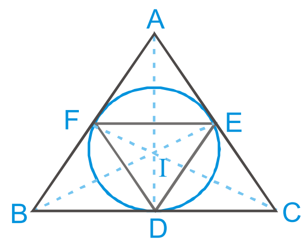
$\therefore$ $r^2+r_1^2+r_2^2+r_3^2 = 16R^2-4s^2+2(ab+bc+ca)$

$= 16R^2-(a+b+c)^2+2(ab+bc+ca)$

$= 16R^2-a^2-b^2-c^2$

8. **(i)** EIFA is a cyclic quadrilateral

$\therefore$ $\dfrac{EF}{\sin A} = AI$



$\because$ AI = r cosec A/2

$\therefore$ EF = r cosec A/2.sin A

= 2 r cos A/2

similarly DF= 2 r cos B/2 and DE = 2r cos C/2.

(ii) IECD is a cyclic quadrilateral

$\therefore \quad \angle ICE = \angle IDE = \frac{C}{2}$

similarly $\angle IDF = \angle IBF = \frac{B}{2}$

$\therefore \quad \angle FDE = \frac{B}{2}+\frac{C}{2} = \frac{\pi - A}{2} = \frac{\pi}{2} - \frac{A}{2}$

(iii) area of $\Delta DEF = \frac{1}{2}$ FD . DE sin $\angle FDE$

$= \frac{1}{2}\left(2r\cos\frac{B}{2}\right)\left(2r\cos\frac{C}{2}\right)\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)$

$= 2r^2 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$= 2r^2\left(\frac{\sin A+\sin B+\sin C}{4}\right) = \frac{r^2}{2}\left(\frac{2\Delta}{bc}+\frac{2\Delta}{ca}+\frac{2\Delta}{ab}\right)$

$= \frac{r^2}{2}\left[\frac{2\Delta(a+b+c)}{abc}\right] = \frac{r^2\Delta.2s}{abc} = \frac{2r^2.\Delta s^2}{(abc)s}$

$= \frac{2\Delta(rs)^2}{(abc)s} \Rightarrow = \frac{2\Delta^3}{(abc)s} = \frac{1}{2}\frac{r\Delta}{R}.$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

1. $\tan\left(\frac{\pi}{n}\right) = \frac{a}{2r}$; $\sin\left(\frac{\pi}{n}\right) = \frac{a}{2R}$

$\Rightarrow \quad r+R = \frac{a}{2}\left[\cot\frac{\pi}{n}+\operatorname{cosec}\frac{\pi}{n}\right]$

$\Rightarrow \quad r+R = \frac{a}{2}.\cot\left(\frac{\pi}{2n}\right)$

2. $a\frac{s(s-c)}{ab}+c.\frac{s(s-a)}{bc} = \frac{3b}{2}$

$\Rightarrow \quad \frac{s}{b}(s-c+s-a) = \frac{3b}{2}$

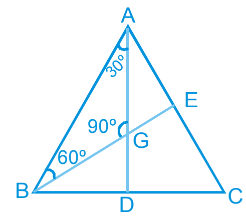
$\Rightarrow \quad a+b+c = 3b. \qquad \Rightarrow \quad a+c = 2b$

$\Rightarrow$ a, b, c are in A.P.

3. $AD = 4$

$\because \quad AG = \frac{2}{3}\times 4 = \frac{8}{3}$

$\therefore \quad$ Area of Δ ABG $= \frac{1}{2}\times AB\times AG\sin 30^o$



$\because \quad = \frac{1}{2}\times\frac{16}{3\sqrt{3}}\times\frac{8}{3}\times\frac{1}{2} = \frac{32}{9\sqrt{3}} \qquad \because \quad \text{Sin } 60^o = \frac{AG}{AB}$

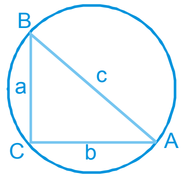
$\Rightarrow \quad AB = \frac{2AG}{\sqrt{3}} = \frac{16}{3\sqrt{3}}$

$\therefore \quad$ Area of Δ ABC $= 3$(Area of Δ ABG) $= \frac{32}{3\sqrt{3}}$

4. $\cos\beta = \frac{\sin^2\alpha+\cos^2\alpha-1-\sin\alpha\cos\alpha}{2\sin\alpha.\cos\alpha} = -\frac{1}{2}$

$\Rightarrow \quad \beta = 120^o$

5. $\angle C = \pi/2$



$r = (s - c)\tan\dfrac{C}{2} \qquad \therefore \quad C = 90^o$

$r = s - 2R$

$\because \quad 2r + 2R = 2(s - 2R) + 2R.$

$\quad = 2s - 2R$

$\quad = (a + b + c) - \dfrac{c}{\sin C} \qquad \because \quad C = 90^o$

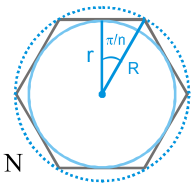
$\quad = a + b + c - c = a + b$

6. $\dfrac{2\Delta}{a}, \dfrac{2\Delta}{b}, \dfrac{2\Delta}{c}$ are in H.P.

$\Rightarrow \quad \dfrac{a}{2\Delta}, \dfrac{b}{2\Delta}, \dfrac{c}{2\Delta}$ are in A.P.

$\Rightarrow$ a,b,c are in A.P.

7. $\dfrac{r}{R} = \cos\left(\dfrac{\pi}{n}\right)$



Let $\cos\dfrac{\pi}{n} = \dfrac{2}{3}$ for some $n \geq 3, n \in N$

As $\dfrac{1}{2} < \dfrac{2}{3} < \dfrac{1}{\sqrt{2}} \quad \Rightarrow \quad \cos\dfrac{\pi}{3} < \cos\dfrac{\pi}{n} < \cos\dfrac{\pi}{4}$

$\Rightarrow \quad \dfrac{\pi}{3} > \dfrac{\pi}{n} > \dfrac{\pi}{4}$

$\Rightarrow \quad 3 < n < 4$, which is not possible

so option (2) is the false statement

so it will be the right choice

Hence correct option is (2)

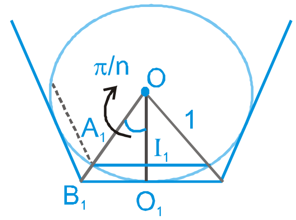
Part # II : IIT-JEE ADVANCED

1. $I_n = 2n \times$ area of ΔOA_1I_1

$\Rightarrow \quad I_n = 2n \times \dfrac{1}{2} \times A_1I_1 \times OI_1$

$\Rightarrow \quad I_n = n \times \sin\dfrac{\pi}{n} \times \cos\dfrac{\pi}{n}$

$\Rightarrow \quad I_n = \dfrac{n}{2}\sin\dfrac{2\pi}{n}. \qquad$ .....**(i)**



$O_n = 2n \times$ area of ΔOB_1O_1

$\Rightarrow \quad O_n = 2n \times \dfrac{1}{2} \times B_1O_1 \times O_1O$

$\quad = n \times \tan\dfrac{\pi}{n} \times 1 = n\tan\dfrac{\pi}{n}$

$\Rightarrow \quad O_n = n\tan\dfrac{\pi}{n} \qquad$ ......**(ii)**

Now

R.H.S. $= \dfrac{O_n}{2}\left[1 + \sqrt{1 - \left(\dfrac{2I_n}{n}\right)^2}\right] = \dfrac{O_n}{2}\left[1 + \cos\dfrac{2\pi}{n}\right]$

$= \dfrac{O_n}{2} \times 2\cos^2\dfrac{\pi}{n} = O_n.\cos^2\dfrac{\pi}{n} = n\tan\dfrac{\pi}{n}.\cos^2\dfrac{\pi}{n}$

$= \dfrac{n}{2}\sin\dfrac{2\pi}{n} = I_n =$ L.H.S

2. Let angle of the triangle be 4x, x and x .

Then $\quad 4x + x + x = 180°$

$\Rightarrow \quad x = 30°$

Longest side is opposite to the largest angle.

Using the law of sines

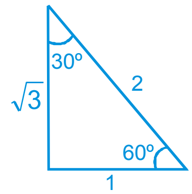
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$\therefore \quad a = R, b = R, c = \sqrt{3}R$

$\therefore \quad 2S = (2 + \sqrt{3})R$

$\therefore \quad \dfrac{c}{2S} = \dfrac{\sqrt{3}}{2 + \sqrt{3}}$

3. Clearly the triangle is right angled. Hence angles are 30º, 60º and 90º are in ratio 1 : 2 : 3



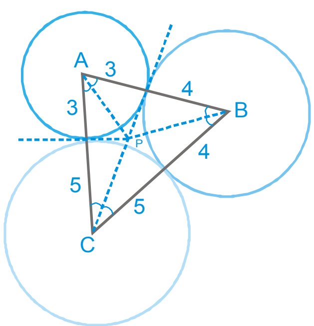
4. Consider $\dfrac{b-c}{a} = \dfrac{k(\sin B - \sin C)}{k \sin A}$

$$= \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \frac{\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)\sin\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)} = \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$$

5. Clearly P is the incentre of triangle ABC.

$$r = \frac{\Delta}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$



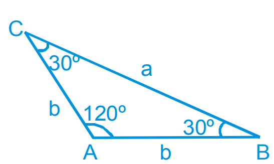
Here $2s = 7 + 8 + 9 \Rightarrow s = 12$

Here $r = \sqrt{\dfrac{5.4.3}{12}} = \sqrt{5}$

6. $\Delta = \dfrac{1}{2} \cdot b \cdot b \cdot \sin 120^\circ = \dfrac{\sqrt{3}}{4} b^2$ ......**(i)**

Also $\dfrac{\sin 120^\circ}{a} = \dfrac{\sin 30^\circ}{b} \Rightarrow a = \sqrt{3}b$ ......**(ii)**

and $\Delta = \sqrt{3}s$ and $s = \dfrac{1}{2}(a+2b)$

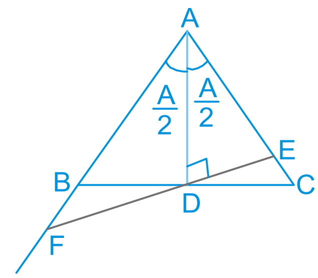


$\Rightarrow \Delta = \dfrac{\sqrt{3}}{2}(a+2b)$ ....**(3)**

From (1), (2) and (3), we get $\Delta = \left(12 + 7\sqrt{3}\right)$

7. We have $\Delta ABC = \Delta ABD + \Delta ACD$

$\Rightarrow \dfrac{1}{2} bc\sin A = \dfrac{1}{2} cAD\sin\dfrac{A}{2} + \dfrac{1}{2} b \times AD\sin\dfrac{A}{2}$



$\Rightarrow AD = \dfrac{2bc}{b+c}\cos\dfrac{A}{2}$

Again $AE = AD \sec\dfrac{A}{2}$

$= \dfrac{2bc}{b+c} \Rightarrow$ AE is HM of b and c.

$EF = ED + DF = 2DE = 2 \times AD \tan\dfrac{A}{2} = \dfrac{2 \times 2bc}{b+c} \times \cos\dfrac{A}{2}$

$\times \tan\dfrac{A}{2} = \dfrac{4bc}{b+c}\sin\dfrac{A}{2}$

As $AD \perp EF$ and $DE = DF$ and AD is bisector

$\Rightarrow \Delta AEF$ is isosceles.

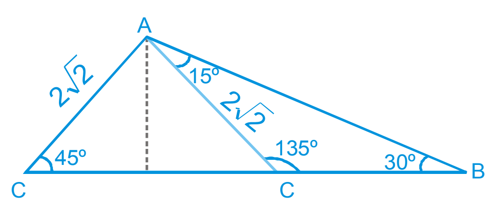
Hence A, B, C and D are correct answers.

8. In ΔABC, by sine rule

$\dfrac{a}{\sin A} = \dfrac{2\sqrt{2}}{\sin 30^\circ} = \dfrac{4}{\sin C} \Rightarrow C = 45^\circ, C' = 135^\circ$

When $C = 45^\circ \Rightarrow A = 180^\circ - (45^\circ + 30^\circ) = 105^\circ$

When $C' = 135^\circ \Rightarrow A = 180^\circ - (135^\circ + 30^\circ) = 15^\circ$



Area of $\Delta ABC' = \dfrac{1}{2} AB \cdot AC' \cdot \sin\angle BAC' = \dfrac{1}{2} \times 4 \times$

$2\sqrt{2}\ \sin(15^\circ) = 4\sqrt{2} \times \dfrac{\sqrt{3}-1}{2\sqrt{2}} = 2\left(\sqrt{3}-1\right)$

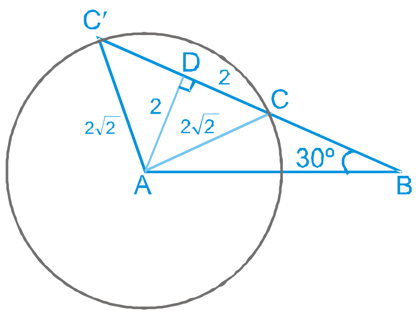
Area of ΔABC

$= \frac{1}{2} AB.AC.\sin A = \frac{1}{2} \times 4 \times 2\sqrt{2}\ \sin(105^\circ) = 2\left(\sqrt{3}+1\right)$

Absolute difference of areas of triangles

$= |2\left(\sqrt{3}+1\right) - 2\left(\sqrt{3}-1\right)| = 4$

Aliter



$AD = 2,\ DC = 2$

Difference of Areas of triangle ABC and ABC′ = Area of triangle ACC′

$= \frac{1}{2} AD \times CC' = \frac{1}{2} \times 2 \times 4 = 4$

9. $\cos B + \cos C = 4\sin^2\frac{A}{2}$

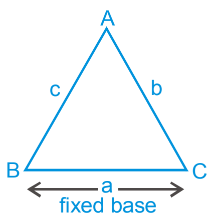
$\Rightarrow\ 2\cos\frac{B+C}{2}\cos\frac{B-C}{2} = 4\sin^2\frac{A}{2}$

$\Rightarrow\ 2\sin\frac{A}{2}\left[\cos\frac{B-C}{2} - 2\sin\frac{A}{2}\right] = 0$

$\Rightarrow\ \cos\left(\frac{B-C}{2}\right) - 2\cos\left(\frac{B+C}{2}\right) = 0$ as $\sin\frac{A}{2} \neq 0$

$\Rightarrow\ -\cos\frac{B}{2}\cos\frac{C}{2} + 3\sin\frac{B}{2}\sin\frac{C}{2} = 0$

$\Rightarrow\ \tan\frac{B}{2}\tan\frac{C}{2} = \frac{1}{3}$



$\Rightarrow\ \sqrt{\frac{(s-a)(s-c)}{s(s-b)}\cdot\frac{(s-b)(s-a)}{s(s-c)}} = \frac{1}{3}$

$\Rightarrow\ \frac{s-a}{s} = \frac{1}{3}\ \Rightarrow\ 2s = 3a\ \Rightarrow\ b + c = 2a$

$\therefore$ Locus of A is an ellipse

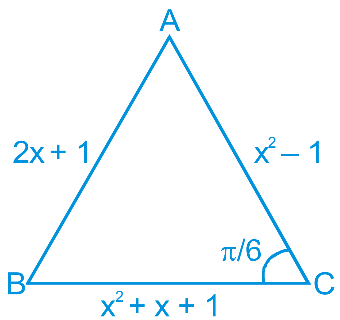
10. $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A = \frac{2}{2R}(a\cos C + c\cos A)$

$= \frac{b}{R} = 2\sin B = 2\sin 60^\circ = \sqrt{3}$

11. $\cos\frac{\pi}{6} = \frac{(x^2-1)^2 + (x^2+x+1)^2 - (2x+1)^2}{2(x^2+x+1)(x^2-1)}$

$\frac{\sqrt{3}}{2} = \frac{(x^2-1)^2 + (x^2+3x+2)(x^2-x)}{2(x^2+x+1)(x^2-1)}$

$\frac{\sqrt{3}}{2} = \frac{(x^2-1)^2 + (x+1)(x+2)x(x-1)}{2(x^2+x+1)(x^2-1)}$



$\Rightarrow\ \sqrt{3} = \frac{x^2 - 1 + x(x+2)}{x^2+x+1}$

$\Rightarrow\ \sqrt{3}(x^2+x+1) = 2x^2 + 2x - 1$

$\Rightarrow\ (\sqrt{3}-2)x^2 + (\sqrt{3}-2)x + (\sqrt{3}+1) = 0$

on solving

$x^2 + x - (3\sqrt{3}+5) = 0$ we get $x = \sqrt{3}+1, -(2+\sqrt{3})$

$\because$ At $x = -(2+\sqrt{3})$, Side c becomes negative.

$\therefore\ x = \sqrt{3}+1$

12. Area of triangle $= \frac{1}{2}ab\sin C = 15\sqrt{3}$

$\Rightarrow\ \frac{1}{2}.6.10\sin C = 15\sqrt{3}\ \Rightarrow\ \sin C = \frac{\sqrt{3}}{2}$

$\Rightarrow\ C = \frac{2\pi}{3}$ (C is obtuse angle)

Now $\cos C = \frac{a^2+b^2-c^2}{2ab}$

$\Rightarrow\ -\frac{1}{2} = \frac{36+100-c^2}{2.6.10}\ \Rightarrow\ c = 14$

$\therefore\ r = \frac{\Delta}{s} = \frac{15\sqrt{3}}{\frac{6+10+14}{2}} = \sqrt{3}\ \Rightarrow\ r^2 = 3$

13. $a = 2 = QR$

$b = \frac{7}{2} = PR$

$c = \frac{5}{2} = PQ$

$s = \frac{a+b+c}{2} = \frac{8}{4} = 4$

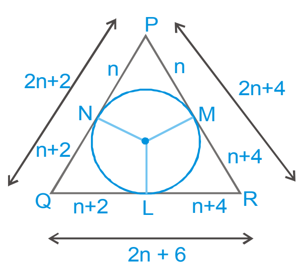
$\frac{2\sin P - 2\sin P\cos P}{2\sin P + 2\sin P\cos P} = \frac{2\sin P(1-\cos P)}{2\sin P(1+\cos P)} = \frac{1-\cos P}{1+\cos P}$

$= \frac{2\sin^2\frac{P}{2}}{2\cos^2\frac{P}{2}} = \tan^2\frac{P}{2}$

$= \frac{(s-b)(s-c)}{s(s-a)} = \frac{(s-b)^2(s-c)^2}{\Delta^2}$

$= \frac{\left(4-\frac{7}{2}\right)^2\left(4-\frac{5}{2}\right)^2}{\Delta^2} = \left(\frac{3}{4\Delta}\right)^2$

14. $\cos P = \frac{(2n+2)^2+(2n+4)^2-(2n+6)^2}{2(2n+2)(2n+4)} = \frac{1}{3}$



$\Rightarrow \frac{4n^2-16}{8(n+1)(n+2)} = \frac{1}{3}$

$= \frac{n^2-4}{2(n+1)(n+2)} = \frac{1}{3}$

$\Rightarrow \frac{n-2}{2(n+1)} = \frac{1}{3}$

$= 3n - 6 = 2n + 2$

$\Rightarrow n = 8$

$\Rightarrow 2n + 2 = 18$

$\Rightarrow 2n + 4 = 20$

$\Rightarrow 2n + 6 = 22$

MOCK TEST

1. Using properties of pedal triangle,

we have

$\angle MLN = 180° - 2A$

$\Rightarrow \angle LMN = 180° - 2B$

$\Rightarrow \angle MNL = 180° - 2C$

Hence the required sum

$= \sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

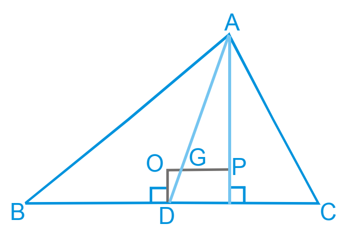
2. **(D)**

$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{64+81-49}{2.8.9} = \frac{145-49}{144} = \frac{96}{144}$

$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{49+81-64}{2.7.9} = \frac{66}{126} = \frac{11}{21}$

$\therefore \quad \cos A : \cos B = \frac{14}{11}$

3.



From figure, we can observe that ΔOGD is directly similar to ΔPGA

4. **(A)**

$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$

$= \frac{s(s-a)}{\Delta} + \frac{s(s-b)}{\Delta} + \frac{s(s-c)}{\Delta}$

$= \frac{s}{\Delta}[3s-(a+b+c)] = \frac{s[3s-2s]}{\Delta} = \frac{s^2}{\Delta}$

$= \left(\frac{a+b+c}{2}\right)^2 \times \frac{4R}{abc} = \frac{(a+b+c)^2 R}{abc} \quad [\because \Delta = \frac{abc}{4R}]$

also $\frac{s^2}{\Delta} = \frac{\Delta^2}{r^2\Delta} = \frac{\Delta}{r^2}$

5. $\tan\frac{C}{2}(a\tan A + b\tan B) = a + b$

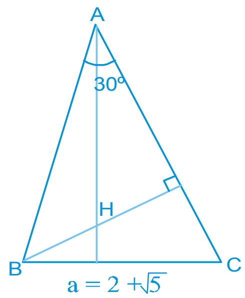
$\Rightarrow \quad a\cos B\cos\left(A + \frac{C}{2}\right) + b\cos A\cos\left(B + \frac{C}{2}\right) = 0$

$\Rightarrow \quad (a\cos B - b\cos A)\cos\left(A + \frac{C}{2}\right) = 0,$

as $\quad \cos\left(B + \frac{C}{2}\right) = \cos\left(\pi - A - \frac{C}{2}\right)$

$\Rightarrow \quad A = B$, in either case

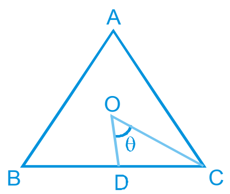
6. (B)



$$R = \frac{a}{2\sin A} = \frac{2+\sqrt{5}}{2\sin 30^\circ} = \frac{2+\sqrt{5}}{2\times\frac{1}{2}} = (2+\sqrt{5})$$

Now $AH = 2R\cos A = 2(2+\sqrt{5})\cos 30^\circ = (2+\sqrt{5})\sqrt{3}$

7. (A) In the adjacent figure we have $\angle OCB = \frac{\pi}{2} - A$



$\Rightarrow \quad \angle ODC = \pi - \left(\frac{\pi}{2} - A + \theta\right) = \frac{\pi}{2} + (A - \theta)$

if R_1 be the circumradius of ΔOCD, then

$$\frac{OC}{\sin\left(\frac{\pi}{2} + (A-\theta)\right)} = 2R_1$$

$\Rightarrow \quad 2R_1 = \frac{R}{\cos(A-\theta)}$

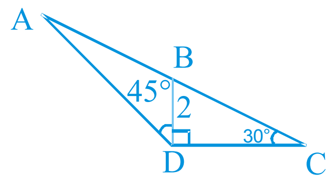
As $D \to C \quad \Rightarrow \quad \theta \to 0$

$\Rightarrow \quad 2R_1 \to \frac{R}{\cos A}$

$\therefore \quad R_1 \to \frac{R}{2\cos A}$

8. In ΔABD, $\frac{AD}{\sin 120^\circ} = \frac{BD}{\sin 15^\circ}$

$$\Rightarrow \quad AD = \frac{2\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = 3\sqrt{2} + \sqrt{6}$$



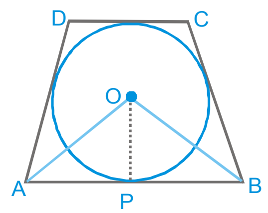
9. (B)

Let 'O' be the centre of circle and 'P' be its point of contact with side AB. We have

$AP = OP\,.\cot\frac{A}{2} = \cot\frac{A}{2}$ and

$PB = OP\,.\cot\frac{B}{2} = \cot\frac{B}{2}$

$\Rightarrow \quad AP + PB = \cot\frac{A}{2} + \cot\frac{B}{2}$



$$= \frac{\sin\left(\frac{A+B}{2}\right)}{\sin\frac{A}{2}.\sin\frac{B}{2}} = AB$$

Similarly

$$CD = \frac{\sin\left(\frac{C+D}{2}\right)}{\sin\frac{C}{2}.\sin\frac{D}{2}}$$

Since $A + B + C + D = 2\pi \Rightarrow \frac{A+B}{2} = \pi - \frac{C+D}{2}$

$\Rightarrow \quad \sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{C+D}{2}\right)$

$\Rightarrow \quad AB.\sin\frac{A}{2}\,.\sin\frac{B}{2} = \sin\frac{C}{2}\,.\sin\frac{D}{2}\,.\,CD$

10. (D)

S_1 :

$$\frac{\Delta}{s-b}+\frac{\Delta}{s-c}=\frac{\Delta}{s-a}-\frac{\Delta}{s} \Rightarrow \frac{a}{(s-b)(s-c)}=\frac{a}{s(s-a)}$$

or $\frac{s(s-a)}{(s-b)(s-c)}=1 \Rightarrow \cot\frac{A}{2}=1 \Rightarrow A=90^0$

S_2 : $4R^2(\sum \sin^2 A) = 8R^2$

$\Rightarrow \sin^2A + \sin^2B + \sin^2C = 2$

$\Rightarrow 2\cos A\cos B\cos C = 0$

$\Rightarrow$ A or B or C = 90^0

S_3 : $2r_1 = 2s \Rightarrow s\tan\frac{A}{2} = s \Rightarrow \tan\frac{A}{2} = 1 \Rightarrow A = 90^0$

S_4 : $2R = 4R\left[\left(\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)\right.$

$\left.-\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]$

$1 = 2\sin\frac{A}{2}\cos\left(\frac{B+C}{2}\right)$ or $1 = 2\sin^2 A/2$

$\Rightarrow \cos A = 0 \Rightarrow A = 90^0$

11. (A, C)

$\Delta = \Delta_1 + \Delta_2$

$$\frac{1}{2}bc\sin A=\frac{1}{2}cx\sin\frac{A}{2}+\frac{1}{2}xb\sin\frac{A}{2}$$

12. (A, C, D)

$$p=\frac{2\Delta}{a},\ q=\frac{2\Delta}{b},\ r=\frac{2\Delta}{c}$$

(A) $(\Sigma p)\left(\Sigma\frac{1}{p}\right)=2\Delta\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{a+b+c}{2\Delta}\right)=(\Sigma a)\left(\Sigma\frac{1}{a}\right)$

(C) $(\Sigma p)(\Sigma pq)(\Pi a)$

$$=\left(\frac{2\Delta}{a}+\frac{2\Delta}{b}+\frac{2\Delta}{c}\right)\left(\frac{4\Delta^2}{ab}+\frac{4\Delta^2}{bc}+\frac{4\Delta^2}{ca}\right)abc$$

$$=\frac{2\Delta(ab+bc+ca)}{abc}\cdot 4\Delta^2\frac{(a+b+c)}{abc}\cdot abc$$

$= (\Sigma a)\ (\Sigma ab)\ (\Pi p)$

(D) $\left(\Sigma\frac{1}{p}\right)\Pi\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)\Pi a^2$

$$=\left(\frac{a+b+c}{2\Delta}\right)\left(\frac{a+b-c}{2\Delta}\right)\left(\frac{a-b+c}{2\Delta}\right)\left(\frac{b+c-a}{2\Delta}\right).a^2b^2c^2$$

$$=\frac{s}{\Delta}\cdot\frac{(s-c)}{\Delta}\cdot\frac{(s-b)}{\Delta}\cdot\frac{(s-a)}{\Delta}(abc)^2=\left(\frac{abc}{\Delta}\right)^2$$

$= (4R)^2 = 16R^2$.

13. (A, C, D)

given expression $(a-c)^2 + (a-2b)^2 = 0$

$\Rightarrow$ a = 2b and c = a. Sides are 2b, b, 2b

$\Rightarrow$ isosceles,

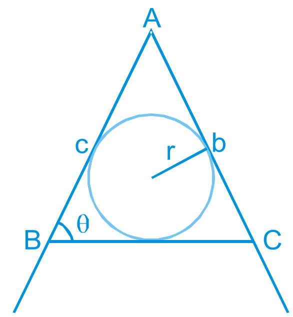
$\because \cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{7b^2}{8b^2} = \frac{7}{8}$

and $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{1}{4}$

14. (A)

$r = \frac{\Delta}{s}$; $\rho = \frac{\Delta}{s-a}$

$$\rho r = \frac{\Delta^2}{s(s-a)} = \frac{s(s-a)(s-b)(s-c)}{s(s-a)}$$



$= (s-b)(s-c) = (s-b)^2 \quad (\because b = c)$

$= \frac{(2s-2b)^2}{4} = \frac{(a+b+c-2b)^2}{4} \quad (\because b=c)$

$= \frac{a^2}{4} = \frac{4R^2\sin^2 A}{4} = R^2\sin^2 A$

Also if $\angle B = \theta \Rightarrow \angle A = \pi - 2\theta$

$\therefore \rho r = R^2\sin^2(\pi-2\theta) = R^2\sin^2 2\theta = R^2\sin^2 2B$

15. We have $\frac{\Delta}{s-a} = \frac{2\Delta}{s-b} = \frac{3\Delta}{s-c}$

$\Rightarrow$ a : b : c = 5 : 4 : 3

16. (A)

Statement-II is true.

Statement-I tanA = tanB = tanC

(By using statement-1)

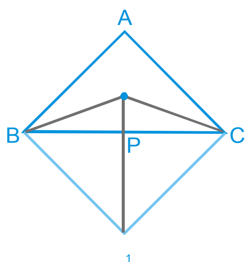
A = B = C i.e. a = b = c

$r_1 = r_2 = r_3$

$$\therefore \frac{r_1 + r_2 + r_3}{r} = 3 . \frac{r_1}{r} = 3 . \frac{\frac{\Delta}{s-a}}{\frac{\Delta}{s}} = 3\left(\frac{a+b+c}{b+c-a}\right) = 9$$

17. (C)

$\angle ICI_1 = \frac{\pi}{2}, \ \angle IBI_1 = \frac{\pi}{2}$



$\therefore$ $BICI_1$ is cyclic quadrilateral

$\therefore$ BP . PC = IP . I_1P

18. (C)

$$\frac{\cos^2 \frac{A}{2}}{a} = \frac{s(s-a)}{abc}$$

$$\therefore \sum \frac{\cos^2 \frac{A}{2}}{a} = \frac{s^2}{abc}$$

19. (A)

s = 21

$$\Delta = \sqrt{21.8.7.6} = \sqrt{3.7.2^4.7.3} = 3 . 7 . 4 = 84$$

$$r = \frac{\Delta}{s} = \frac{84}{21} = 4$$

20. (B)

Statement-I :

a < b < c

s – a > s – b > s – c

s > s – a > s – b > s – c

$$\frac{\Delta}{s} < \frac{\Delta}{s-a} < \frac{\Delta}{s-b} < \frac{\Delta}{s-c}$$

$r < r_1 < r_2 < r_3$

Statement-II :

$$r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}, r = \frac{\Delta}{s}$$

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s}{\Delta} = \frac{1}{r}$$

21. (A)→(p), (B)→(q), (C)→(r), (D)→(s)

(A) $(b + c)^2 - a^2 = \lambda bc$

or $b^2 + c^2 - a^2 = (\lambda - 2)\, bc$

$$\frac{b^2 + c^2 - a^2}{2bc} = \frac{\lambda - 2}{2}$$

$$\cos A = \frac{\lambda - 2}{2} < 1$$

or $\lambda - 2 < 2$

$\lambda < 4$

$\therefore$ greatest value of λ is 3.

(B) tanA + tanB + tanC = 9

in any triangle tanA + tanB + tanC = tan A tan B tan C

$$\frac{\tan^2 A + \tan^2 B + \tan^2 C}{3} \geq (\tan A \tan B \tan C)^{2/3}$$

$k \geq 3\,(9)^{2/3}$

$k \geq 9 . 3^{1/3}$

(C) Since the line joining the circumcenter to the incentre is parallel to BC

$\therefore$ r = RcosA

$$\therefore 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = R\cos A$$

$\therefore$ – 1 + cos A + cosB + cosC = cosA

$\therefore$ cosB + cosC = 1

(D) $a = 5, b = 4$

$$\cos(A - B) = \frac{31}{32}$$

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2} = \frac{1}{9}\cot\frac{C}{2}$$

$$\because \quad \cos(A-B) = \frac{1-\tan^2\frac{A-B}{2}}{1+\tan^2\frac{A-B}{2}}$$

$$\frac{31}{32} = \frac{1-\frac{1}{81}\cot^2\frac{C}{2}}{1+\frac{1}{81}\cot^2\frac{C}{2}}$$

$$31 + \frac{31}{81}\cot^2\frac{C}{2} = 32 - \frac{32}{81}\cot^2\frac{C}{2}$$

$$\frac{7}{9}\cot^2\frac{C}{2} = 1$$

$$\cot^2\frac{C}{2} = \frac{9}{7}$$

$$\because \quad \cos C = \frac{1-\tan^2\frac{C}{2}}{1+\tan^2\frac{C}{2}} = \frac{1-\frac{7}{9}}{1+\frac{7}{9}} = \frac{2}{16} = \frac{1}{8}$$

$$\because \quad \cos C = \frac{a^2+b^2-c^2}{2ab} \Rightarrow \frac{25+16-c^2}{2\times 20} = \frac{1}{8}$$

$25 + 16 - c^2 = 5$

$c^2 = 36$

$c = 6$

22. (A)→(p), (B)→(p,r), (C)→(r), (D)→(p,r)

(A) Since $\cos A = \frac{\sin B}{2\sin C}$, we have $\frac{b^2+c^2-a^2}{2bc} = \frac{b}{2c}$

or $b^2 + c^2 - a^2 = b^2$ or $c^2 = a^2$

Hence c = a and so the ΔABC is isosceles

(B) $\cos A(\sin B - \sin C) + (\sin 2B - \sin 2C) = 0$

or $\cos A(\sin B - \sin C) + 2\sin(B - C)\cos(B + C) = 0$

or $\cos A(\sin B - \sin C) - 2\cos A\sin(B - C) = 0$

∴ either $\cos A = 0 \Rightarrow A = 90^\circ$

or $(\sin B - \sin C) - 2(\sin B\cos C - \cos B\sin C) = 0$

$$\therefore \quad (b-c) - 2\left[b\cdot\frac{a^2+b^2-c^2}{2ab} - c\cdot\frac{c^2+a^2-b^2}{2ca}\right] = 0$$

or $a(b - c) - 2(b^2 - c^2) = 0$

$\Rightarrow (b - c)[a - 2(b + c)] = 0$

∴ $b - c = 0$ or $b = c$

∴ isosceles

(C) Combine first and third terms and put the value of cos B, we get

$$\therefore \quad \frac{2}{ac}\cdot(B) + \frac{1}{b}\cdot\frac{c^2+a^2-b^2}{2ca} = \frac{a^2+b^2}{abc}$$

or $4b^2 + c^2 + a^2 - b^2 = 2a^2 + 2b^2$

∴ $b^2 + c^2 = a^2$

∴ $\angle A = 90^\circ$

(D) $$\frac{\sin(A-B)}{\sin(A+B)} = \frac{k^2(\sin^2 A - \sin^2 B)}{k^2(\sin^2 A + \sin^2 B)}$$ by sine formula

or $$\frac{\sin(A-B)}{\sin C} = \frac{\sin(A-B)\sin(A+B)}{\sin^2 A + \sin^2 B}$$

or $$\sin(A-B)\left[\frac{1}{\sin C} - \frac{\sin C}{\sin^2 A + \sin^2 B}\right] = 0$$

∴ either $\sin(A - B) = 0$

∴ A = B i.e. Δ is isosceles

or $\sin^2 A + \sin^2 B = \sin^2 C$ or $a^2 + b^2 = c^2$

∴ Δ is right angled

23.

1. $$PG = \frac{1}{3}AD = \frac{1}{3}\cdot\frac{2\Delta}{a} = \frac{2}{3a}\cdot\frac{1}{2}.absinC \quad \text{or} \quad = \frac{1}{3}bsinC$$

$$(\because \ \Delta = \frac{1}{2}acsinB)$$

$$\therefore \ PG = \frac{2}{3a}\cdot\frac{1}{2}acsinB = \frac{1}{3}csinB$$

2. Area of ΔGPL $= \frac{1}{2}$(PL) (PG) and Area of ΔALD

$= \frac{1}{2}$(DL) (AD)

$\because$ PL $= \frac{1}{3}$ DL and PG $= \frac{1}{3}$ AD

$$\therefore \quad \frac{\text{Area of } \Delta GPL}{\text{Area of } \Delta ALD} = \frac{\frac{1}{2}(PL)(PG)}{\frac{1}{2}(DL)(AD)}$$

$$= \frac{\frac{1}{3}(DL)\times\frac{1}{3}(AD)}{(DL)(AD)} = \frac{1}{9}$$

3. Area of ΔPQR = Area of ΔPGQ + Area of ΔQGR + Area of ΔRGP ......**(i)**

$\because$ Area of $\Delta PGQ = \frac{1}{2}$ PG.GQ.sin($\angle$ PGQ)

$= \frac{1}{2} \times \frac{1}{3} AD \times \frac{1}{3} BE \sin(\pi - C)$

$= \frac{1}{18} \times \frac{2\Delta}{a} \times \frac{2\Delta}{b} \sin C$

$= \frac{2}{9ab} \times \frac{1}{2} bc\sin A \times \frac{1}{2} ac\sin B \times \sin C$

$= \frac{c^2}{18} \sin A.\sin B.\sin C$

Similarly Area of $\Delta QGR = \frac{a^2}{18} \sin A.\sin B.\sin C$ and

Area of $\Delta RGP = \frac{b^2}{18} \sin A.\sin B.\sin C$

$\therefore$ From equation **(i)**, we get Area of $\Delta PQR = \frac{1}{18}$ $(a^2 + b^2 + c^2) \sin A.\sin B.\sin C$

24.

1. $\frac{bx}{c} + \frac{cy}{a} + \frac{az}{b} = b\sin B + c\sin C + a\sin A = \frac{b^2 + c^2 + a^2}{2R}$

$\therefore \quad k = 2R$

2. $\cot A + \cot B + \cot C = \frac{R}{abc}(b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2)$

$= \frac{R}{abc}(b^2 + c^2 + a^2) = \frac{R}{abc}\left(\frac{4\Delta^2}{x^2} + \frac{4\Delta^2}{y^2} + \frac{4\Delta^2}{z^2}\right)$

$= \frac{4\Delta^2 R}{abc}\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)$

$= \frac{4\Delta R}{abc} . \Delta \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)$

$= \Delta\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)$

$\therefore \quad k = \Delta$

3. $\sum \frac{c\sin B + b\sin C}{x} = \sum \frac{x + x}{x} = 6$

25. (B, B, B)

$\cos(A - B) = \frac{4}{5} \quad \Rightarrow \quad \frac{1 - \tan^2 \frac{A-B}{2}}{1 + \tan^2 \frac{A-B}{2}} = \frac{4}{5}$

$\Rightarrow \quad \frac{2\tan^2 \frac{A-B}{2}}{2} = \frac{1}{9}$

$\Rightarrow \quad \tan \frac{A-B}{2} = \frac{1}{3} \quad \Rightarrow \quad \frac{1}{3} = \frac{6-3}{6+3} \cot \frac{C}{2}$

$\Rightarrow \quad \cot \frac{C}{2} = 1 \quad \Rightarrow \quad C = 90°$

Area of triangle $= \frac{1}{2}$ absinC

$\Rightarrow \quad \text{Area} = \frac{1}{2} \times 6 \times 3 \times 1 = 9$

$\frac{a}{\sin A} = \frac{\sqrt{a^2 + b^2}}{1}$

$\Rightarrow \quad \frac{6}{\sin A} = \sqrt{45}$

$\Rightarrow \quad \sin A = \frac{2}{\sqrt{5}}$

26. (8)

$r = \frac{\Delta}{s}$

$s = 5$ or $a + b + c = 10$

$\Delta = \frac{abc}{4R}$ or $abc = 60$

Now $\quad \Delta^2 = s(s - a)(s - b)(s - c)$

or $\quad 5 = (5 - a)(5 - b)(5 - c)$

$= 125 - 25(a + b + c) + 5(ab + bc + ca) - abc$

$\therefore \quad ab + bc + ca = 38$

or $\quad a^2 + b^2 + c^2 = (a + b + c)^2 - 2(38) = 24$

27. (3)

$a + b - c = 2$

and $\quad 2ab - c^2 = 4$

$\Rightarrow \quad a^2 + b^2 + c^2 + 2ab - 2bc - 2ca = 4 = 2ab - c^2$

$\Rightarrow \quad (b - c)^2 + (a - c)^2 = 0$

$\Rightarrow \quad a = b = c$

Triangle is equilateral ;

hence $\quad a = 2$

$\Rightarrow \quad \Delta = \sqrt{3}$

28. (2)

We have $\cos^2\frac{C}{2}=\frac{s(s-c)}{ab}$

So that

$$\frac{2ab}{(a+b+c)\Delta}\cos^2\frac{C}{2}=\frac{2ab}{2s\Delta}\cdot\frac{s(s-c)}{ab}=\frac{s-c}{\Delta}$$

Now, the area of triangle ABC is $\Delta=\frac{1}{2}ap_1$,

i.e., $p_1=2\Delta/a$. Similarly, $p_2=2\Delta/b$ and $p_3=2\Delta/c$.

$$\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=\frac{a+b-c}{2\Delta}=\frac{2s-2c}{2\Delta}=\frac{s-c}{\Delta}$$

29. (2)

$$\frac{a+c}{\sqrt{a^2-ac+c^2}}=\frac{\sin A+\sin C}{\sqrt{\sin^2 A-\sin A\sin C+\sin^2 C}}$$

$$=\frac{2\sin\frac{A+C}{2}\cos\frac{A-C}{2}}{\sqrt{\frac{1-\cos 2A}{2}-\frac{\cos(A-C)-\cos(A+C)}{2}+\frac{1-\cos 2C}{2}}}$$

$$=\frac{2\sqrt{2}\cdot\frac{\sqrt{3}}{2}\cos\frac{A-C}{2}}{\sqrt{2-(\cos 2A+\cos 2C)-\cos(A-C)+\cos(A+C)}}$$

$$=\frac{\sqrt{6}\cos\frac{A-C}{2}}{\sqrt{\frac{3}{2}-2\cos(A+C)\cos(A-C)-\cos(A-C)}}$$

$$=\frac{\sqrt{6}\cos\frac{A-C}{2}}{\sqrt{\frac{3}{2}+\cos(A-C)-\cos(A-C)}}=2\cos\frac{A-C}{2}$$

30. (6)

$$\frac{h_1+r}{h_1-r}+\frac{h_2+r}{h_2-r}+\frac{h_3+r}{h_3-r}\ge 6$$

$$\Delta=\frac{1}{2}ah \Rightarrow h_1=\frac{2\Delta}{a}.$$

Similarly $h_2=\frac{2\Delta}{b}$, $h_3=\frac{2\Delta}{c}$

So $\frac{h_1+r}{h_1-r}+\frac{h_2+r}{h_2-r}+\frac{h_3+r}{h_3-r}$

$$=\frac{2\Delta/a+\Delta/s}{2\Delta/a-\Delta/s}+\frac{2\Delta/b+\Delta/s}{2\Delta/b-\Delta/s}+\frac{2\Delta/c+\Delta/s}{2\Delta/c-\Delta/s}$$

$$=\frac{2s+a}{2s-a}+\frac{2s+b}{2s-b}+\frac{2s+c}{2s-c}=\frac{4s}{2s-a}+\frac{4s}{2s-b}+\frac{4s}{2s-c}-3$$

$$=3\left[\frac{1}{3}\left\{\frac{4s}{2s-a}+\frac{4s}{2s-b}+\frac{4s}{2s-c}\right\}\right]-3$$

$$\ge 3\left(\frac{3}{\frac{2s-a}{4s}+\frac{2s-b}{4s}+\frac{2s-c}{4s}}\right]-3,$$

Since $(AM\ge HM)$ ≥ 6