HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

1. $\sin\frac{6\pi}{5} + i\left(1+\cos\frac{6\pi}{5}\right)$

lies in 2^{nd} quadrant and

$$\left|\frac{1+\cos\frac{6\pi}{5}}{\sin\frac{6\pi}{5}}\right| = \left|\cot\left(\frac{3\pi}{5}\right)\right| = \left|\cot\left(\frac{\pi}{2}+\frac{\pi}{10}\right)\right| = \tan\frac{\pi}{10}$$

2^{nd} quadrant $\Rightarrow \pi - \frac{\pi}{10}$

2. $z = 1 + \cos\frac{11\pi}{9} + i\sin\frac{11\pi}{9}$

$$= \left(-2\cos\frac{11\pi}{18}\right)\left[\cos\frac{11\pi}{18} + i\sin\frac{11\pi}{18}\right](-1)$$

$$|z| = -2\cos\frac{11\pi}{18} = 2\cos\frac{7\pi}{18}$$

$$\arg(z) = \frac{11\pi}{18} - \pi = \frac{-7\pi}{18}$$

4. $S = i + 2i^2 + 3i^3 + ... + 100\, i^{100}$

$iS = i^2 + 2i^3 + ... + 100\, i^{101}$

$S(1-i) \quad = i + i^2 + i^3 + ... + i^{100} - 100\, i^{101}$

$$S = \frac{-100i}{1-i} = \frac{-100i(1+i)}{2} = -50(i-1) = 50(1-i)$$

7. We have $\frac{3}{|z_2 - z_3|} = \frac{4}{|z_3 - z_1|} = \frac{5}{|z_1 - z_2|} = k$ (let)

$$\Rightarrow \frac{9}{|z_2 - z_3|^2} = \frac{16}{|z_3 - z_1|^2} = \frac{25}{|z_1 - z_2|^2} = k^2$$

Now $\frac{9}{|z_2 - z_3|^2} = k^2$

$$\Rightarrow \frac{9}{z_2 - z_3} = k^2\left(\bar{z}_2 - \bar{z}_3\right) \quad ....(i)$$

[As $|z|^2 = z\bar{z}$]

ly $\frac{16}{|z_3 - z_1|^2} = k^2 \Rightarrow \frac{16}{z_3 - z_1} = k^2\left(\bar{z}_3 - \bar{z}_1\right) \quad ....(ii)$

ly $\frac{25}{|z_1 - z_2|^2} = k^2 \Rightarrow \frac{25}{z_1 - z_2} = k^2\left(\bar{z}_1 - \bar{z}_2\right) \quad ....(iii)$

$\therefore$ On adding (1), (2) and (3), we get

$$= \frac{9}{z_2 - z_3} + \frac{16}{z_3 - z_1} + \frac{25}{z_1 - z_2}$$

$$= k^2\left(\bar{z}_2 - \bar{z}_3 + \bar{z}_3 - \bar{z}_1 + \bar{z}_1 - \bar{z}_2\right) = 0$$

11. $G \rightarrow$ Centroid of $\Delta = \frac{z_1 + z_2 + z_3}{3}$

$H \rightarrow$ Orthocentre = z say, $O \rightarrow$ Circum centre = 0

$\because$ G divides HO in ratio 2 : 1 reckening from

$$\frac{z_1 + z_2 + z_3}{3} = \frac{2\cdot 0 + 1\cdot z}{2+1} \Rightarrow z = z_1 + z_2 + z_3$$

14. $z = \frac{\pi}{4}(1+i)^4\left[\frac{1+\pi+\pi+1}{(\sqrt{\pi}+i)(1+\sqrt{\pi}\,i)}\right]$

$$= \frac{\pi}{4}(1+i)^4\,\frac{2}{i}$$

$$= \frac{\pi}{2}\frac{(1+i)^4}{i} = \frac{\pi}{2}\frac{4e^{i\pi}}{e^{i\pi/2}} = 2\pi\, e^{i\pi/2}$$

$|z| = 2\pi \qquad \text{amp}(z) = \frac{\pi}{2}$

$$\left(\frac{|z|}{\text{amp}(z)}\right) = \frac{2\pi}{\frac{\pi}{2}} = 4$$

15. Let $S = 1(\alpha_1 + \alpha_{2008}) + 2(\alpha_2 + \alpha_{2007}) + 3(\alpha_3 + \alpha_{2006}) +$ .... $+ 2008(\alpha_{2008} + \alpha_1)$ ....(i)

Also $S = 2008(\alpha_{2008} + \alpha_1) + 2007(\alpha_2 + \alpha_{2007}) +$ .... $+ 2(\alpha_2 + \alpha_{2007}) + 1(\alpha_1 + \alpha_{2008})$ ....(ii)

(writing in reverse order)

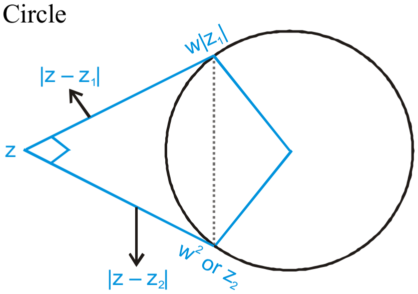
$\therefore$ On adding (1) and (2), we get

$$2S = 2009[2(\alpha_1 + \alpha_2 + \alpha_3 + ...\alpha_{2008})]$$

$$2S = 2009[2(\underbrace{1 + \alpha_1 + \alpha_2 + \alpha_3 + ...\alpha_{2008}}_{\text{zero}} - 1)]$$

Hence $S = -2009$

19. $\because$ Circle

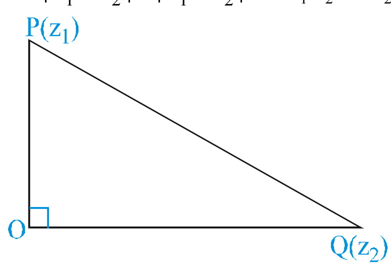


so by pythogorous theorem

$\lambda = |w - w^2|^2 = \left|\sqrt{3}\right|^2 = 3$

21. We have $\left|\dfrac{1}{z_2} + \dfrac{1}{z_1}\right| = \left|\dfrac{1}{z_2} - \dfrac{1}{z_1}\right|$

$\Rightarrow |z_1 + z_2| = |z_1 - z_2| \Rightarrow z_1\bar{z}_2 + z_2\bar{z}_1 = 0$



$\Rightarrow \dfrac{z_1}{z_2}$ is purely imaginary.

Hence ΔPQR is right angled at O.

$\therefore$ Circumcentre of ΔPOQ is the mid point of PQ i.e.

$\dfrac{1}{2}(z_1 + z_2)$

24. $z^2 + z + 1$ is real so

$z^2 + z + 1 = \bar{z}^2 + \bar{z} + 1$

$z^2 - \bar{z}^2 + z - \bar{z} = 0$

$(z - \bar{z})(z + \bar{z} + 1) = 0$

either $z = \bar{z}$ or $z + \bar{z} + 1 = 0$

$\Rightarrow$ Im(z) = 0 Let $z = \alpha + i\beta$

$\Rightarrow$ z is purely real

So $\alpha + i\beta + \alpha - i\beta + 1 = 0$

$\Rightarrow 2\alpha + 1 = 0$

$\Rightarrow \alpha = -\dfrac{1}{2}$

Also $(\alpha + i\beta)^2 + (\alpha + i\beta) + 1 > 0$

$\alpha^2 + \alpha + 1 - \beta^2 + i(2\alpha\beta + \beta) > 0$

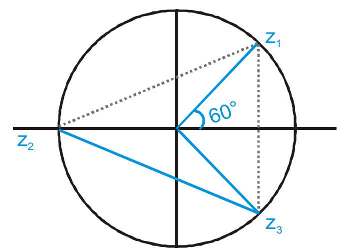
if $\alpha = -1/2$ then

$\dfrac{1}{4} - \dfrac{1}{2} + 1 - \beta^2 > 0$

$\Rightarrow \beta^2 - \dfrac{3}{4} < 0 \Rightarrow -\dfrac{\sqrt{3}}{2} < \beta < \dfrac{\sqrt{3}}{2}$

25. $z_1 = 1 + i\sqrt{3} = 2e^{\frac{i\pi}{3}}$

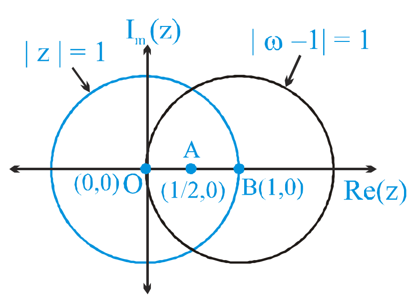
$z_2 = z_1 e^{\frac{2\pi}{3}i} = \dfrac{-1-3}{2} = -2$



$z_3 = z_1 e^{-\frac{2\pi}{3}i} = 2e^{-\frac{\pi}{3}i} = 1 - i\sqrt{3}$

27. Least distance and greatest distance of any z and ω from the point $\left(\dfrac{1}{2}, 0\right)$ are $\dfrac{1}{2}$ and $\dfrac{3}{2}$ respectively.

$\therefore \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{2}\right)^2 \le \left|z - \dfrac{1}{2}\right|^2 + \left|\omega - \dfrac{1}{2}\right|^2 \le \left(\dfrac{3}{2}\right)^2 + \left(\dfrac{3}{2}\right)^2$



Hence $2 \le |2z - 1|^2 + |2\omega - 1|^2 \le 18$

31. Let centre be origin & A_1 be Z_0 & $OA_1 = OA_2 = ... = a$

So $A_2 = z_0\, e^{i\frac{2\pi}{n}}$, $A_3 = z_0 e^{i\frac{4\pi}{n}}$ ......

Now $A_1A_2 = \left|z_0 - z_0 e^{i\frac{2\pi}{n}}\right| = |z_0| \left|1 - e^{\frac{i2\pi}{n}}\right| = a\left|e^{\frac{i\pi}{n}}\right|$

$$\left|e^{-\frac{i\pi}{n}} - e^{\frac{i\pi}{n}}\right| = 2a\sin\left(\frac{\pi}{n}\right)$$

similarlly $A_1A_3 = 2a\sin\frac{2\pi}{n}$, $A_1A_4 = 2a\sin\frac{3\pi}{n}$

Given $\quad \dfrac{1}{2a\sin\frac{\pi}{n}} = \dfrac{1}{2a\sin\frac{2\pi}{n}} + \dfrac{1}{2a\sin\frac{3\pi}{n}}$

$\Rightarrow \sin\frac{3\pi}{n} = \sin\frac{4\pi}{n} \Rightarrow \frac{3\pi}{n} = \pi - \frac{4\pi}{n}$

$\Rightarrow$ n = 7 is only possible value.

33. Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

So $\omega = \frac{1}{r}[\cos(\theta - \pi/2) + i\sin(\theta - \pi/2)] = \frac{1}{r}e^{i(\theta - \pi/2)}$

So $\bar{z}\ \omega = r\,e^{-i\theta} \times \frac{1}{r}\,e^{i(\theta - \pi/2)} = e^{-i\pi/2} = -i$

35. $(z-1)(z-\alpha_1)\,.........(z-\alpha_4) = z^5 - 1$

Put $z = \omega$, $z = \omega^2$ and divide

$$\frac{(\omega-1)(\omega-\alpha_1)(\omega-\alpha_2)(\omega-\alpha_3)(\omega-\alpha_4)}{(\omega^2-1)(\omega^2-\alpha_1)(\omega^2-\alpha_2)(\omega^2-\alpha_3)(\omega^2-\alpha_4)}$$

$$= \frac{\omega^5-1}{\omega^{10}-1}\ \frac{(\omega-\alpha_1)(\omega-\alpha_2)(\omega-\alpha_3)(\omega-\alpha_4)}{(\omega^2-\alpha_1)(\omega^2-\alpha_2)(\omega^2-\alpha_3)(\omega^2-\alpha_4)}$$

$$= \frac{(\omega^2-1)^2}{(\omega-1)^2} = (\omega+1)^2 = \omega^4 = \omega$$

EXERCISE - 2

Part # I : Multiple Choice

2. $\alpha = -2 + 3z$

$\alpha + 2 = 3z$

$|\alpha + 2| = 3|z|$

$(x+2)^2 + y^2 = 9$

Similarly $\beta = -2 - 3z$

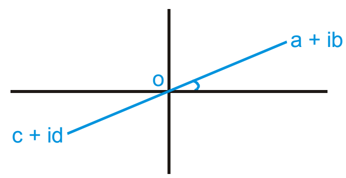
$\Rightarrow \beta + 2 = -3z \Rightarrow |\beta + 2| = |-3z|$

$(x+2)^2 + y^2 = 9$

Now $\quad \alpha - \beta = 6z \Rightarrow |\alpha - \beta| = 6|z|$

so $(\alpha - \beta)$ moves on a circle with centre as origin and radius 6.

3.



$|a + ib| = |(c + id)|$

$a + c = b + d$

4. $z^3 + (1+i)z^2 + (1+i)z + i = 0$

$\Rightarrow (z+i)(z^2+z+1) = 0$

$\Rightarrow (z+i)(z-\omega)(z-\omega^2) = 0 \Rightarrow z = -i, \omega, \omega^2$

Now ω, and ω^2 satisfies the equation

$z^{1993} + z^{1994} + 1 = 0$

So ω and ω^2 are common roots

We have $|z|(z+1)^8 = z^8|z+1|$ ....**(i)**

Taking modulus on both sides, we get

$|z||z+1|^8 = |z|^8|z+1|$

$\therefore |z+1|^7 = |z|^7$

$\Rightarrow |z+1| = |z|$

which represents the locus of z will be a straight line which is perpendicular bisector of the line segment joining $(-1, 0)$ and $(0, 0)$ i.e. $Re(z) = \frac{-1}{2}$. Also there will be exactly 7 distinct 'z' satisfying given equation,

i.e. $z = \frac{-1}{2}, \frac{-1}{2} \pm ki$ where k has 3 distinct positive values k_1, k_2 and k_3.

$\therefore \sum_{r=1}^{n} Re(z_r) = \sum_{r=1}^{7} z_r = \frac{-1}{2} \times 7 = -\frac{7}{2}$

and $\sum_{r=1}^{n} Im(z_r) = 0 + (k_1 + k_2 + k_3) + (-k_1 - k_2 - k_3) = 0$

9. $\prod_{r=1}^{n} x_r = e^{i\frac{\pi}{2}} \cdot e^{i\frac{\pi}{2^2}} \cdot e^{i\frac{\pi}{2^3}} \ldots\ldots e^{i\frac{\pi}{2^n}}$

$$\lim_{n\to\infty} \prod_{r=1}^{n} x_r = e^{i\frac{\pi}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots\right)}$$

$$\lim_{n\to\infty} \text{Re}\left(\prod_{r=1}^{n} x_r\right) = \text{Re}\left(e^{i\left(\frac{\pi}{2}\right)2}\right) = -1$$

$$\lim_{n\to\infty} \text{Im}\left(\prod_{r=1}^{n} x_r\right) = \text{Im}\left(e^{i\left(\frac{\pi}{2}\right)2}\right) = 0$$

10. All the three vertices lies on circle $|z| = 1$

take $z_1 = z_1, z_2 = z_1\omega, z_3 = z_1\omega^2$

So $z_1 + z_2 + z_3 = z_1(1 + \omega + \omega^2) = 0$

$z_1z_2z_3 = z_1^{\,3}$

$z_1z_2 + z_2z_3 + z_3z_1 = z_1^{\,2}(\omega + \omega^3 + \omega^2) = 0$

11. $1 - \log_2 \dfrac{|x+1+2i|-2}{\sqrt{2}-1} \ge 0$

$\Rightarrow \dfrac{|x+1+2i|-2}{\sqrt{2}-1} \le 2 \quad \Rightarrow \quad |x+1+2i| \le 2\sqrt{2}$

$\Rightarrow \sqrt{(x+1)^2+4} \le 2\sqrt{2} \quad \Rightarrow \quad (x+1)^2 + 4 \le 8$

$\Rightarrow (x+1)^2 \le 4 \quad \Rightarrow \quad -3 \le x \le 1$

But $x = -1$ not lie in the domain of function.

12. Since z_1 and z_2 lie on $|z| = 1$ and $|z| = 2$

then $|z_1| = 1$ and $|z_2| = 2$

$|2z_1 + z_2| \le 2|z_1| + |z_2| \le 4$

max $|2z_1 + z_2| = 4$

$|z_1 - z_2| \ge |z_1| - |z_2\| = |1-2| = 1$

min $|z_1 - z_2| = 1$

$$\left|z_2 + \frac{1}{z_1}\right| \le |z_2| + \frac{1}{|z_1|} = 2 + 1 = 3$$

$$\left|z_2 + \frac{1}{z_1}\right| \le 3$$

13. $z = re^{i\theta}$

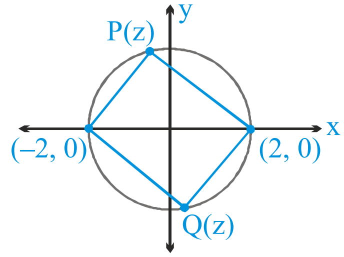
$r^2e^{i\theta 2} + r^2e^{i\theta} + r^2 = 0$

$r^2\,[e^{i2\theta} + e^{i\theta} + 1] = 0$

$\theta = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

$z = k\omega$ or $k\omega^2$ where $k > 0$

17. **(A)** For P(z), $\arg\left(\dfrac{z-2}{z+2}\right) = \dfrac{\pi}{2}$



and for Q(z), $\arg\left(\dfrac{z-2}{z+2}\right) = \dfrac{-\pi}{2}$

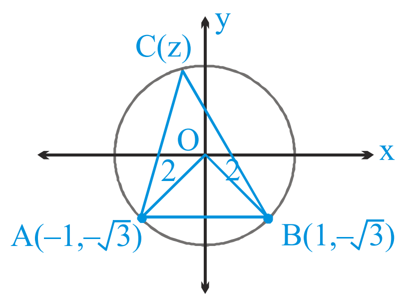
$\Rightarrow$ (A) is true.

(B) ΔAOB is equilateral.

$\therefore \angle AOB = \dfrac{\pi}{3}$

and $\angle ACB = \dfrac{\pi}{6}$

$\therefore \arg\left(\dfrac{z-1+i\sqrt{3}}{z+1+i\sqrt{3}}\right) = \dfrac{\pi}{6}$ (By rotation)



$\Rightarrow \arg\left(\dfrac{z+1+i\sqrt{3}}{z-1+i\sqrt{3}}\right) = -\dfrac{\pi}{6}$

$\Rightarrow$ **(B)** is not true.

(C) and **(D)**

We have $|z^2 - 1| = (z^2-1)\,(\bar{z}^2 - 1) = z^2\,\bar{z}^2 - z^2 - \bar{z}^2 + 1$

$|z|^4 - 2(x^2 - y^2 + 1) = 17 - 2(x^2 - y^2)$

$\because x^2 + y^2 = 4 = 25 - 4x^2$

Now given $|z| = 2$

$\Rightarrow$ range of x is $[-2, 2]$

$\Rightarrow 9 \le |z^2 - 1|^2 \le 25$

Hence $3 \le |z^2 - 1| \le 5$

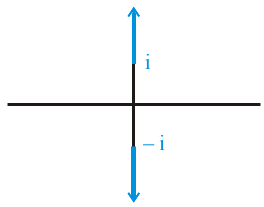
$\Rightarrow$ **(C)** and **(D)** are true.

19. $||z+i|-|z-i||=k$

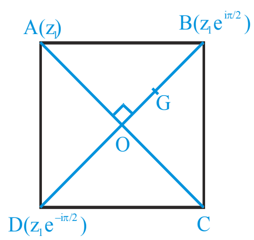
for $0<k<2$ its hyperbola having foci as i & $-i$.

for $k=0$ $|z+i|=|z-i|$ which is perpendicular bisector of line joining $i, -i$

for $k=2$ a pair of ray.



21.



ΔABC is isosceles triangle

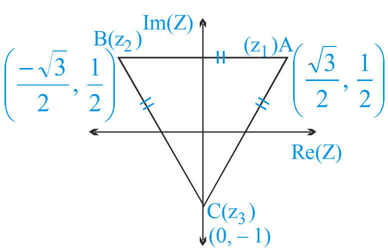
So centroid divide median BO in ratio 2 : 1

$$\text{centroid } G=\frac{z_1e^{i\pi/2}}{3}=\frac{z_1}{3}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

$$\text{Also centroid } G=\frac{z_1e^{-i\pi/2}}{3}=\frac{z_1}{3}\left(\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}\right)$$

22. We have $z^4=iz \Rightarrow z^3=i \Rightarrow z=e^{i(4k+1)\frac{\pi}{6}}$

(Using D.M.T.)



Put $k=0, 1, 2$, we get

$z_1=e^{i\frac{\pi}{6}}, z_2=e^{i\frac{5\pi}{6}}$ and $z_3=e^{i\frac{3\pi}{2}}$

Clearly triangle formed by z_1, z_2 and z_3 is equilateral.

$\therefore$ centroid of ΔABC is (0, 0) and Area $(\Delta ABC)=\frac{3\sqrt{3}}{4}$

24. $\Rightarrow \frac{\pi}{2}$

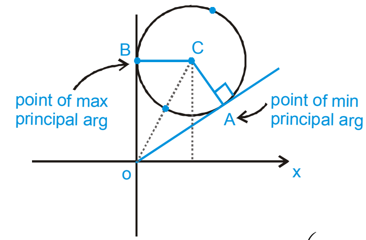
$\max|z|=d+r$

$\min|z|=d-r$

$d=OC=\sqrt{5}$

$r=1$

$\theta=\angle OCX=\tan^{-1}\frac{2}{1}$



$\alpha=\angle OCA=\tan^{-1}\frac{1}{2}$ $\left(\because \sin\alpha=\frac{1}{\sqrt{5}}\right)$

So principal Arg of A $=\theta-\alpha=\tan^{-1}2-\tan^{-1}\frac{1}{2}$

$$=\tan^{-1}\frac{2-\frac{1}{2}}{1+1}=\tan^{-1}\frac{3}{4}$$

Part # II : Assertion & Reason

1. $|z-4-5i|=4$ represents a circle with centre (4, 5) and radius 4 and $\arg(z-3-4i)=\frac{\pi}{4}$ represents a ray emanating from point (3, 4). Ray will intersect the circle at only one point.

So statement **(I)** is false and statement **(II)** is true.

2. For statement-1

$$\frac{1}{z_1-z_2}+\frac{1}{(z_2-z_3)}+\frac{1}{(z_3-z_1)}=0$$

$\Rightarrow$ $z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$

$\Rightarrow$ $z_1\, z_2\, z_3$ are vertices of equilateral triangle

For statement-2

$|z_1-z_0|=|z_2-z_0|=|z_3-z_0|$

$\therefore$ z_0 is circum centre

4. Statement -1 $(1+z)^6=-z^6$

take modulus

$|1+z|^6=|z|^6$

$\left|\frac{1+z}{z}\right|=1$ which is straight line

Statement -2 $z_2=\frac{z_1+z_3}{2}$

$\therefore$ z_2 is mid point of line joining z_1 & z_3. Hence $z_1\, z_2\, z_3$ are collinear

EXERCISE - 3

Part # I : Matrix Match Type

1.(A) $\left||z|-\frac{1}{|z|}\right| \le \left|z+\frac{1}{z}\right|$

$-2 \le |z| - \frac{1}{|z|} \le 2$

$|z|^2 + 2|z| - 1 \ge 0$ and $|z|^2 - 2|z| - 1 \le 0$

$|z| \ge \sqrt{2}-1, \ \ |z| \le \sqrt{2}+1$

$|z|_{min} = \sqrt{2}-1$

so minimum value of $\dfrac{|z|}{\tan\frac{\pi}{8}} = 1$

(B) $|z| = 1$

Let $z = \cos\theta + i\sin\theta$

$$\frac{z^n}{z^{2n}+1} - \frac{\bar{z}^n}{\bar{z}^{2n}+1}$$

$$= \frac{\cos n\theta + i\sin n\theta}{1+\cos 2n\theta + i\sin 2n\theta} - \frac{\cos n\theta - i\sin n\theta}{1+\cos 2n\theta - i\sin 2n\theta}$$

$$= \frac{\cos n\theta + i\sin n\theta}{2\cos n\theta(\cos n\theta + i\sin n\theta)}$$

$$- \frac{\cos n\theta - i\sin n\theta}{2\cos n\theta(\cos n\theta - i\sin n\theta)}$$

$$= \frac{1}{2\cos n\theta} - \frac{1}{2\cos n\theta} = 0$$

(C) $8iz^3 + 12z^2 - 18z + 27i = 0$

$\Rightarrow \ (2iz+3)(4z^2+9i) = 0$

$\Rightarrow \ z = \frac{3}{2}i, \ z^2 = -\frac{9}{4}i \ \Rightarrow \ \ 2|z| = 3$

(D) $z^4 + z^3 + z^2 + z + 1$

$= (z-z_1)(z-z_2)(z-z_3)(z-z_4)$

Put $z = -2$

$\prod_{i=1}^{4}(z_i+2) = (-2)^4 + (-2)^3 + (-2)^2 + (-2) + 1 = 11$

3. (A) Let $f(x) \equiv (x-x_1)(x-x_2)(x-x_3)(x-x_4)$

$f(-i) = (i+x_1)(i+x_2)(i+x_3)(i+x_4)$

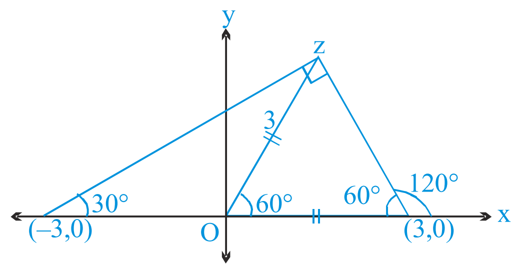
$|f(-i)| = |(i+x_1)(i+x_2)(i+x_3)(i+x_4)|$

$= \sqrt{(x_1^2+1)}\sqrt{(x_2^2+1)}\sqrt{(x_3^2+1)}\sqrt{(x_4^2+1)} = 1$

$\Rightarrow \ x_1 = x_2 = x_3 = x_4 = 0$

$\Rightarrow \ a = b = c = d = 0$

(B) From the figure $z = 3e^{i\frac{\pi}{3}}$



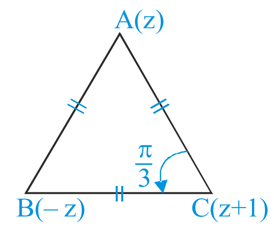
$\therefore \ \arg z = \frac{\pi}{3}$

Hence $\tan^2(\arg z) - 2\cos(\arg z)$

$= \tan^2 60° - 2\cos 60° = 3 - 1 = 2$

(C) By rotation, we get

$$\frac{(z+1)-(-z)}{(z+1)-z} = e^{\frac{i\pi}{3}}$$



$\Rightarrow \ 2z+1 = \left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \ \Rightarrow \ 2z = \left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)$

$\therefore \ z = \left(\frac{-1}{4}+i\frac{\sqrt{3}}{4}\right) \ \Rightarrow \ \mathrm{Re}(z) = \frac{-1}{4}$

$\Rightarrow \ 5 + 4\left(\frac{-1}{4}\right) = 5 - 1 = 4$

(D) We have $z_1^{\,x} + z_2^{\,x} = 2^x$

$\Rightarrow \ (-2\omega^2)^x + (-2\omega)^x = 2^x$

$\Rightarrow \ (-1)^x[\omega^{2x} + \omega^x] = 1$

$\Rightarrow \ (-1)^x[\omega^{2x} + \omega^x + 1^x - 1] = 1$

Clearly $x \ne 3n, n \in I$

(Because if x is an integral multiple of 3 then LHS $\ne$ RHS)

$\Rightarrow \ (-1)^x[\underbrace{\omega^{2x} + \omega^x + 1^x}_{\text{zero}} - 1] = 1$

Now verify $\Rightarrow \ x = 1, 3$

Part # II : Comprehension

Comprehension # 2

Let $z = z_1 + z_2 + z_3 + z_4$

$z_2 + z_3 + z_4 = z - z_1$

$\therefore$ $(1-d)\, z_1 + z_2 + z_3 + z_4 = (1-d)\, z_1 + z - z_1 = z - dz_1$

Similarly $z_1 + (1-d)\, z_2 + z_3 + z_4 \quad = z - dz_2$

and $z_1 + z_2 + (1-d)\, z_3 + z_4 = z - dz_3$

$\therefore$ $|z - dz_1| = |z - dz_2| = |z - dz_3|$

$\therefore$ z is equidistant from dz_1, dz_2, dz_3

but dz_1, dz_2, dz_3 lie on a circle with centre O and radius $|dz_1| = |dz_2| = |dz_3|$

$\because$ $|dz_1| = |d| \cdot |z_1|$ $\therefore |dz_1| = |dz_2| = |dz_3|$

$|dz_2| = |d| \cdot |z_2|$ as $|z_1| = |z_2| = |z_3|$

$|dz_3| = |d| \cdot |z_3|$

$\therefore$ $z = 0$

$\therefore$ Arg z not defined, Also $|z| = 0$

(C) and **(D)**

We have

$|z^2 - 1| = (z^2 - 1)\,(\overline{z}^2 - 1) = z^2\,\overline{z}^2 - z^2 - \overline{z}^2 + 1$

$|z|^4 - 2(x^2 - y^2 + 1) = 17 - 2(x^2 - y^2)$

$\because$ $x^2 + y^2 = 4 = 25 - 4x^2$

$\Rightarrow$ $9 \le |z^2 - 1|^2 \le 25$

$(z\overline{z})^2 - \left((z+\overline{z})^2 - 2z\overline{z}\right) + 1 = |z|^4 + 2|z|^2 + 1 - \left(2\,\mathrm{Re}(z)\right)^2$

$= 2^4 + 2.2^2 + 1 - 4\,|\mathrm{Re}(z)|^2 = 25 - 4\,|\mathrm{Re}(z)|^2$

Since $\quad 0 \le |\mathrm{Re}(z)| \le 2$

$\Rightarrow$ $25 - 4 \times 4 \le |z^2 - 1|^2 \le 25 - 4 \times 0$

$\Rightarrow$ $9 \le |z^2 - 1|^2 \le 25$

Hence $\quad 3 \le |z^2 - 1| \le 5$

$\Rightarrow$ **(C)** and **(D)** are true.

Comprehension # 4

For A, $|z + 1| \le 2 + \mathrm{Re}(z)$

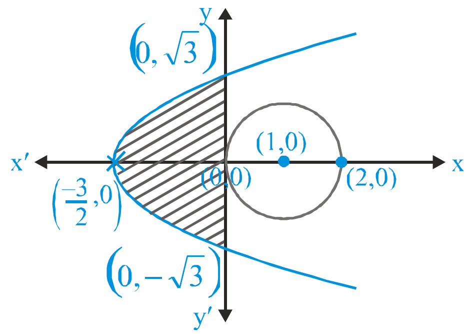
$\Rightarrow$ $(x+1)^2 + y^2 \le 4 + 4x + x^2$

$\Rightarrow$ $y^2 \le 3 + 2x$

$\Rightarrow$ $y^2 \le 2\left(x + \frac{3}{2}\right)$ .....**(i)**

For B, $\quad |z - 1| \ge 1$

$\Rightarrow$ $(x-1)^2 + y^2 \ge 1$ ......**(ii)**



For C, $\quad |z - 1|^2 \ge |z + 1|^2$

$\Rightarrow$ $(z-1)(\overline{z} - 1) \ge (z+1)(\overline{z} + 1)$

$\Rightarrow$ $(z\,\overline{z} - \overline{z} - z + 1) \ge (z\,\overline{z} + \overline{z} + z + 1)$

$\Rightarrow$ $z + \overline{z} \le 0$

i.e. $x \le 0$ .....**(3)**

(i) (–1,0), (–1, 1), (–1,–1), (0,0), (0,1), (0,–1)

$\therefore$ Total number of point(s) having integral coordinates in the region $A \cap B \cap C$ is 6.

(ii) Required area $= 2\int_{-3/2}^{0} \sqrt{2\left(x + \frac{3}{2}\right)}\, dx = 2\sqrt{3}$ (square units)

(iii) Clearly $\quad z = \frac{-3}{2} + i\,0$ is the complex number in the region $A \cap B \cap C$ and having maximum amplitude.

$\therefore$ $\mathrm{Re}(z) = \frac{-3}{2}$

Comprehension # 5

1. $AD = x$, $\angle ADC = 180 - (C + \theta)$

Area of $\Delta ABC = 2$ area $\Delta ADC = \frac{1}{2}\, 2y.x \sin(C + \theta)$

$= xy \sin(C + \theta)$

2. Let affix of M is z_m and $\angle BOM = \pi - 2B$, then

$$\frac{z_m - 0}{z_b - 0} = \frac{OM}{OB} e^{i(\pi - 2B)}$$

$z_m = z_b\, e^{i(\pi - 2B)}$

3. Let affix of L is z_L and $\angle BOL = 2(A - \theta)$, then

$$\frac{z_L - 0}{z_b - 0} = e^{i(2A - 2\theta)}$$

$z_L = z_b\, e^{i(2A - 2\theta)}$

EXERCISE - 4
Subjective Type

2. **(A)** The region between the concentric circles with centre at (0 , 2) & radii 1 & 3 units

(B) region outside or on the circle with centre $\frac{1}{2} + 2i$ and radius $\frac{1}{2}$

(C) semi circle (in the 1st & 4th quadrant) $x^2 + y^2 = 1$

(D) a ray emanating from the point (3 + 4i) directed away from the origin & having equation

$$\sqrt{3}x - y + 4 - 3\sqrt{3} = 0$$

3. **(i)** $z = 1 + e^{i\frac{18\pi}{25}} = e^{i\frac{9\pi}{25}}\left[e^{i\frac{9\pi}{25}} + e^{-i\frac{9\pi}{25}}\right]$

$$z = 2\cos\left(\frac{9\pi}{25}\right) e^{i\frac{9\pi}{25}}$$

$$|z| = 2\cos\left(\frac{9\pi}{25}\right) \qquad \text{Arg } z = \frac{9\pi}{25}$$

(ii) $z = 2e^{i\pi} e^{i\pi/6} = 2e^{-i5\pi/6}$

$$|z| = 2 \qquad \text{Arg } z = -\frac{5\pi}{6}.$$

(iii) $|z| = \left(\sqrt{1+\tan^2 1}\right)^2 = \sec^2 1$

$$\text{Arg } z = 2\,\text{Arg}(\tan 1 - i) = 2\left(1 - \frac{\pi}{2}\right) = 2 - \pi$$

(iv) $z = \dfrac{(i-1)}{2\sin\left(\frac{\pi}{5}\right)\left[\sin\left(\frac{\pi}{5}\right)i + \cos\left(\frac{\pi}{5}\right)\right]}$

$$|z| = \frac{\sqrt{2}}{2\sin\left(\frac{\pi}{5}\right)} = \frac{1}{\sqrt{2}}\text{cosec}\left(\frac{\pi}{5}\right)$$

$$\text{Arg}(z) = \pi - \frac{\pi}{4} - \frac{\pi}{5} = \frac{11\pi}{20}$$

4. If $x = \alpha + i\beta$ is a root then

$$\frac{A_1^2}{\alpha - a_1 + i\beta} + \frac{A_2^2}{\alpha - a_2 + i\beta} + \ldots\ldots\ldots + \frac{A_n^2}{\alpha - a_n + i\beta} = K$$

& taking conjugate

$$\frac{A_1^2}{\alpha - a_1 - i\beta} + \frac{A_2^2}{\alpha - a_2 - i\beta} + \ldots\ldots\ldots + \frac{A_n^2}{\alpha - a_n - i\beta} = K$$

Substracting

$$\frac{2\beta A_1^2}{(\alpha - a_1)^2 + \beta^2} + \frac{2\beta A_2^2}{(\alpha - a_2)^2 + \beta^2} + \ldots + \frac{2\beta A_n^2}{(\alpha - a_n)^2 + \beta^2} = 0$$

$\Rightarrow \beta = 0 \quad \Rightarrow x = \alpha + i\,0$

which is purely real. Hence true.

5. $|z|^2\omega - |\omega|^2 z = z - \omega$ .....**(i)**

Put $z = \omega$ & $z = \frac{1}{\bar{\omega}}$

we get L.H.S = R.H.S

Now, equation **(i)** be written as

$\omega(1 + |z|^2) = z(1 + |\omega|^2)$

$$\Rightarrow \frac{\omega}{z} = \frac{(1+|\omega|^2)}{1+|z|^2} = \lambda \Rightarrow \omega = \lambda z$$

But this is equation **(i)**

$|z|^2\lambda z - \lambda^2|z|^2 z = z - \lambda z$

$\Rightarrow z\lambda|z|^2(1 - \lambda) = z(1 - \lambda)$

$\Rightarrow (1 - \lambda)(\lambda|z|^2 - 1) = 0$

$$\Rightarrow \lambda = 1 \quad ; \quad \lambda = \frac{1}{|z|^2}$$

from $\lambda = 1$ we get $z = \omega$

$$\lambda = \frac{1}{|z|^2} \text{ we get } \omega = \frac{1}{\bar{z}}$$

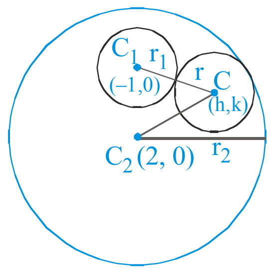
6. **(i)** $z_1\bar{z}_1 = 1 \Rightarrow \bar{z}_1 = \frac{1}{z_1}$

(ii) $|z_1 + z_2 .... z_n| = |\bar{z}_1 + \bar{z}_2 .....\bar{z}_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + ... \frac{1}{z_n}\right|$

7. (a) $\pi - 2$ (b) 1/2

8. We have $C_1 : (x + 1)^2 + y^2 = 9$

$C_2 : (x-2)^2 + y^2 = 49$



Now $CC_1 = r + r_1$

and $CC_2 = r_2 - r$

$\Rightarrow \quad CC_1 + CC_2 = r_1 + r_2$

$\therefore$ Locus of C is an ellipse with focus at C_1 and C_2

Now $\quad r_1 + r_2 = 2a = 10 \qquad$ ....**(1)**

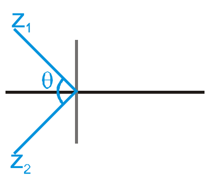
and $d_{C_1C_2}$ (focal length) $= 2ae = 3 \qquad$ ....**(2)**

(1) and **(2)** $\Rightarrow$ eccentricity 'e' is $\dfrac{3}{10} \Rightarrow p + q = 13$

9. $\dfrac{z_1}{z_2} = e^{i\theta}$

$$\frac{z_1+z_2}{z_1-z_2} = \frac{e^{i\theta}+1}{e^{i\theta}-1} = \frac{\cos\theta + 1 + i\sin\theta}{\cos\theta - 1 + i\sin\theta}$$

$$= \frac{2\cos\frac{\theta}{2}e^{i\frac{\theta}{2}}}{-2\sin^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$



$$\frac{z_1+z_2}{z_1-z_2} = \frac{\cot(\theta/2)}{i}$$

$$i\tan\frac{\theta}{2} = \left(\frac{z_1-z_2}{z_1+z_2}\right)$$

$$-\tan^2\frac{\theta}{2} = \left(\frac{z_1-z_2}{z_1+z_2}\right)^2 \Rightarrow 1-\sec^2\frac{\theta}{2} = \frac{\frac{b^2}{a^2}-\frac{4c}{a}}{\frac{b^2}{a^2}}$$

$$\Rightarrow 1-\sec^2\frac{\theta}{2} = 1 - \frac{4ac}{b^2} \Rightarrow \cos^2\frac{\theta}{2} = \frac{b^2}{4ac}$$

$$\Rightarrow \cos\frac{\theta}{2} = \sqrt{\frac{b^2}{4ac}} \Rightarrow \theta = 2\cos^{-1}\sqrt{\frac{b^2}{4ac}}$$

11. 51

12. $D = \begin{vmatrix} e^{-2iA} & e^{iC} & e^{iB} \\ e^{iC} & e^{-2iB} & e^{iA} \\ e^{iB} & e^{iA} & e^{-2iC} \end{vmatrix}$

$$= e^{-iA}e^{-iB}e^{-iC}\begin{vmatrix} e^{-iA} & e^{i(A+C)} & e^{i(A+B)} \\ e^{i(B+C)} & e^{-iB} & e^{i(A+B)} \\ e^{i(B+C)} & e^{i(A+C)} & e^{-iC} \end{vmatrix}$$

As $A + B + C = \pi$

So $A + C = \pi - B, B + C = \pi - A, A + B = \pi - C$

$$D = e^{-i\pi}\begin{vmatrix} e^{-iA} & e^{i(\pi-B)} & e^{i(\pi-C)} \\ e^{i(\pi-A)} & e^{-iB} & e^{i(\pi-C)} \\ e^{i(\pi-A)} & e^{i(\pi-B)} & e^{-iC} \end{vmatrix}$$

$$D = -\begin{vmatrix} e^{-iA} & -e^{iB} & -e^{-iC} \\ -e^{iA} & e^{-iB} & -e^{iC} \\ -e^{-iA} & -e^{-iB} & e^{-iC} \end{vmatrix}$$

$$= -e^{-iA}e^{-iB}e^{-iC}\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = -e^{-i\pi}(-4) = -4$$

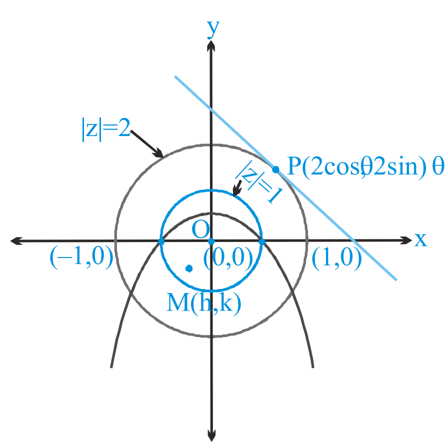
14. 18

15. Clearly the parabola should pass through (1, 0) and (–1,0). Let directrix of this parabola be x cosθ + y sinθ = 2. If M (h,k) be the focus of this parabola, then distance of (±1, 0) from 'M' and from the directrix should be same.

$\Rightarrow \quad (h-1)^2 + k^2 = (\cos\theta - 2)^2 \qquad$ ....**(i)**

and $(h+1)^2 + k^2 = (\cos\theta + 2)^2 \qquad$ ....**(ii)**

Now $(2) - (1) \Rightarrow \cos\theta = \dfrac{h}{2} \qquad$ ....**(iii)**



Also $(2) + (1) \Rightarrow (h^2 + k^2 + 1) = (\cos^2\theta + 4) \qquad$ ....**(iv)**

$\therefore$ From **(iii)** and **(iv)**, we get

$$h^2 + k^2 + 1 = 4 + \frac{h^2}{4} \Rightarrow \frac{3h^2}{4} + k^2 = 3$$

Hence locus of focus M(h, k) is $\frac{x^2}{4} + \frac{y^2}{3} = 1$ (Ellipse)

Also we know that area of the quadrilateral formed by the tangents at the ends of the latus-rectum is $\frac{2a^2}{e}$

(where e is eccentricity of ellipse)

$$\therefore \text{ Requred area} = \frac{2(4)}{\frac{1}{2}} = 16 \text{ (square units)}$$

$$\left(\text{As } e^2 = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow e = \frac{1}{2}\right.$$

16. $|z_1 - 2z_2| = |2 - z_1\bar{z}_2|$

$|z_1 - 2z_2|^2 = |2 - z_1\bar{z}_2|^2$

$(z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1 z_2)$

$z_1\bar{z} - 2\bar{z}_1 z_2 - 2z_1\bar{z}_2 + 4z_2\bar{z}_2$

$= 4 - 2z_1\bar{z}_2 - 2\bar{z}_1 z_2 + z_1\bar{z}_1 z_2\bar{z}_2$

$|z_1|^2 + 4|z_2|^2 - 4 - |z_1|^2|z_2|^2 = 0$

$\Rightarrow (|z_1|^2 - 4)(1 - |z_2|^2) = 0$

$\Rightarrow |z_1| = 2 \quad (\text{as } |z_2| \neq 1)$

17. **(i)** $|z| = 20$ **(ii)** $OP = OQ = PR = QR = 20$

18. Let $z^{2m} + z^{2m-1} - 1 ... + z + 1 = (z - z_1)(z - z_2) ..... (z - z_{2m})$

Taking log on both the sides & differentiating w.r.t.z

$$\frac{2mz^{2m-1} + (2m-1)z^{2m-2} + ... + 2z + 1}{z^{2m} + z^{2m-1} + ... + z^2 + z + 1}$$

$$= \frac{1}{z - z_1} + \frac{1}{z - z_2} + ... + \frac{1}{z - z_{2m}}$$

$$\Rightarrow \frac{1 + 2 + 3 + ... + 2m}{(2m+1)} \quad (\text{put } z = 1)$$

$$= \frac{1}{1 - z_1} + \frac{1}{1 - z_2} + ... + \frac{1}{1 - z_{2m}}$$

$$\Rightarrow \sum_{r=1}^{2m} \frac{1}{z_r - 1} = -\left[\frac{2m(2m+1)}{2(2m+1)}\right] = -m$$

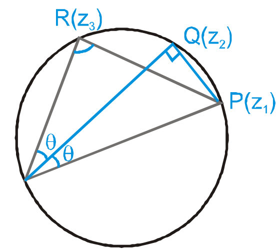
20. $\overline{az} + az = 0$

21. From the fig.

we have

$z_2 = z_1 (\cos\theta\, e^{i\theta})$

and $z_3 = z_1 (\cos 2\theta\, e^{i2\theta})$



$$\Rightarrow \frac{z_2^2}{z_3} = \frac{(z_1\cos\theta)^2}{(z_1\cos 2\theta)} \Rightarrow z_2^2\cos 2\theta = z_1 z_3\cos^2\theta$$

22. We have $f(w) = 2 + i\sqrt{3}$, $f(w^2) = 2 - i\sqrt{3}$

Let $f(z) = (z^2 + z + 1)g(z) + (az + b)$

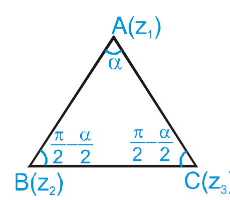
$\therefore \quad aw + b = 2 + i\sqrt{3}$ ....**(i)**

and $aw^2 + b = 2 - i\sqrt{3}$ ....**(ii)**

$\therefore$ On solving (1) and (2), we get

$a = 2,\ b = 3 \Rightarrow a + b = 5$

23. $\frac{z_3 - z_1}{z_2 - z_1} = e^{i\alpha}$



$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} = \cos\alpha + i\sin\alpha$$

$$\Rightarrow \frac{z_3 - z_1}{z_2 - z_1} - 1 = \cos\alpha - 1 + i\sin\alpha$$

$$\Rightarrow \frac{z_3 - z_2}{z_2 - z_1} = -2\sin^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2} . \cos\frac{\alpha}{2}$$

$$\Rightarrow \frac{z_3 - z_2}{z_2 - z_1} = 2i\sin\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$$

squaring both sides

$$\frac{(z_3 - z_2)^2}{(z_2 - z_1)^2} = -4\sin^2\frac{\alpha}{2}(\cos\alpha + i\sin\alpha)$$

$$\frac{(z_3 - z_2)^2}{(z_2 - z_1)^2} = -4\sin^2\frac{\alpha}{2}\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

$$\Rightarrow (z_3 - z_2)^2 = 4\sin^2(\alpha/2)(z_3 - z_1)(z_1 - z_2)$$

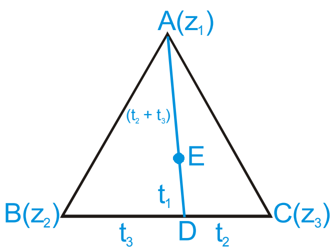
24. Affixes of a point D which divides z_2, z_3 in the ratio $t_3 : t_2$

is $\dfrac{t_2z_2 + t_3z_3}{t_2 + t_3}$ (Internal division)

Affies of a point E which divides. AD in the ratio

$(t_2 + t_3) : t_1$ is $\left(\dfrac{t_1z_1 + t_2z_2 + t_3z_3}{t_1 + t_2 + t_3}\right)$

(Internal division)



Hence E always lies in or on the ΔABC

26. **(i)** $a + b + c = \cos\alpha + i\sin\alpha + \cos\beta + i\sin\beta + \cos\gamma + i\sin\gamma$

$= (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\gamma + \sin\gamma)$

$= 0 + i\,0 = 0$

(ii) $\because$ $a + b + c = 0$ $\Rightarrow$ $\bar{a} + \bar{b} + \bar{c} = 0$

$\bar{a} = \dfrac{1}{a}$

$\bar{b} = \dfrac{1}{b}$

$\bar{c} = \dfrac{1}{c}$

$L.H.S = abc\left(\dfrac{1}{c} + \dfrac{1}{a} + \dfrac{1}{b}\right) = abc\,(\bar{a} + \bar{b} + \bar{c}) = abc(0) = 0$

(iii) Squaring and using

(ii) $a^2 + b^2 + c^2 + 2(ab + bc + ca) = 0$

(iv) by **(iii)** $\Rightarrow$ $e^{2i\alpha} + e^{2i\beta} + e^{2i\gamma} = 0$

$\cos2\alpha + i\sin2\alpha + \cos2\beta + i\sin2\beta + \cos2\gamma + i\sin2\gamma = 0$

$\Rightarrow$ $\cos2\alpha + \cos2\beta + \cos2\gamma = 0$

and $\sin2\alpha + \sin2\beta + \sin2\gamma = 0$

(v) $2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1 = 0$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{3}{2}$

Similarly

$1 - 2\sin^2\alpha + 1 - 2\sin^2\beta + 1 - 2\sin^2\gamma = 0$

$\dfrac{3}{2} = \sin^2\alpha + \sin^2\beta + \sin^2\gamma.$

27. **(B)** one if n is even ; $-\omega^2$ if n is odd

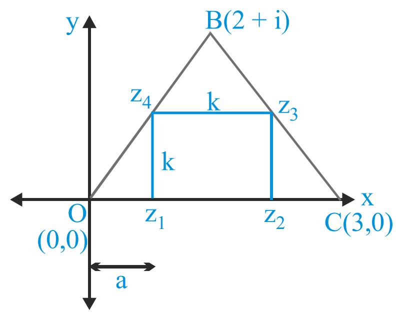
28. Let $|z_1| = a$,

and $|z_1 - z_2| = |z_1 - z_4| = k$

$\Rightarrow$ $\dfrac{z_4 - z_1}{z_2 - z_1} = e^{\frac{i\pi}{2}}$

$\Rightarrow$ $z_4 = a + ik$

$\Rightarrow$ $z_3 = a + ik + k = (a + k) + ik$



Now, O, z_4, z_B are collinear.

$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ a & k & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0$ $\Rightarrow$ $a - 2k = 0$ .....**(i)**

Also z_B, z_3, z_C are collinear.

$\Rightarrow \begin{vmatrix} 3 & 0 & 1 \\ a+k & k & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0$

$\Rightarrow$ $2k + a - 3 = 0$ .....**(ii)**

$\therefore$ From (1) and (2), we get

$a = \dfrac{3}{2},\ k = \dfrac{3}{4}$

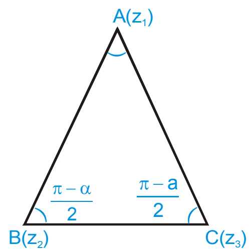
Hence $z_3 = \dfrac{9}{4} + \dfrac{3}{4}i$ and $z_4 = \dfrac{3}{2} + \dfrac{3}{4}i$

$\therefore$ Required area of triangle $= \dfrac{1}{2}\left\|\begin{matrix} 0 & 0 & 1 \\ 9/4 & 3/4 & 1 \\ 3/2 & 3/4 & 1 \end{matrix}\right\| = \dfrac{9}{32}$

(square units)

29. $\dfrac{z_1 - z_2}{z_3 - z_2} = \dfrac{AB}{BC}\ e^{i\left(\frac{\pi - \alpha}{2}\right)}$

$\dfrac{z_2 - z_3}{(z_1 - z_3)} = \dfrac{BC}{AC}\ e^{i\left(\frac{\pi - \alpha}{2}\right)}$



$$\Rightarrow \frac{(z_1-z_2)(-z_1+z_3)}{(z_3-z_2)^2} = \left(\frac{AB}{BC}\right)^2 .1$$

$$\Rightarrow (z_2-z_3)^2 = (z_3-z_1)(z_1-z_2).\left(\frac{BC}{AB}\right)^2$$

$$= (z_1-z_2)(z_1-z_2)\left[2\cos\left(\frac{\pi-\alpha}{2}\right)\right]^2$$

$$= 4(z_3-z_1)(z_1-z_2)\sin^2\alpha/2$$

30. $\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11} = -i\left[\cos\frac{2q\pi}{11} + i\sin\frac{2q\pi}{11}\right]$

$$= -i\,e^{i\frac{2q\pi}{11}}$$

$$\therefore \sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right) = -i\sum_{k=1}^{10} e^{i\frac{2q\pi}{11}}$$

$$= -i\left[e^{\frac{2\pi i}{11}} + e^{\frac{4\pi i}{11}} + \ldots + e^{\frac{20\pi i}{11}}\right] = -i(-1) = i$$

Given expression

$$\sum_{p=1}^{32}(3p+2)\left\{\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right\}^p$$

$$= \sum_{p=1}^{32}(3p+2)i^p = 3\sum_{p=1}^{32} pi^p + 2\sum_{p=1}^{32} i^p = 3S_1 + 2S_2$$

where $S_1 = \sum_{p=1}^{32} pi^p$

$S_1 = i + 2i^2 + 3i^3 + \ldots + 32i^{32}$

$iS_1 = i^2 + 2i^3 + \ldots + 32i^{33}$

$S_1(1-i) = i + i^2 + i^3 + \ldots + i^{32} - 32i^{33}$

$$S_1 = \frac{-32i}{(1-i)} = 16(1-i)$$

$S_2 = 0$

$$\therefore \sum_{p=1}^{32}(3p+2)\left\{\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right\}^p$$

$= 3S_1 + 2S_2 \quad = 48(1-i) + 0 = 48(1-i)$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

6. Given thar $\arg z\omega = \pi$ ... **(i)**

$\bar{z} + i\bar{\omega} = 0 \Rightarrow \bar{z} = -i\bar{\omega}$

$\Rightarrow z = i\omega \Rightarrow \omega = -iz$

From **(i)** $\arg(-iz)^2 = \pi$

$\arg(-i) + 2\arg(z) = \pi$; $\frac{-\pi}{2} + 2\arg(z) = \pi$

$2\arg(z) = \frac{3\pi}{2}$; $\arg(z) = \frac{3\pi}{4}$

10. Given that $\omega = \frac{z}{z-\frac{i}{3}}$ and $|\omega| = 1$

$$\therefore |\omega| = \left|\frac{z}{z-\frac{i}{3}}\right| \Rightarrow \frac{|z|}{\left|z-\frac{i}{3}\right|} = 1$$

$$\Rightarrow |z| = \left|z-\frac{i}{3}\right| \Rightarrow -\frac{2}{3}y + \frac{1}{9} = 0$$

Which is a straight line.

13. $\left|Z - \frac{4}{Z}\right| \geq |Z| - \left|\frac{4}{Z}\right|$

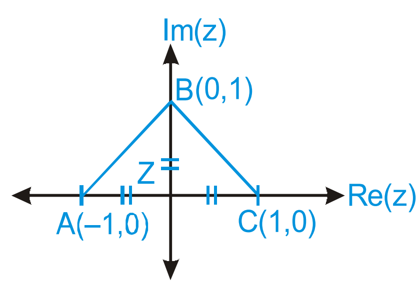
$2 \geq |Z| - \frac{4}{|Z|}$

$2|Z| \geq |Z|^2 - 4$

$|Z|^2 - 2|Z| - 4 \leq 0$

$|Z| \leq \sqrt{5} + 1$

14. z is the circumcentre (0, 0) of triangle ABC so their exist only one complex number.



15. Let $z^2 + \alpha z + \beta = 0$ has $(1 + iy_1)$ and $(1 + iy_2)$

so $z_1z_2 = \beta$

$(1 + iy_1)(1 + iy_2) = \beta$

$\beta = 1 - y_1y_2 + i(y_1 + y_2)$ ($\because$ β is purely real)

here $y_1 + y_2 = 0$

$y_1 = -y_2$

$\beta = 1 - y_1y_2$

$\beta = 1 + y_1^2$

$\beta > 1$

$\Rightarrow \quad \beta \in (1, \infty)$

16. $(1 + \omega)^7 = A + B\omega$

$(-\omega^2)^7 = A + B\omega$

$-\omega^2 = A + B\omega$

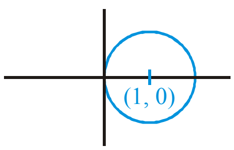
$1 + \omega = A + B\omega$

$A = 1$

$B = 1 \qquad (1, 1)$

17. $\dfrac{z^2}{z-1}$ is purely real where $(Z \neq 1)$

So $\dfrac{\bar{z}^2}{\bar{z}-1} = \dfrac{z^2}{z-1}$



$z\bar{z}^2 - \bar{z}^2 = \bar{z}z^2 - z^2$

$z\bar{z}(z - \bar{z}) = z^2 - \bar{z}^2$

$z\bar{z}(z - \bar{z}) = (z + \bar{z})(z - \bar{z})$

$\Rightarrow \ \bar{z} - z = 0$ or $z + \bar{z} = z\bar{z}$

$\Rightarrow \ \bar{z} = z$ or $x^2 + y^2 - 2x = 0$

$(x - 1)^2 + y^2 = 1$

so either lie on z real axis or on a circle passing through the origin.

18. $\bar{z} = \dfrac{1}{z}$

$\arg\left(\dfrac{1+z}{1+\frac{1}{z}}\right) \Rightarrow \arg z \Rightarrow \theta$

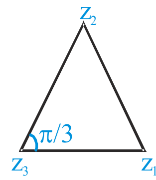
21. $\text{Re}((2 + 3i\sin\theta)(1 + 2i\sin\theta)) = 2 - 6\sin^2\theta = 0$

$\Rightarrow \ \sin^2\theta = \dfrac{1}{3}$

Part # II : IIT-JEE ADVANCED

2. **(A)** $\dfrac{z_1 - z_3}{z_2 - z_3} = \dfrac{1 - i\sqrt{3}}{2} = \dfrac{(1 - i\sqrt{3})(1 + i\sqrt{3})}{2(1 + i\sqrt{3})}$

$= \dfrac{1 - i^2.3}{2(1 + i\sqrt{3})}$



$= \dfrac{4}{2(1 + i\sqrt{3})} = \dfrac{2}{(1 + i\sqrt{3})}$

$\Rightarrow \ \dfrac{z_2 - z_3}{z_1 - z_3} = \dfrac{1 + i\sqrt{3}}{2} = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$

$\Rightarrow \ \left|\dfrac{z_2 - z_3}{z_1 - z_3}\right| = 1$ and $\arg\left(\dfrac{z_2 - z_3}{z_1 - z_3}\right) = \dfrac{\pi}{3}$

Hence the Δ is equilateral,

(B) $\arg\dfrac{z_1}{z_2} = \dfrac{\pi}{2} \Rightarrow \dfrac{z_1}{z_2} = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = i$

($\because$ $|z_2| = |z_1| = 1$)

$\therefore \ \dfrac{z_1^n}{z_2^n} = (i)^n$

Hence $i^n = 1 \quad \Rightarrow \quad n = 4k$

3. **(C)** $z^{p+q} - z^p - z^q + 1 = 0$

$\Rightarrow \ (z^p - 1)(z^q - 1) = 0$

as α is root of (1), either $\alpha^p - 1 = 0$

or $\alpha^q - 1 = 0$

$\Rightarrow$ either $\dfrac{\alpha^p - 1}{\alpha - 1} = 0$ or $\dfrac{\alpha^q - 1}{\alpha - 1} = 0$ (as $\alpha \neq 1$)

$\Rightarrow$ either $1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0$

or $1 + \alpha + \ldots + \alpha^{q-1} = 0$

But $\alpha^p - 1 = 0$ and $\alpha^q - 1 = 0$

cannot occur simultaneously as p and q are distinct primes, so neither p divides q nor q divides p, which is the requirement for $1 = \alpha^p = \alpha^q$.

5. Given $|z_1| < 1$ and $|z_2| > 1$

Then to prove $\left|\dfrac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$ $\left\{\text{using } \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\right\}$

$\Rightarrow \ |1 - z_1\bar{z}_2| < |z_1 - z_2|$

Squaring both sides, we get

$$(1-z_1\bar{z}_2)(1-\bar{z}_1 z_2) < (z_1-z_2)(\bar{z}_1-\bar{z}_2)$$

{using $|z|^2 = z\,\bar{z}$ }

$\Rightarrow 1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_1\bar{z}_1 z_2\bar{z}_2 < z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2$

$\Rightarrow 1 + |z_1|^2\,|z_2|^2 < |z_1|^2 + |z_2|^2$

$\Rightarrow 1 - |z_1|^2 - |z_2|^2 + |z_1|^2\,|z_2|^2 < 0$

$\Rightarrow (1-|z_1|^2)\,(1-|z_2|^2) < 0$ ... **(2)**

which is true by (1) as $|z_1| < 1$ and $|z_2| > 1$

$\therefore$ $(1-|z_1|^2) > 0$ and $(1-|z_2|^2) < 0$

$\therefore$ (2) is true whenever (1) is true.

$\Rightarrow \left|\dfrac{1-z_1\bar{z}_2}{z_1-z_2}\right| < 1$

6. Given : $a_1z + a_2z^2 + ... + a_nz^n = 1$

and $|z| < 1/3$ ... **(i)**

{using $|z_1 + z_2| \le |z_1| + |z_2|$}

$\Rightarrow$ $|a_1z| + |a_2z^2| + |a_3z^3| + ... + |a_nz^n| \ge 1$

$\Rightarrow$ $2\{(|z| + |z|^2 + |z|^3 + ... + |z|^n)\} > 1$ (using $|a_r| < 2$)

$\Rightarrow$ $\dfrac{2\,|z|\,(1-|z|^n)}{1-|z|} > 1$

{using sum of n terms of G.P.}

$\Rightarrow$ $2|z| - 2|z|^{n+1} > 1 - |z|$

$\Rightarrow$ $3|z| > 1 + 2|z|^{n+1}$ $\Rightarrow$ $|z| > \dfrac{1}{3} + \dfrac{2}{3}|z|^{n+1}$

$\Rightarrow$ $|z| > \dfrac{1}{3}$, which contradicts .....**(i)**

$\therefore$ There exists no complex number z such that

$|z| < \dfrac{1}{3}$ and $\sum\limits_{r=1}^{n} a_r z^r = 1$

8. As we know; $|z|^2 = z.\,\bar{z} \Rightarrow \dfrac{|z-\alpha|^2}{|z-\beta|^2} = k^2$

$\Rightarrow$ $(z-a)\,(\bar{z}-\bar{\alpha}) = k^2\,(z-\beta)\,(\bar{z}-\bar{\beta})$

$|z|^2 - \alpha\,\bar{z} - \bar{\alpha}\,z + |\alpha|^2 = k^2\,(|z|^2 - \beta\,\bar{z} - \bar{\beta}\,z + |\beta|^2)$

or $|z|^2(1-k^2) - (\alpha - k^2\beta)\,\bar{z} - (\bar{\alpha} - \bar{\beta}\,k^2)z + (|\alpha|^2 - k^2|\beta|^2) = 0$

$\Rightarrow$ $|z|^2 - \dfrac{(\alpha-k^2\beta)}{(1-k^2)}\bar{z} - \dfrac{(\bar{\alpha}-\bar{\beta}k^2)}{(1-k^2)}z + \dfrac{|\alpha|^2 - k^2|\beta|^2}{(1-k^2)} = 0$

On comparing with equation of circle.

$|z|^2 + a\,\bar{z} + \bar{a}\,z + b = 0$

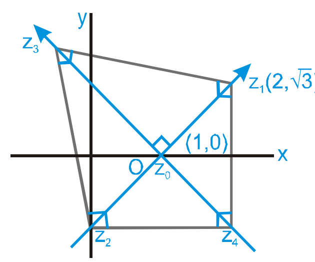
whose centre is (–a) and radius = $\sqrt{|a|^2 - b}$

$\therefore$ centre for **(i)**

$= \dfrac{\alpha - k^2\beta}{1-k^2}$ and radius

$= \sqrt{\left(\dfrac{\alpha-k^2\beta}{1-k^2}\right)\left(\dfrac{\bar{\alpha}-k^2\bar{\beta}}{1-k^2}\right) - \dfrac{\alpha\bar{\alpha}-k^2\beta\bar{\beta}}{1-k^2}}$

11. Here, centre of circle is (1, 0) is also the mid-point of diagonals of square



$\Rightarrow$ $\dfrac{z_1+z_2}{2} = z_0$

$\Rightarrow$ $z_2 = -\sqrt{3}\,i$,

(where $z_0 = 1 + 0\,i$)

and $\dfrac{z_3-1}{z_1-1} = e^{\pm i\pi/2}$

$\Rightarrow$ $z_3 = 1 + (1+\sqrt{3}i).\left(\cos\dfrac{\pi}{2} \pm i\sin\dfrac{\pi}{2}\right)$, as $z_1 = 2+\sqrt{3}i$

$= 1 \pm i(1 + \sqrt{3}\,i) = (1 \mp \sqrt{3}) \pm i$

$z_3 = (1 - \sqrt{3}) + i$ and $z_4 = (1 + \sqrt{3}) - i$

12. Let, $z_1 = \dfrac{w - \bar{w}z}{1-z}$, be purely real

$\Rightarrow$ $z_1 = \bar{z}_1$

$\therefore$ $\dfrac{w-\bar{w}z}{1-z} = \dfrac{\bar{w}-w\bar{z}}{1-\bar{z}}$

$\Rightarrow$ $w - w\bar{z} - \bar{w}z + \bar{w}z.\bar{z} = \bar{w} - z\bar{w} - w\bar{z} + wz.\bar{z}$

$\Rightarrow$ $(w-\bar{w}) - (\bar{w}-w)\,|z|^2 = 0$

$\Rightarrow$ $(w-\bar{w})(1-|z|^2) = 0$

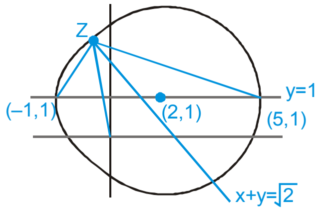
$\Rightarrow$ $|z|^2 = 1$ {as, $w - \bar{w} \neq 0$, since $\beta \neq 0$}

$\Rightarrow$ $|z| = 1$ and $z \neq 1$.

15. A = {z : Im z $\ge$ 1} $y \ge 1$

B = {z : |z–2–i| = 3} $(x-2)^2 + (y-1)^2 = 9$

C = {z : Re((1 – i)z} = $\sqrt{2}$ } $x + y = \sqrt{2}$



As we can see 3 curves intersects at only one point
So $A \cap B \cap C$ contains exactly one element

16. $|z+1-i|^2+|z-5-i|^2=(-1-5)^2+(1-1)^2=36$
so exactly 36

17. As $3-\sqrt{5} \le |z| \le 3+\sqrt{5}$

As $-3+\sqrt{5} \le |\omega| \le 3+\sqrt{5}$

$-3-\sqrt{5} \le -|\omega| \le 3-\sqrt{5}$

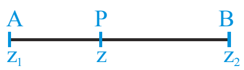
$-\sqrt{5} \le -|\omega|+3 \le 6-\sqrt{5}$

$-3 \le |z|-|\omega|+3 \le 9$

22. $z = z_1 + t(z_2 - z_1)$

$\dfrac{z-z_1}{z_2-z_1} = t\,,\ t \in (0,1) \quad \Rightarrow \quad z = \dfrac{z_1(1-t)+tz_2}{(1-t)+t}$

point P(z) divides point $A(z_1)$ & $B(z_2)$ internally in ratio $(1-t) : t$
Hence locus is a line segment such that P(z) lies between $A(z_1)$ & $B(z_2)$ as shown in figure.



Hence options A,C & D are correct.

24. (A) $|z-i|z||^2 = |z+i|z||^2$

$\Rightarrow (z-i|z|)\left(\bar{z}+i|z|\right) = (z+i|z|)(\bar{z}-i|z|)$

$\Rightarrow 2i|z|z = 2i|z|\bar{z}$

$\Rightarrow z = \bar{z}$ $\therefore$ z is purely real.

$\therefore$ z lies on real axis.

(B) Locus is ellipse having focii (–4, 0) & (4, 0)
$2ae = 8$ & $2a = 10$
$\Rightarrow a = 5$ & $e = 4/5$
It is ellipse having eccentricity 4/5.

(C) $w = 2(\cos\theta + i\sin\theta)$

$z = 2\left(\cos\theta + i\sin\theta\right) - \dfrac{1}{2(\cos\theta + i\sin\theta)}$

$x + iy = \dfrac{3}{2}\cos\theta + \dfrac{i5}{2}\sin\theta$

$\Rightarrow \quad x = \dfrac{3}{2}\cos\theta$ & $y = \dfrac{5}{2}\sin\theta$

It is a locus $\dfrac{x^2}{9/4} + \dfrac{y^2}{25/4} = 1$

$\dfrac{9}{4} = \dfrac{25}{4}(1-e^2) \quad \Rightarrow \quad e = \dfrac{4}{5}$

since $x = \dfrac{3}{2}\cos\theta \quad \Rightarrow \quad |\mathrm{Re}(z)| \le \dfrac{3}{2}$

$|\mathrm{Re}(z)| \le \dfrac{3}{2} \quad \Rightarrow \quad |\mathrm{Re}(z)| \le 2$

Consider the circle $x^2 + y^2 - 9 = 0$

By putting $x = \dfrac{3}{2}\cos\theta$

& $y = \dfrac{5}{2}\sin\theta$ into $x^2 + y^2 - 9$

$\dfrac{9\cos^2\theta}{4} + \dfrac{25}{4}\sin^2\theta - 9 < 0$

(D) $z = (\cos\theta + i\sin\theta) + \dfrac{1}{(\cos\theta + i\sin\theta)}$

$z = 2\cos\theta$

where z is real value & $z \in [-2, 2]$

25. Comprehension (3 questions together)

$a + 8b + 7c = 0$
$9a + 2b + 3c = 0$
$7a + 7b + 7c = 0$
$\Rightarrow a = K,\ b = 6K\ ,\ c = -7K$

(i) (K, 6K, –7K)
$2x + y + z = 1$
$2K + 6K - 7K = 1$
($\because$ point lies on the plane)
$\Rightarrow K = 1$
$\Rightarrow 7a + b + c = 7K + 6K - 7K = 6$

(ii) $x^3 - 1 = 0$
$\Rightarrow x = 1, \omega, \omega^2$

$\omega = -\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}$ since $\mathrm{Im}(\omega) > 0$

If $a = 2 = K \Rightarrow b = 12$ & $c = -14$

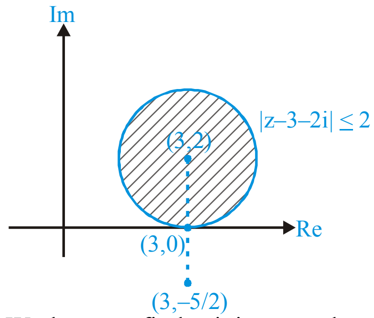
Hence $\dfrac{3}{\omega^a} + \dfrac{1}{\omega^b} + \dfrac{3}{\omega^c} = \dfrac{3}{\omega^2} + \dfrac{1}{\omega^{12}} + \dfrac{3}{\omega^{-14}}$

$= 3\omega + 1 + 3\omega^2 = -3 + 1 = -2$

(iii) $\because$ $b = 6 \Rightarrow 6K = 6 \Rightarrow K = 1$
$\Rightarrow a = 1,\ b = 6$ & $c = -7$
$x^2 + 6x - 7 = 0$
$\Rightarrow \alpha + \beta = -6\ ,\ \alpha\beta = -7$

$\Rightarrow \displaystyle\sum_{n=0}^{\infty}\left(\frac{\alpha+\beta}{\alpha\beta}\right)^n = \sum_{n=0}^{\infty}\left(\frac{6}{7}\right)^n = \frac{1}{1-\frac{6}{7}} = 7$

26.



We have to find minimum value of

$$2\left|z-\left(3-\frac{5}{2}i\right)\right|$$

= 2 × (minimum distance between z and point $\left(3,-\frac{5}{2}\right)$)

= 2 × (distance between (3,0) and $\left(3,-\frac{5}{2}\right)$

$= 2 \times \frac{5}{2} = 5$ units.

27. Ans. 3 (Bonus)

(Comment : If $\omega = e^{i\pi/3}$ then $\frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}$ is not always an integer. For example if a = b = c = 1 then the value of $\frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}$ is $\frac{17}{3}$. Now if we consider $\omega = e^{i2\pi/3}$ then the solution is)

$$|x|^2 = (a+b+c)\left(\bar{a}+\bar{b}+\bar{c}\right)$$

$$=|a|^2+|b|^2+|c|^2+a\bar{b}+a\bar{c}+b\bar{a}+b\bar{c}+c\bar{a}+c\bar{b}$$

$$|y|^2 = (a+b\omega+c\omega^2)\left(\bar{a}+\bar{b}\omega^2+\bar{c}\omega\right)$$

$$=|a|^2+|b|^2+|c|^2+a\bar{b}\omega^2+a\bar{c}\omega+b\bar{a}\omega+b\bar{c}\omega^2+c\bar{a}\omega^2+c\bar{b}\omega$$

$$|z|^2=(a+b\omega^2+c\omega)(\bar{a}+\bar{b}\omega+\bar{c}\omega^2)$$

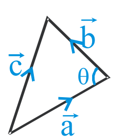
$= |a|^2 + |b|^2 + |c^2|$

$+\ a\bar{b}\omega+a\bar{c}\omega^2+b\bar{a}\omega^2+b\bar{c}\omega+c\bar{a}\omega+c\bar{b}\omega^2$

$\therefore |x|^2 + |y|^2 + |z|^2 = 3\ (|a|^2 + |b|^2 + |c|^2)$

$$\Rightarrow \frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}=3$$

28. (A)



$|\vec{a}|=2$
$|\vec{b}|=2$
$|\vec{c}|=2\sqrt{3}$

$$\cos\theta=\frac{|\vec{a}|^2+|\vec{b}|^2-|\vec{c}|^2}{2|\vec{a}|\,|\vec{b}|}$$

$$=\frac{4+4-12}{2.2.2}=\frac{-1}{2} \quad \Rightarrow \quad \theta=\frac{2\pi}{3}$$

(B) $\int_a^b (f(x)-3x)dx = a^2 - b^2 = \int_a^b (-2x)dx$

$\Rightarrow \int_a^b (f(x)-x)dx = 0$

$\Rightarrow$ one of the possible solution of this equation is

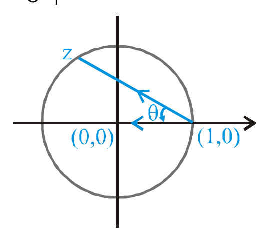
$f(x)=x \quad \Rightarrow \quad f\left(\frac{\pi}{6}\right)=\frac{\pi}{6}$

(C) $\frac{\pi^2}{\ell n3}\int_{7/6}^{5/6}(\sec \pi x)dx$

$$=\frac{\pi^2}{\ell n3}\frac{1}{\pi}\left[\ell n|\sec\pi x+\tan\pi x|\right]_{7/6}^{5/6}$$

$$=\frac{\pi}{\ell n3}\ell n\left|\frac{\sec\frac{5\pi}{6}+\tan\frac{5\pi}{6}}{\sec\frac{7\pi}{6}+\tan\frac{7\pi}{6}}\right|=\frac{\pi}{\ell n3}\ell n\,3=\pi$$

(D) Let $\theta = \text{Arg}\left(\frac{1}{1-z}\right)$



$\Rightarrow \theta = \text{Arg}\left(\frac{0-1}{z-1}\right)$

which is shown in adjacent diagram.

$\Rightarrow$ Maximum value of θ is approaching to $\frac{\pi}{2}$ but θ will never obtained the value equal to $\frac{\pi}{2}$.

Hence there is an error in aksing the problem.

29. (A) Let z = cosθ + isinθ

$$\text{Re}\left(\frac{2i(\cos\theta+i\sin\theta)}{1-(\cos\theta+i\sin\theta)^2}\right)=\text{Re}\left(\frac{\cos\theta i-\sin\theta}{\sin^2\theta-i\cos\theta\sin\theta}\right)$$

$= \text{Re}\left(-\frac{1}{\sin\theta}\right) = \frac{-1}{\sin\theta}$

$\therefore$ Set will be $(-\infty, -1] \cup [1, \infty)$

(B) $-1 \le \frac{8.3^{(x-2)}}{1-3^{2(x-1)}} \le 1 \qquad x \ne 1$

$\Rightarrow \quad -1 \le \frac{8.3^x}{(3-3^x)(3+3^x)} \le 1$

$3^x = t \quad \therefore \quad t > 0$

$\frac{8t}{(3-t)(t+3)} \ge -1$

$\Rightarrow \quad t \in (0,3) \cup [9, \infty)$

$\Rightarrow \quad x \in (-\infty, 1) \cup [2, \infty)$

$\frac{8t}{(3-t)(t+3)} \le 1$

$\Rightarrow \quad t \in (0, 1] \cup (3, \infty)$

$\Rightarrow \quad x \in (-\infty, 0] \cup (1, \infty)$

Taking intersection,

$x \in (-\infty, 0] \cup [2, \infty)$

(C) $f(\theta) = \begin{vmatrix} 1 & \tan & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$

$C_1 \to C_1 + C_3$

$\Rightarrow \quad f(\theta) = \begin{vmatrix} 2 & \tan\theta & 1 \\ 0 & 1 & \tan\theta \\ 0 & -\tan\theta & 1 \end{vmatrix}$

$\Rightarrow \quad f(\theta) = 2\sec^2\theta$

$\Rightarrow \quad f(\theta) \in [2, \infty)$

(D) $f(x) = 3x^{5/2} - 10x^{3/2}$

$f'(x) = \frac{15}{2}x^{3/2} - \frac{30}{2}x^{1/2} > 0$

$\Rightarrow \quad \frac{15}{2}\sqrt{x}(x-2) \ge 0 \qquad \Rightarrow \quad x \ge 2$

30. $z^2 + z + 1 - a = 0$

$\because$ z is imaginary $\quad \Rightarrow \quad D < 0$

$1 - 4(1-a) < 0$

$4a < 3$

$a < \frac{3}{4}.$

Aliter : $a = z^2 + z + 1$

$\because \quad a = \bar{a}$ (given a is real)

$\therefore \quad z^2 + z = \bar{z}^2 + \bar{z} \quad \Rightarrow \quad z^2 - \bar{z}^2 = \bar{z} - z$

$\Rightarrow \quad z + \bar{z} = -1$ ($\because$ Im(z) is non zero)

$\Rightarrow \quad \text{Re}(z) = -\frac{1}{2}$

$\therefore$ z can be taken as $-\frac{1}{2} + iy$

where $y \in R$

$\therefore \quad a = \left(-\frac{1}{2} + iy\right)^2 + \left(\frac{-1}{2} + iy\right) + 1$

$\Rightarrow \quad a = \frac{1}{4} - \frac{1}{2} + 1 - iy + iy - y^2$

$\Rightarrow \quad a = \frac{3}{4} - y^2 \quad \Rightarrow \quad a < \frac{3}{4}$

$\therefore \quad a \ne \frac{3}{4}$

31. Given : α satisfies $|z - z_0| = r$

$\Rightarrow \quad |\alpha - z_0| = r \qquad$ ...(i)

& $\frac{1}{\alpha}$ satisfies $|z - z_0| = 2r$

$\Rightarrow \quad \left|\frac{1}{\alpha} - z_0\right| = 2r \qquad$ ...(ii)

squaring (i) and (ii) we get $(\alpha - z_0)(\bar{\alpha} - \bar{z}_0) = r^2$

$\Rightarrow \quad \alpha\bar{\alpha} - z_0\bar{\alpha} - \alpha\bar{z}_0 + z_0\bar{z}_0 = r^2 = 2|z_0|^2 - 2 \qquad$ ...(iii)

& $\left(\frac{1}{\bar{\alpha}} - z_0\right)\left(\frac{1}{\alpha} - \bar{z}_0\right) = 4r^2$

$\Rightarrow \quad \frac{1}{\alpha\bar{\alpha}} - \frac{z_0}{\alpha} - \frac{\bar{z}_0}{\bar{\alpha}} + z_0\bar{z}_0 = 4r^2$

$\Rightarrow \quad 1 - z_0\bar{\alpha} - \bar{z}_0\alpha + |z_0|^2|\alpha|^2 = 4\left(2|z_0|^2 - 2\right)|\alpha|^2$

$\Rightarrow \quad 1 + 2|z_0|^2 - 2 - |\alpha|^2 - |z_0|^2 + |z_0|^2|\alpha|^2$

$= 8|z_0|^2|\alpha|^2 - 8|\alpha|^2$

$\Rightarrow \quad -1 + |z_0|^2 - 7|z_0|^2|\alpha|^2 + 7|\alpha|^2 = 0$

$\Rightarrow \quad (|z_0|^2 - 1)(7|\alpha^2| - 1) = 0$

$\Rightarrow \quad |z_0| = 1$ (rejected as $r = 0$) & $|\alpha| = \frac{1}{\sqrt{7}}$

32. $P^2 = [\alpha_{ij}]_{n \times n}$

$\alpha_{ij} = \sum_{k=1}^{n} p_{ik}.p_{kj}$

$$= \sum_{k=1}^{n} \omega^{i+k}.\omega^{k+j} = \omega^{i+j}\sum_{k=1}^{n}\omega^{2k}$$

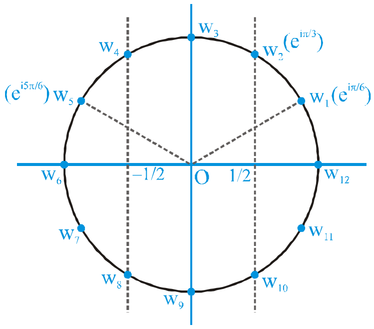
$$= \omega^{i+j}(\omega^2 + \omega^4 + \omega^6 + ... + \omega^{2n})$$

If n is a multiple of 3 then $P^2 = 0$

$\Rightarrow$ n is not a multiple of 3

$\Rightarrow$ n can be 55, 58, 56

33.



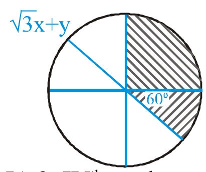
$z_1 = \{w_1, w_{11}, w_{12}\}$

$z_2 = \{w_5, w_6, w_7\}$

$\angle w_1Ow_5 = \frac{2\pi}{3}$ & $\angle w_1Ow_6 = \frac{5\pi}{6}$

Paragraph for Question 34 and 35

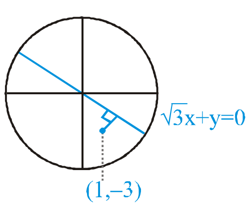
34. S_1 is interior of circle centred at (0,1) & radius = 4.



Re(z) > 0 is in I[st] & IV[th]quadrant.

$$\frac{\left(z-\left(1-i\sqrt{3}\right)\right)}{\left(1-i\sqrt{3}\right)} = \frac{\left(\left(x-1\right)+i\left(y-\sqrt{3}\right)\right)}{\left(1-i\sqrt{3}\right)}$$

$$= \frac{\left(\left(x-1\right)+i\left(y-\sqrt{3}\right)\right)\left(1+i\sqrt{3}\right)}{2}$$



$I_m\left(S_2\right) = \sqrt{3}x + y > 0$

erpendicular distance from (1,–3) to the line is

$$P = \left|\frac{\sqrt{3}-3}{2}\right| = \left(\frac{3-\sqrt{3}}{2}\right)$$

35. Ans. (B)

$$\text{Area of } S = \frac{\pi(4)^2}{4} + \frac{1}{2}(4)^2\,\frac{\pi}{3}$$

$$\frac{8\pi}{3} + 4\pi = \frac{20\pi}{3}$$

38. $Z = \frac{-1+i\sqrt{3}}{2} = \omega$

$$p = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$$

$$p^2 = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}\begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$$

$$= \begin{bmatrix} (-\omega)^r + \left(\omega^{2s}\right)^2 & \omega^{2s}(-\omega)^r + \omega^r\omega^{2s} \\ \omega^{2s}(-\omega)^r + \omega^r\omega^{2s} & \omega^{4s} + \omega^{2r} \end{bmatrix}$$

$$= \begin{bmatrix} \omega^{4s} + \omega^{2r} & \omega^{2s}\left(\omega^r + (-\omega)^r\right) \\ \omega^{2s}\left(\omega^r + (-\omega)^r\right) & \omega^{4s} + \omega^{2r} \end{bmatrix}$$

$= -I$ (Given)

$\omega^{4s} + \omega^{2r} = -1$ and

$\omega^{2s} + \omega^{2s}\left(\omega^r + (-\omega)^r\right) = 0$

$\omega^r + (-\omega)^r = 0$

r	s	r	s
1	1	1	1
2	2	3	3

Total no. pairs = 1

39. $z = \frac{1}{a+ibt}$

$\Rightarrow$ $x + iy = \frac{a-ibt}{a^2+b^2t^2}$

$\Rightarrow$ $x = \frac{a}{a^2+b^2t^2}, y = \frac{-bt}{a^2+b^2t^2}$

Eliminating t, we get

$$x^2 + y^2 = \frac{x}{a} \Rightarrow \left(x - \frac{1}{2a}\right)^2 + y^2 = \left(\frac{1}{2a}\right)^2$$

$\therefore$ (A) is correct.

(C), (D) can be verifiedby putting b = 0 and a = 0 respectively.

MOCK TEST

1. **(D)** $e^{\frac{2\pi r}{p}i} = e^{\frac{2\pi m}{q}i}$

$r = 0, 1, ... p-1$

$m = 0, 1, ... q-1$

This is possible iff $r = m = 0$

but for $r = m = 0$ we get 1 which is not an imaginary number.

2. **(D)** $z^3 + \dfrac{3(\bar{z})^2}{|z|} = 0$ Let $z = re^{i\theta}$

$\Rightarrow$ $r^3 e^{i3\theta} + 3re^{-i2\theta} = 0$

Since 'r' cannot be zero

$\Rightarrow$ $r^2 e^{i5\theta} = -3$ which will hold for

$r = \sqrt{3}$ and 5 distinct values of 'θ'

Thus there are five solution.

3. **(B)**

$(x-1)(x-\alpha_1)(x-\alpha_2)\,.............(x-\alpha_8) \equiv x^9 - 1$

$\therefore$ $(2-\alpha_1)(2-\alpha_2)...............(2-\alpha_8) = 2^9 - 1$

Now since $2-\alpha_1$ and $2-\alpha_8$ are conjugates of each other

$\therefore$ $|2-\alpha_1| = |2-\alpha_8|$

similarly

$|2-\alpha_2| = |2-\alpha_7|$, $|2-\alpha_3| = |2-\alpha_6|$

and $|2-\alpha_4| = |2-\alpha_5|$

$\therefore$ $|(2-\alpha_1)(2-\alpha_3)(2-\alpha_5)(2-\alpha_7)| = \sqrt{2^9-1} = \sqrt{511}$

4. **(D)**

$\arg(z-i+2) = \dfrac{\pi}{6}$ $\Rightarrow$ $\tan\dfrac{\pi}{6} = \dfrac{y-1}{x+2}$

$\Rightarrow$ $x - \sqrt{3}y = -(\sqrt{3}+2)$, $x > -2, y > 1$ ....**(i)**

$\arg(z+4-3i) = -\dfrac{\pi}{4}$ $\Rightarrow$ $\tan\left(-\dfrac{\pi}{4}\right) = \dfrac{y-3}{x+4}$

$\Rightarrow$ $y + x = -1$, $x > -4, y < 3$ ....**(ii)**

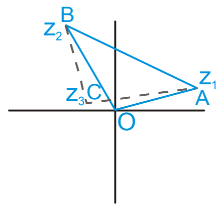
so, there is no point of intersection.

5. **(D)**

$|z_2 + iz_1| = |z_1| + |z_2|$

$\Rightarrow$ $\text{Arg}(iz_1) = \text{Arg}(z_2)$

$\Rightarrow$ $\text{Arg}(z_2) - \text{Arg}(z_1) = \dfrac{\pi}{2}$



Let $z_3 = \dfrac{z_2 - iz_1}{1-i}$

$(1-i)z_3 = z_2 - iz_1$

$\Rightarrow$ $(z_3 - z_2) = i(z_3 - z_1)$ $\Rightarrow$ $(z_2 - z_3) = i(z_1 - z_3)$

$\therefore$ $\angle ACB = \dfrac{\pi}{2}$ and $|z_2 - z_3| = |z_1 - z_3|$

$\Rightarrow$ $AC = BC$

$\because$ $AB^2 = AC^2 + BC^2$ $\Rightarrow$ $AC = \dfrac{5}{\sqrt{2}}$.

$(\because$ $AB = 5)$

$\therefore$ $\Delta ABC = \dfrac{1}{2}AC.BC = \dfrac{AC^2}{2} = \dfrac{25}{4}$ square unit

6. **(C)**

$\dfrac{(1+i)^5(1+\sqrt{3}\,i)^2}{-2i(-\sqrt{3}+i)}$

$= \dfrac{\left(\sqrt{2}\right)^5\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^5 .2^2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2}{2i\;2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)}$

$\therefore$ argument $= \dfrac{5\pi}{4} + \dfrac{2\pi}{3} - \dfrac{\pi}{2} + \dfrac{\pi}{6} = \dfrac{19\pi}{12}$

$\therefore$ principal argument is $-\dfrac{5\pi}{12}$

7. **(C)**

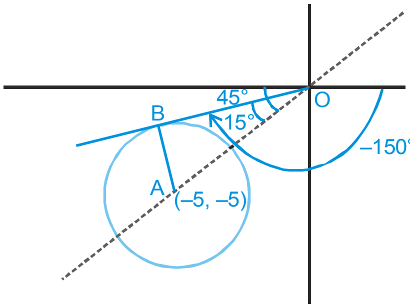
Equation of the line is $2x - 2y = 0$ i.e $y = x$

Now $\dfrac{2-i}{3+i} = \dfrac{(2-i)(3-i)}{10} = \dfrac{1-i}{2}$

$\therefore$ image is the point whose affix is $\dfrac{-1+i}{2}$

8. (C)

Point B has least principal argument



$$AB = \frac{5(\sqrt{3}-1)}{2}$$

$$OA = 5\sqrt{2}$$

$$\angle AOB = \frac{\pi}{12}$$

$$\therefore \quad \text{Arg}(z) = -\frac{5\pi}{6}$$

9. (C)

$$z = \frac{at+b}{t-c} \quad \Rightarrow \quad t = \frac{b+cz}{z-a}$$

$$|t| = 1 \quad \Rightarrow \quad t\bar{t} = 1$$

$$\Rightarrow \quad \frac{(b+cz)(\bar{b}+\bar{c}\bar{z})}{(\bar{z}-\bar{a})(z-a)} = 1$$

$$\Rightarrow \quad z\bar{z}\,(c\bar{c}-1) + (c\bar{b}+\bar{a})z + (\bar{c}b+a)\,\bar{z} + b\bar{b} - a\bar{a} = 0$$

10. (B)

S_1. $(z+1)^7 + z^7 = 0$

$$z^7 + z^7 + {}^7C_1 z^6 + {}^7C_2 z^5 + \ldots + 1 = 0$$

$$\therefore \quad \sum_{k=0}^{6} \text{Re}(z_k) = -\frac{7}{2}$$

S_2. $\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 1 + i$

squaring both the sides

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} + 2\left(\frac{\alpha\beta}{ab} + \frac{\beta\gamma}{bc} + \frac{\gamma\alpha}{ca}\right) = 2i$$

i.e. $$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} + 2\left(\frac{c\alpha\beta + a\beta\gamma + b\gamma\alpha}{abc}\right) = 2i$$

$$\therefore \quad \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = 2i \qquad \because \quad \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 0$$

S_3. $z^6 - z^5 + z^4 - z^3 + z^2 - z + 1 = \prod_{i=1}^{6}(z - z_i)$

Put $z = -1$

$$\therefore \quad 7 = \prod_{i=1}^{6}(-1-z_i) = \prod_{i=1}^{6}(1+z_i)$$

S_4. $z^3 = \bar{z}\,i\,|z| \quad \Rightarrow \quad |z| = 1$ or $|z| = 0$

Thus $z = 0$ is a solution.

If $|z| = 1$, Let $z = e^{i\theta}$ then $e^{i3\theta} = e^{-i\theta}\, i$

$\Rightarrow \quad e^{i4\theta} = i$

$$\Rightarrow \quad 4\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}$$

$$\therefore \quad \theta = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8} \text{ are solutions.}$$

$\therefore$ In all there are 5 solutions.

11. (A, B)

$n = 1 \quad \Rightarrow \quad b = 0$ not possible

$n = 2 \quad \Rightarrow$ either $a = 0$ or $b = 0$ not possible

$n = 3 \quad \Rightarrow \quad a^3 - ib^3 + 3a^2 bi - 3ab^2$

$\qquad = a^3 + ib^3 - 3a^2bi - 3ab^2$

$\Rightarrow \quad 2ib^3 = 3a^2 bi$

$$\Rightarrow \quad \frac{b^2}{a^2} = 3 \quad \Rightarrow \quad \frac{b}{a} = \sqrt{3} = \tan\frac{\pi}{3}$$

12. (A, B)

$|z - i\,\text{Re}(z)| = |z - \text{Im}(z)|$

Let $z = x + iy$, then

$|x + iy - ix| = |x + iy - y|$

i.e. $x^2 + (y-x)^2 = (x-y)^2 + y^2$

i.e. $x^2 = y^2$ i.e. $y = \pm x$

13. (A, D)

$z_1 = 5 + 12i$, $|z_2| = 4$

$|z_1 + iz_2| \le |z_1| + |z_2| = 13 + 4 = 17$

$\therefore \quad |z_1 + (1+i)z_2| \ge ||z_1| - |1+i|\,|z_2||$

$= 13 - 4\sqrt{2}$

$\therefore \quad \min(|z_1 + (1+i)z_2|) = 13 - 4\sqrt{2}$

$$\left|z_2 + \frac{4}{z_2}\right| \le |z_2| + \frac{4}{|z_2|} = 4 + 1 = 5$$

$$\left|z_2 + \frac{4}{z_2}\right| \geq |z_2| - \frac{4}{|z_2|} = 4 - 1 = 3$$

$$\therefore \max\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{3} \text{ and } \min\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{5}$$

14. (A, D)

$\mu^2 - 2\mu + 2 = 0$

$\therefore \quad \alpha = 1 + i$ and $\beta = 1 - i$

$x = \cot\theta - 1$

$\therefore \quad x + \alpha = \cot\theta + i = \text{cosec}\,\theta \,.\, e^{i\theta}$

and $x + \beta = \cot\theta - i = \text{cosec}\,\theta \,.\, e^{-i\theta}$

$\alpha - \beta = 2i$

$$\frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta} = \frac{\text{cosec}^n\theta.[e^{in\theta} - e^{-in\theta}]}{2i}$$

$$= \text{cosec}^n\theta.\frac{2i\sin n\theta}{2i} = \frac{\sin n\theta}{\sin^n\theta}$$

15. (A, B, C, D)

$|z_1| = 1, \ |z_2| = 2$

(A) $||z_1| - 2|-z_2|| \leq |z_1 - 2z_2| \leq |z_1| + 2|-z_2|$

$|1 - 2(2)| \leq |z_1 - 2z_2| \leq 1 + 2(2)$

$3 \leq |z_1 - 2z_2| \leq 5$

(B) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

$|1 - 2| \leq |z_1 + z_2| \leq 1 + 2$

$1 \leq |z_1 + z_2| \leq 3$

(C) $||z_1| - 3|z_2|| \leq |z_1 - 3z_2| \leq |z_1| + 3|-z_2|$

$5 \leq |z_1 - 3z_2| \leq 7$

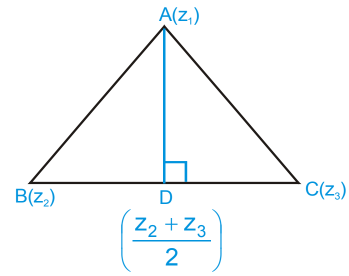
(D) $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$

$1 \leq |z_1 - z_2| \leq 3$

16. (D)

$$\because \arg\left(\frac{z_2 + z_3 - 2z_1}{z_3 - z_2}\right) = \arg\left(\frac{\frac{z_2+z_3}{2} - z_1}{z_3 - z_2}\right)$$

$$= \frac{\pi}{2} \text{ (as AD} \perp \text{BC)}$$



17. (D)

$x + \frac{1}{x} = 1$

$\Rightarrow \quad x^2 - x + 1 = 0$

$\therefore \quad x = -\omega, -\omega^2$

Now for $x = -\omega$, $P = \omega^{4000} + \frac{1}{\omega^{4000}} = \omega + \frac{1}{\omega} = -1$

Similarly for $x = -\omega^2$ also $p = -1$

for $n > 1$, $2^n = 4k$

$\therefore \quad 2^{2^n} = 2^{4k} = (16)^k$ = a number with last digit = 6

$\Rightarrow \quad q = 6 + 1 = 7$

Hence $\quad p + q = -1 + 7 = 6$

18. (B)

$$\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3} \Rightarrow \frac{1}{z_1} - \frac{1}{z_2} = \frac{1}{z_3} - \frac{1}{z_1}$$

$$\Rightarrow \frac{z_2 - z_1}{z_1 z_2} = \frac{z_1 - z_3}{z_1 z_3}$$

$$\Rightarrow \frac{z_2 - z_1}{z_3 - z_1} = -\frac{z_2}{z_3} \Rightarrow \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg\left(-\frac{z_2}{z_3}\right)$$

$$\Rightarrow \arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \pm\pi + \arg\left(\frac{z_2}{z_3}\right) = \pm\pi + \arg\left(\frac{z_2 - O}{z_3 - O}\right)$$

(+ or – as applicable)

$\Rightarrow$ Points O, A, B, C are concylic.

19. (D)

If $\quad 3 + ix^2y$ and $x^2 + y + 4i$ are conjugate

then $x^2y = -4$ and $x^2 + y = 3$

$\Rightarrow \quad x^2 = 4, y = -1$

$\Rightarrow \quad x^2 + y^2 = 5$

20. (A)

$|z^2 + 2z\cos\alpha| \leq |z^2| + |2z\cos\alpha| < |z|^2 + 2|z|\,|\cos\alpha| < (\sqrt{2} - 1)^2$

$+2(\sqrt{2} - 1) = 1$

$(\because |\cos\alpha| \leq 1)$

21. (A) → s ; (B) → q, r ; (C) → q, t ; (D) → t

(A) Put $z = x + iy$

$\therefore$ $Re(x + iy)^2 = Re(x + iy + x - iy)$

$x^2 - y^2 = 2x$

or $x^2 - y^2 - 2x = 0$

Rectangular hyperbola, eccentricity = $\sqrt{2}$

(B) For ellipse $\lambda > |z_1 - z_2|$ and for straight line $\lambda = |z_1 - z_2|$

(C) $\left|\frac{2z - i}{z + i}\right| = m \Rightarrow \left|\frac{z - \frac{i}{2}}{z + 1}\right| = \frac{m}{2}$

for $m = 2$, $\left|\frac{z - \frac{i}{2}}{z + 1}\right| = 1 \Rightarrow \left|z - \frac{i}{2}\right| = |z + 1|$

ie, a straight line and for $m \neq 2$, locus is circle

(D) Let $z = x + iy$

$\Rightarrow$ $x^2 + y^2 = 25^2$

$-1 + 75\bar{z} = 75x - 1 + i75y = h + ik$

$\Rightarrow \left(\frac{h+1}{75}\right)^2 + \left(\frac{k}{75}\right)^2 = 25^2$

$\Rightarrow$ Locus of (h, k) is a circle

(D) $|z_1 + z_2| = \sqrt{2 + 2\cos 144°}$ for minimum

$= 2\cos 72° = \frac{\sqrt{5} - 1}{2}$ whose greatest integer is 0.

22. (A) → (r), (B) → (q, t), (C) → (s), (D) → (p)

The given equation is $\frac{z^5 - 1}{z - 1} = 0$ which means that z_1, z_2, z_3, z_4 are four out of five roots of unity except 1.

(A) $z_1^4 + z_2^4 + z_3^4 + z_4^4 + 1^4 = 0 \quad \Rightarrow \left|\sum_{i=1}^{4} z_i^4\right| = 1$

(B) $z_1^5 + z_2^5 + z_3^5 + z_4^5 + 1^5 = 5 \quad \Rightarrow \sum_{i=1}^{4} z_i^5 = 4$

(C) $z^4 + z^3 + z^2 + z + 1 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$.

Putting $z = -2$ both the sides and we get $\prod_{i=1}^{4}(z_i + 2) = 11$

23.

1. (B)

$\omega_1 = \omega_2 \, e^{i\frac{4\pi}{3}}$

$\Rightarrow$ $\omega_1^3 = \omega_2^3$

$\Rightarrow$ $\omega_1^3 \, \bar{\omega}_1^2 \, \bar{\omega}_2^2 = \omega_2^3 \, \bar{\omega}_1^2 \, \bar{\omega}_2^2$

$\Rightarrow$ $\omega_2 \bar{\omega}_1^2 = \omega_1 \bar{\omega}_2^2$

2. (B)

Since $i\beta$ is real

$\therefore$ β is pure imaginary

3. (C)

$-\frac{\alpha}{\bar{\alpha}} = e^{\pm i\frac{\pi}{2}} = \pm i$

$\therefore$ $(1 + i)\left(-\frac{2\alpha}{\bar{\alpha}}\right) = \pm 2(-1 + i)$

25. 1 (B) 2. (C) 3. (D)

25. (Q. 1. to 3.)

$BM \equiv y - 0 = -1(x - 1)$

$x + y = 1$

$\therefore$ $\sqrt{u - 1} = t + i(1 - t)$

$u = 2t + 2it(1 - t)$

$x = 2t$ and $y = 2t(1 - t)$, where $u = x + iy$

$\Rightarrow$ $(x - 1)^2 = -2(y - \frac{1}{2})$ which is a parabola

axis is $x = 1$ i.e. $z + \bar{z} = 2$

directrix is $y = 1$ i.e. $z - \bar{z} = 2i$

26. (4)

$\left(\frac{1+i}{1-i}\right)^n = \frac{2}{\pi} \times \frac{\pi}{2} = 1$

Now $\left(\frac{1+i}{1-i}\right)^n = 1$

$\Rightarrow$ $i^n = 1 \quad \Rightarrow \quad n = 4, 8, 12, 16$

27. (1)

$$f_p(\alpha) = e^{\frac{i\alpha}{p^2}(1+2+\ldots+p)} = e^{\frac{i\alpha}{2}\left(1+\frac{1}{p}\right)}$$

$$\lim_{n\to\infty} f_n(\pi) = \lim_{n\to\infty} e^{\frac{i\alpha}{2}\left(1+\frac{1}{n}\right)} = e^{\frac{i\alpha}{2}}$$

$$\left|\lim_{n\to\infty} f_n(\pi)\right| = \left|e^{\frac{i\alpha}{2}}\right| = 1$$

28. (1)

If $|z| = |z-1| \Rightarrow |z|^2 = |z-1|^2$

$\Rightarrow z\bar{z} = (z-1)(\bar{z}-1)$

$\Rightarrow z + \bar{z} = 1$

again if $|z| = |z+1|$

$\Rightarrow |z|^2 = |z+1|^2 \Rightarrow z + \bar{z} = -1 \Rightarrow |z+\bar{z}| = 1$

29. (30)

$|z| + |z-1| + |2z-3| = |z| + |z-1| + |3-2z| \geq |z + z - 1 + 3 - 2z| = 2$

$\therefore |z| + |z-1| + |2z-3| \geq 2$

$\therefore \lambda = 2$

then $2[x] + 3 = 3[x-\lambda]$

$= 3[x-2]$

$\Rightarrow 2[x] + 3 = 3([x]-2)$

or $[x] = 9$, then $y = 2.9 + 3 = 21$

$\therefore [x+y] = [x+21] = [x] + 21 = 9 + 21 = 30$

30. (7) $f(x) = A_0 + \sum_{k=1}^{20} A_k x^k$

$$= \sum_{k=0}^{20} A_k x^k$$

$$\because \sum_{r=0}^{6} f(\alpha^r x) = f(x) + f(\alpha x) + f(\alpha^2 x) + f(\alpha^3 x) + f(\alpha^4 x) + f(\alpha^5 x) + f(\alpha^6 x)$$

$$= \sum_{k=0}^{20} \{A_k x^k + A_k(\alpha x)^k + A_k(\alpha^2 x)^k + A_k(\alpha^3 x)^k + A_k(\alpha^4 x)^k + A_k(\alpha^5 x)^k + A_k(\alpha^6 x)^k\}$$

$$= \sum_{k=0}^{20} A_k x^k (1 + (\alpha)^k + (\alpha^2)^k + (\alpha^3)^k + (\alpha^4)^k + (\alpha^5)^k + (\alpha^6)^k)$$

$= A_0 x^0 (7) + A_7 x^7 (7) + A_{14} x^{14} (7)$

$= 7(A_0 + A_7 x^7 + A_{14} x^{14})$

$\therefore n = 7$