SIMPLE HARMONIC MOTION

INTRODUCTION

Our heart beats. Our hung oscillate, we shiver when we are cold, we sometimes snore, we can hear and speak because our eardrums and larynges vibrate. We cannot even say "vibration" properly without the tip of the tongue oscillating.

PERIODIC MOTION

(a) The motion of the hands of a clock is periodic, the period of the motion of the minute hand being one hour, or 3600 seconds.

(b) The bob of a pendulum moves periodically.

(c) Motion of planets around the Sun.

The period being equal to the time of one complete (to and fro) oscillation. Periodic motion can be along any path.

OSCILLATORY MOTION

(i) The motion of body is said to be oscillatory or vibratory motion if it moves back and forth (to and fro) about a fixed point after regular interval of time.

(ii) The fixed point about which the body oscillates is called mean position or equilibrium position.

Examples : **(i)** Vibration of the wire of 'Sitar'. **(ii)** Oscillation of the mass suspended from spring.

Note : Every oscillatory motion is periodic but every periodic motion is not oscillatory.

So a periodic motion may not be oscilatory. For example planetary motion, the hands of a clock.

SIMPLE HARMONIC MOTION (SHM)

If the restoring force/torque acting on the body in oscillatory motion is directly proportional to the displacement of body/particle and is always directed towards equilibrium position then the motion is called Simple Harmonic Motion (SHM). In this kind of motion the acceleration of particle is always proportional to negative of its displacement. It is the simplest (easy to analyze) form of oscillatory motion.

Important characteristics of SHM

(i) SHM can take place only about stable equilibrium position.

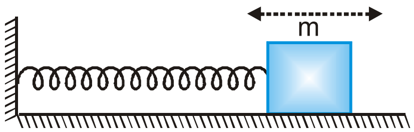
(ii) The force/torque should always be directed towards mean position.

(iii) Energy should remain conserved.

TYPES OF SHM

(i) Linear S.H.M.

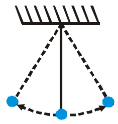
When a particle moves to and fro about a fixed point (called equilibrium position) along a straight line then its motion is called linear simple harmonic motion.



Example : Motion of a mass connected to spring.

(ii) Angular S.H.M.

When a system oscillates angularly with respect to a fixed axis then its motion is called angular simple harmonic motion. Example :- Motion of a bob of simple pendulum.



Necessary Condition to execute S.H.M.

(i) In linear S.H.M. :-

The restoring force (or acceleration) acting on the particle should always be proportional to the displacement of the particle and directed towards the equilibrium position $\therefore$ $F \propto -x$ or $a \propto -x$

Negative sign shows that direction of force and acceleration is towards equilibrium position and x is displacement of particle from equilibrium position.

(ii) In angular S.H.M. :-

The restoring torque (or angular acceleration) acting on the particle should always be proportional to the angular displacement of the particle and directed towards the equilibrium position

$\therefore$ $\tau \propto -\theta$ or $\alpha \propto -\theta$

Equation of S.H.M.

(i) In linear S.H.M.

The necessary and sufficient condition for SHM is

$$F = -kx$$

where k = positive constant for SHM known as Force constant

x = displacement from mean position

or $m\dfrac{d^2x}{dt^2} = -kx$ $\Rightarrow$ $\dfrac{d^2x}{dt^2} + \dfrac{k}{m}x = 0$

$\Rightarrow$ $\dfrac{d^2x}{dt^2} + \dfrac{k}{m}x = 0$ [differential equation of SHM]

$\Rightarrow$ $\dfrac{d^2x}{dt^2} + \omega^2 x = 0$ where $\omega = \sqrt{\dfrac{k}{m}}$

Its solution is $x = A\sin(\omega t + \phi)$

(ii) In Angular S.H.M.

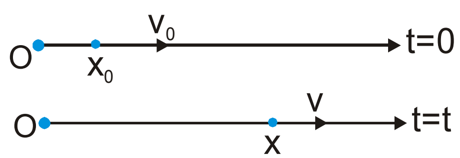
Restoring torque acting on the $\tau = -C\theta$ particle where C is a constant which can be defined as torque per unit angular displacement.

Mathematically, $I\alpha = -C\theta$. Where I is the moment of inertia of the system about the axis of rotation.

$$\Rightarrow I\frac{d^2\theta}{dt^2} + C\theta = 0 \Rightarrow \frac{d^2\theta}{dt^2} + \left(\frac{C}{I}\right)\theta = 0 . \text{ Since, } \frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \Rightarrow \omega = \sqrt{\left(\frac{C}{I}\right)} \qquad \text{.....(i)}$$

Derivation

Consider a particle of mass m moving along the x - axis. Suppose a force $F = -kx$ acts on the particle where k is a positive constant and x is the displacement of the particle from the assumed origin. The particle then executes a simple harmonic motion with the centre of oscillation at the origin. We shall calculate the displacement x and the velocity v as a function of time.



Suppose the position of the particle at t = 0 is x_0 and its velocity is v_0. Thus, at t = 0, $x = x_0$ and $v = v_0$

The acceleration of the particle is any instant is

$$a = \frac{F}{m} = -\frac{k}{m}x = -\omega^2 x \qquad \text{where } \omega = \sqrt{k/m}$$

thus, $$\frac{dv}{dt} = -\omega^2 x \qquad \text{....... (1)}$$

or, $$\frac{dv}{dx}\frac{dx}{dt} = -\omega^2 x \qquad \text{or,} \qquad v\,dv = -\omega^2 x\,dx$$

The velocity of the particle is v_0 when the particle is at $x = x_0$. It becomes v when the displacement becomes x. We can integrate the above equation and write

$$\int_{v_0}^{v} v\,dv = \int_{x_0}^{x} -\omega^2 x \qquad \text{or,} \qquad \left[\frac{v^2}{2}\right]_{v_0}^{v} = -\omega^2\left[\frac{x^2}{2}\right]_{x_0}^{x}$$

or, $$v^2 - v_0^2 = -\omega^2(x^2 - x_0^2) \qquad \text{or,} \qquad v = \sqrt{(v_0^2 + \omega^2 x^2) - \omega^2 x^2}$$

or, $$v = \omega\sqrt{\left(\frac{v_0^2}{\omega^2} + x_0^2\right) - x^2}$$

writing $$\left(\frac{v_0}{\omega}\right) + x_0^2 = A^2 \qquad \text{.......... (2)}$$

the above equation is becomes

$$v = \omega\sqrt{A^2 - x^2} \qquad \text{........... (3)}$$

we can write this equation as

$$\frac{dx}{dt} = \omega\sqrt{A^2 - x^2} \qquad \text{or,} \qquad \frac{dx}{\sqrt{A^2 - x^2}} = \omega dt$$

At time t = 0 the displacement is $x = x_0$ and at time t the displacement becomes x. The above equation can be integrated as

$$\int_{x_0}^{x} \frac{dx}{\sqrt{A^2 - x^2}} = \int_0^t \omega dt \qquad \text{or,} \qquad \left[\sin^{-1}\frac{x}{A}\right]_{x_0}^{x} = [\omega t]_0^t \qquad \text{or,} \qquad \sin^{-1}\frac{x}{A} - \sin^{-1}\frac{x_0}{A} - \omega t$$

writing $\sin^{-1}\frac{x_0}{A} = \delta$, this becomes

$$\sin^{-1}\frac{x}{A} = \omega t + \delta \qquad \text{or,} \qquad x = A\sin(\omega t + \delta) \qquad \text{......... (4)}$$

The velocity at time t is

$$v = \frac{dx}{dt} = A\omega\cos(\omega t + \delta) \qquad \text{........... (5)}$$

and $$a = \frac{dv}{dt} = -\omega^2 A\sin(\omega t + \delta) \qquad \text{........... (6)}$$

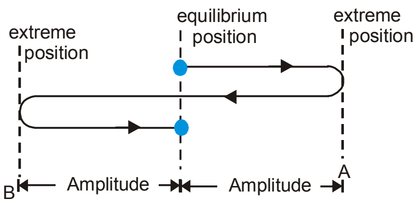
SOME BASIC TERMS

(i) Mean Position : The point at which the restoring force on the particle is zero and potential energy is minimum, is known as its mean position.

(ii) Restoring Force :

- The force acting on the particle which tends to bring the particle towards its mean position, is known as restoring force.
- Restoring force always acts in a direction opposite to that of displacement. Displacement is measured from the mean position.

(iii) Displacement : It is defined as the distance of the particle from the mean position at that instant. Displacement in SHM at time t is given by $x = A\sin(\omega t + \phi)$



(iv) Amplitude : It is the maximum value of displacement of the particle from its equilibrium position.

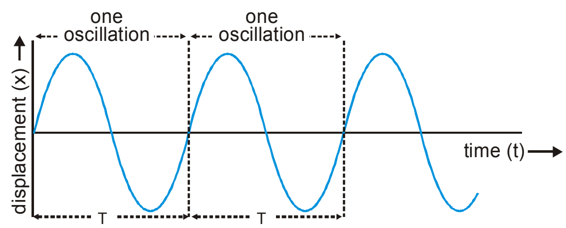
Amplitude = $\frac{1}{2}$ [distance between extreme points or positions]

It depends on energy of the system.

(v) Time period (T)

- The minimum time after which the particle keeps on repeating its motion is known as time period.
- The smallest time taken to complete one oscillation or vibration is also define as time period.
- It is given by $T = \frac{2\pi}{\omega} = \frac{1}{n}$ where ω is angular frequency and n is frequency.

(vi) **Oscillation or Vibration :** When a particle goes on one side from mean position and returns back and then it goes to other side and again returns back to mean position, then this process is known as one oscillation.



(vii) **Frequency (n or f)**

The number of oscillations per second is define as frequency.

It is given by $n = \frac{1}{T}$, $n = \frac{\omega}{2\pi}$

SI unit : hertz (Hz), 1 hertz = 1 cycle per second (cycle is a number not a dimensional quantity).

Dimensions : $M^0 L^0 T^{-1}$.

(viii) **Phase :**

- Phase of a vibrating particle at any instant is the state of the vibrating particle regarding its displacement and direction of vibration at that particular instant.
- In the equation $x = A \sin(\omega t + \theta)$, $(\omega t + \theta)$ is the phase of the particle.
- The phase angle at time t = 0 is known as initial phase or epoch.
- The difference of total phase angles of two particles executing S.H.M. with respect to the mean position is known as phase difference.
- Two vibrating particles are said to be in same phase if the phase difference between them is an even multiple of π, i.e., $\Delta\phi = 2n\pi$ Where n = 0, 1, 2, 3,....
- Two vibrating particle are said to be in opposite phase if the phase difference between them is an odd multiple of π i.e., $\Delta\phi = (2n + 1)\pi$ Where n = 0, 1, 2, 3,....

(ix) **Angular frequency (ω) :**

The rate of change of phase angle of a particle with respect to time is define as its angular frequency.

SI unit **:** radian/second, *Dimensions* **:** $M^0 L^0 T^{-1}$, $\omega = \sqrt{\frac{k}{m}}$ • **Time period (T)**

(i) The minimum time after which the particle keeps on repeating its motion is known as time period.

(ii) The smallest time taken to complete one oscillation or vibration is also define as time period.

(iii) It is given by $T = \frac{2\pi}{\omega} = \frac{1}{n}$ where ω is angular frequency and n is frequency.

(x) **Phase constant (ϕ) :** Constant ϕ in equation of SHM is called phase constant or initial phase. It depends on initial position and direction of velocity.

(xi) **Velocity(v) :** Velocity at an instant is the rate of change of particle's position w.r.t time at that instant.

Let the displacement from mean position is given by

$$x = A \sin(\omega t + \phi)$$

Velocity, $v = \frac{dx}{dt} = \frac{d}{dt}[A\sin(\omega t + \phi)]$

$v = A\omega\cos(\omega t + \phi)$ or, $v = \omega\sqrt{A^2 - x^2}$

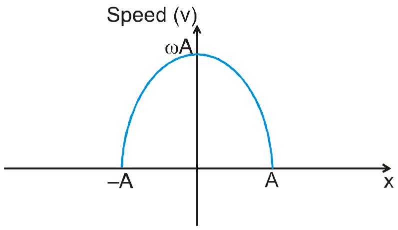
At mean position (x = 0), velocity is maximum.

$v_{max} = \omega A$

At extreme position (x = A), velocity is minimum.

$v_{min} = \text{zero}$

Graph of speed (v) vs displacement (x):



$v = \omega\sqrt{A^2 - x^2}$ $\quad v^2 = \omega^2(A^2 - x^2)$

$v^2 + \omega^2x^2 = \omega^2A^2$ $\quad \frac{v^2}{\omega^2A^2} + \frac{x^2}{A^2} = 1$

Graph would be an ellipse :

- **Acceleration :** Acceleration at an instant is the rate of change of particle's velocity w.r.t. time at that instant.

Acceleration, $a = \frac{dv}{dt} = \frac{d}{dt}[A\omega\cos(\omega t + \phi)]$

$a = -\omega^2A\sin(\omega t + \phi)$

$a = -\omega^2x$

Note : Negative sign shows that acceleration is always directed towards the mean position.

At mean position (x = 0), acceleration is minimum.

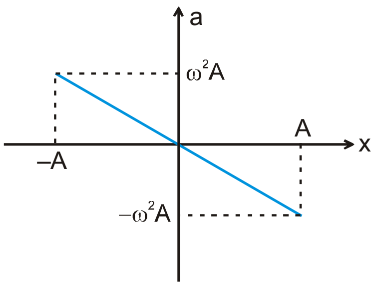
$a_{min} = \text{zero}$

At extreme position (x = A), acceleration is maximum.

$a_{max} = \omega^2A$

Graph of acceleration (a) vs displacement (x)

$a = -\omega^2x$



Ex. If equation of displacement of a particle is y = A sinQt + B cosQt then find the nature of the motion of particle.

Sol. $y = A\sin Qt + B\cos Qt$

Differentiate with respect to t $\frac{dy}{dt} = AQ\cos Qt - BQ\sin Qt$

Again differentiating with respect to t $\frac{d^2y}{dt^2} = -Q^2A\sin Qt - Q^2B\cos Qt$

$$\frac{d^2y}{dt^2} = -Q^2(A\sin Qt + B\cos Qt) \Rightarrow \frac{d^2y}{dt^2} = -Q^2y \Rightarrow \frac{d^2y}{dt^2} + Q^2y = 0$$

It is a differential equation of linear S.H.M. So motion of the particle is simple harmonic

Ex. If two S.H.M.'s are represented by equations $y_1 = 10\sin\left[3\pi t + \frac{\pi}{4}\right]$ and $y_2 = 5\left[\sin(3\pi t) + \sqrt{3}\cos(3\pi t)\right]$ then find the ratio of their amplitudes and phase difference in between them.

Sol. As $y_2 = 5\left[\sin(3\pi t) + \sqrt{3}\cos(3\pi t)\right]$ ...(i)

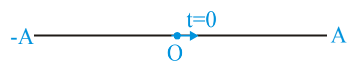
So if $5 = A\cos\phi$ and $5\sqrt{3} = A\sin\phi$

Then $A = \sqrt{5^2 + (5\sqrt{3})^2} = 10$ and $\tan\phi = \frac{5\sqrt{3}}{5} = \sqrt{3}$ so $\phi = \frac{\pi}{3}$

The above equation (i) becomes $y_2 = A\cos\phi\sin(3\pi t) + A\sin\phi\cos(3\pi t) \Rightarrow y_2 = A\sin(3\pi t + \phi) = 10\sin[3\pi t + (\frac{\pi}{3})]$ so,

$\frac{A_1}{A_2} = \frac{10}{10} \Rightarrow A_1 : A_2 = 1 : 1$, Phase difference $= \frac{\pi}{4} - \frac{\pi}{3} = -\frac{\pi}{12}$

Ex. A particle starts from mean position and moves towards positive extreme as shown below. Find the equation of the SHM. Amplitude of SHM is A.

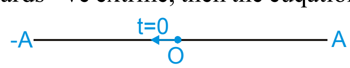


Sol. General equation of SHM can be written as $x = A\sin(\omega t + \phi)$

At $t = 0$, $x = 0$ $\therefore 0 = A\sin\phi$ $\therefore \phi = 0, \pi$ $\phi \in [0, 2\pi]$

Also, at $t = 0$, $v = +ve$ $\therefore A\omega\cos\phi$ = maximum +ve or $\phi = 0$

Hence, if the particle is at mean position at $t = 0$ and is moving towards +ve extrme, then the euqation of SHM is given by $x = A\sin\omega t$



Similarly, for particle moving towards –ve extreme then $\phi = \pi$

$\therefore$ equation of SHM is $x = A\sin(\omega t + \pi)$ or, $x = -A\sin\omega t$

Ex. Write the equation of SHM for the situation show below :

Sol. General equation of SHM can be written as $x = A\sin(\omega t + \phi)$

At $t = 0$, $x = A/2 \Rightarrow \frac{A}{2} = A\sin\phi \Rightarrow \phi = 30°, 150°$

Also at $t = 0$, $v = -ve$; $A\omega\cos\phi = -ve \Rightarrow \phi = 150°$

Ex. The equation of particle executing simple harmonic motion is $x = (5\text{ m})\sin\left[(\pi s^{-1})t + \frac{\pi}{3}\right]$. Write down the amplitude, time period and maximum speed. Also find the velocity at $t = 1$ s.

Sol. : Comparing with equation $x = A\sin(\omega t + \delta)$,
we see that the amplitude = 5 m,

and time period $= \frac{2\pi}{\omega} = \frac{2\pi}{\pi s^{-1}} = 2$s.

The maximum speed $= A\omega = 5\text{ m} \times \pi s^{-1} = 5\pi$ m/s.

The velocity at time $t = \frac{dx}{dt} = A\omega\cos(\omega t + \delta)$

At $t = 1$ s,

$v = (5\text{ m})(\pi s^{-1})\cos\left(\pi + \frac{\pi}{3}\right) = -\frac{5\pi}{2}$ m/s.

Ex. A particle executing simple harmonic motion has angular frequency 6.28 s^{-1} and amplitude 10 cm. Find **(a)** the time period, **(b)** the maximum speed, **(c)** the maximum acceleration, **(d)** the speed when the displacement is 6 cm from the mean position, **(e)** the speed at t = 1/6 s assuming that the motion starts from rest at t = 0.

Sol. : **(a)** Time period $= \frac{2\pi}{\omega} = \frac{2\pi}{6.28}$ s = 1 s.

(b) Maximum speed $= A\omega = (0.1 \text{ m})(6.28 \text{ s}^{-1})$

$= 0.628$ m/s.

(c) Maximum acceleration $= A\omega^2$

$= (0.1 \text{ m})(6.28 \text{ s}^{-1})^2$

$= 4 \text{ m/s}^2$.

(d) $v = \omega\sqrt{A^2 - x^2} = (6.28 \text{ s}^{-1})\sqrt{(10 \text{ cm})^2 - (6 \text{ cm})^2} = 50.2$ cm/s.

(e) At t = 0, the velocity is zero i.e., the particle is at an extreme. The equation for displacement may be written as

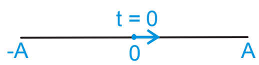
$$x = A\cos\omega t.$$

The velocity is $v = -A\,\omega \sin \omega t$.

At $t = \frac{1}{6}$ s, $v = -(0.1 \text{ m})(6.28 \text{ s}^{-1}) \sin\left(\frac{6.28}{6}\right)$

$= (-0.628 \text{ m/s}) \sin\frac{\pi}{3} = 54.4$ cm/s.

Ex. A particle starts from mean position and moves towards positive extreme as shown. Find the equation of the SHM. Amplitude of SHM is A.



Sol. : General equation of SHM can be written as $x = A\sin(\omega t + \phi)$

At $t = 0, x = 0$

$\therefore$ $0 = A\sin\phi$

$\therefore$ $\phi = 0, \pi$ $\qquad \phi \in [0, 2\pi)$

Also; at t = 0, v = +ve

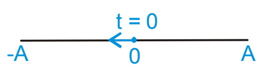
$\therefore$ $A\omega\cos\phi = +ve$

or, $\phi = 0$

Hence, if the particle is at mean position at t = 0 and is moving towards +ve extreme, then the equation of SHM is given by $x = A\sin\omega t$

Similarly

for



$\phi = \pi$

$\therefore$ equation of SHM is $x = A\sin(\omega t + \pi)$

or, $x = -A\sin\omega t$

Note : If mean position is not at the origin, then we can replace x by $x - x_0$ and the eqn. becomes $x - x_0 = A\sin(\omega t + \phi)$, where x_0 is the position co-ordinate of the mean position.

Ex. A particle is performing SHM of amplitude "A" and time period "T". Find the time taken by the particle to go from 0 to A/2.

Sol. Let equation of SHM be $x = A \sin \omega t$

when $x = 0$, $t = 0$

when $x = A/2$; $A/2 = A \sin \omega t$

or $\sin \omega t = 1/2$ $\qquad \omega t = \pi/6$

$$\frac{2\pi}{T} t = \pi/6 \qquad t = T/12$$

Hence , time taken is T/12, where T is time period of SHM.

Ex. A particle of mass 2 kg is moving on a straight line under the action of force $F = (8 - 2x)$ N. It is released at rest from $x = 6$ m.

(a) Is the particle moving simple harmonically.

(b) Find the equilibrium position of the particle.

(c) Write the equation of motion of the particle.

(d) Find the time period of SHM.

Sol. : $F = 8 - 2x$

or $F = -2(x - 4)$

at equilibrium position $F = 0$

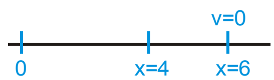
$\Rightarrow$ $x = 4$ is equilibrium position

Hence the motion of particle is SHM with force constant 2 and equilibrium position $x = 4$.

(a) Yes, motion is SHM.

(b) Equilibrium position is $x = 4$

(c) At $x = 6$ m, particle is at rest i.e. it is one of the extreme position



Hence amplitude is $A = 2$ m and initially particle is at the extreme position.

$\therefore$ Equation of SHM can be written as

$$x - 4 = 2 \cos \omega t \quad \text{, where } \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{2}} = 1$$

i.e. $x = 4 + 2 \cos t$

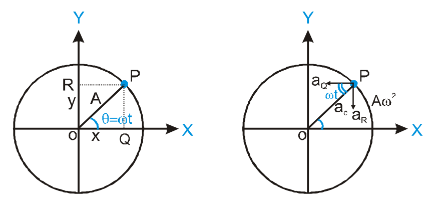
(d) Time period, $T = \dfrac{2\pi}{\omega} = 2\pi$ sec.

GEOMETRICAL MEANING OF S.H.M.

If a particle is moving with uniform speed along the circumference of a circle then the straight line motion of the foot of perpendicular drawn from the particle on the diameter of the circle is called 'S.H.M.'

S.H.M. AS A PROJECTION OF CIRCULAR MOTION

Suppose a particle P is moving uniformly on a circle of radius A with angular speed ω . Q and R be the two feet of the perpendicular drawn from P on two diameters one along X-axis and the other along Y-axis.



(a) (b)

Suppose the particle P is on the X-axis at t=0. Radius OP makes an angle $\theta = \omega t$ with the X-axis at time t then

$$x = A \cos \omega t \text{ , } y = A \sin \omega t$$

Here, x and y are the displacement of Q and R from the origin at time t, which are the displacement equations of SHM. It implies that although P is under uniform circular motion, Q and R are performing SHM about O with the same angular speed ω as that of P. From figure **(b)** centripetal acceleration of P = $a_c = A\omega^2$ (towards the centre) a_c can have two components as shown in the figure $a_R = A\omega^2 \sin\omega t = \omega^2 x$, $a_Q = A\omega^2 \cos\omega t = \omega^2 y$

a_R and a_Q are actually the acceleration corresponding to the points R and Q respectively.

(i) Every periodic motion can be resolved into a number of simple harmonic motions.

(ii) Oscillatory motion can be treated as simple harmonic motion only in the limit of small amplitudes because in this limit the restoring force (or torque) becomes linear.

(iii) Harmonic oscillations is that oscillations which can be expressed in terms of single harmonic function (i.e. sine functions or cosine function)

(iv) The motion of the molecules of a solid, the vibration of the air columns and the vibration of string of music instruments are either simple harmonic or superposition of simple harmonic motions.

Comparison between linear and angular S.H.M.

Linear S.H.M.

$F \propto -x \Rightarrow F = -kx$

where k is the restoring force constant

$$a = -\frac{k}{m}x \Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

It is known as differential equation of linear SHM

$x = A\sin\omega t;\ a = -\omega^2 x$

where ω is the angular frequency

$$\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} = 2\pi n$$

where T is time period and n is frequency

$$T = 2\pi\sqrt{\frac{m}{k}},\ n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

This concept is valid for all types of linear S.H.M.

Angular S.H.M.

$\tau \propto -\theta \Rightarrow \tau = -C\theta$

where C is the restoring torque constant

$$\alpha = -\frac{C}{I}\theta \Rightarrow \frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$$

It is known as differential equation of angular SHM.

$\theta = \theta_0 \sin\omega t;\ \alpha = -\omega^2\theta$

$$\omega^2 = \frac{C}{I} \Rightarrow \omega = \sqrt{\frac{C}{I}} = \frac{2\pi}{T} = 2\pi n$$

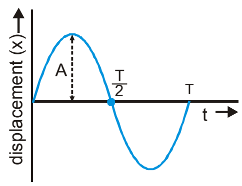
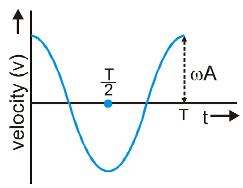
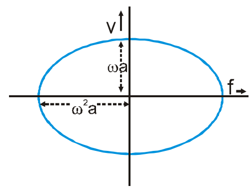
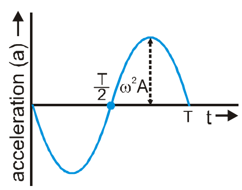
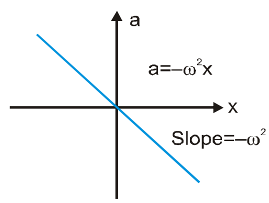
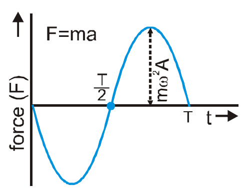
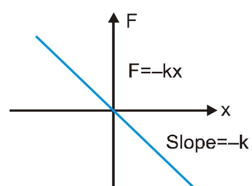
where T is time period and n is frequency

$$T = 2\pi\sqrt{\frac{I}{C}},\ n = \frac{1}{2\pi}\sqrt{\frac{C}{I}}$$

This concept is valid for all types of angular SHM.

Graphical representation

Graphical study of displacement, velocity, acceleration and force in S.H.M.

S. No.	Graph	In form of t	In from of x	Maximum value
1.	Displacement 	$x = A\sin\omega t$	$x = x$	$x = \pm A$
2.	Velocity 	$v = A\omega\cos\omega t$	$v = \pm\omega\sqrt{A^2 - x^2}$ 	$v = \pm\omega A$
3.	Acceleration 	$a = -\omega^2 A\sin\omega t$	$a = -\omega^2 x$  (a) Acceleration v/s displacement	$a = \pm\,\omega^2 A$
4.	Force (F = ma) 	$F = -\,m\omega^2 A\sin\omega t$	$F = -\,m\omega^2 x$  (a) Force v/s displacement	$F = \pm\, m\omega^2 A$

Ex. An object performs S.H.M. of amplitude 5 cm and time period 4 s. If timing is started when the object is at the centre of the oscillation i.e., x = 0 then calculate.

(i) Frequency of oscillation

(ii) The displacement at 0.5 s

(iii) The maximum acceleration of the object

(iv) The velocity at a displacement of 3 cm.

Sol. **(i)** Frequency $f = \dfrac{1}{T} = \dfrac{1}{4} = 0.25\ \text{Hz}$

(ii) The displacement equation of object $x = A\sin\omega t$ at $t = 0.5$ s, $x = 5\sin(2\pi \times 0.25 \times 0.5) = 5\sin\dfrac{\pi}{4} = \dfrac{5}{\sqrt{2}}$ cm

(iii) Maximum acceleration $a_{max} = \omega^2 A = (0.5\,\pi)^2 \times 5 = 12.3$ cm/s^2

(iv) Velocity at x = 3 cm is $v = \pm\omega\sqrt{A^2 - x^2} = \pm 0.5\pi\sqrt{5^2 - 3^2} = \pm 6.28$ cm/s

Ex. Amplitude of a harmonic oscillator is A, when velocity of particle is half of maximum velocity, then determine position of particle.

Sol. $v = \omega\sqrt{A^2 - x^2}$ but $v = \frac{v_{max}}{2} = \frac{A\omega}{2} = \omega\sqrt{A^2 - x^2} \Rightarrow A^2 = 4[A^2 - x^2] \Rightarrow x^2 = \frac{4A^2 - A^2}{4}$

$\Rightarrow x = \pm\frac{\sqrt{3}A}{2}$

Ex. Which of the following functions represent SHM :–

(i) $\sin^2\omega t$ **(ii)** $\sin 2\omega t$ **(iii)** $\sin\omega t + 2\cos\omega t$ **(iv)** $\sin\omega t + \cos 2\omega t$

Sol. A motion will be S.H.M. if acceleration $\propto -y$

(i) $y = \sin^2\omega t \Rightarrow \frac{dy}{dt} = 2(\sin\omega t)(\omega\cos\omega t) = \omega\sin 2\omega t, \frac{d^2y}{dt^2} = 2\omega^2\cos 2\omega t \Rightarrow \frac{d^2y}{dt^2} \propto y = 2\omega^2(1 - 2y)$

(Oscillatory but S.H.M. not possible)

(ii) As $y = \sin 2\omega t \Rightarrow v = \frac{dy}{dt} = 2\,\omega\cos 2\omega t \Rightarrow$ Acceleration $= \frac{d^2y}{dt^2} = -4\omega^2\sin 2\omega t = -4\,\omega^2 y$

so $y = \sin 2\omega t$ represents S.H.M.

(iii) $y = \sin\omega t + 2\cos\omega t \Rightarrow v = \frac{dy}{dt} = \omega\cos\omega t - 2\omega\sin\omega t,$

Acceleration $= \frac{dv}{dt} = -\omega^2\sin\omega t - 2\omega^2\cos\omega t = -\omega^2(\sin\omega t + 2\cos\omega t) = -\omega^2 y$

$\therefore$ The given function represents SHM

(iv) $y = \sin\omega t + \cos 2\omega t \Rightarrow \frac{dy}{dt} = \omega\cos\omega t - 2\omega\sin 2\omega t, \frac{d^2y}{dt^2} = -\omega^2\sin\omega t - 4\omega^2\cos 2\omega t = -\omega^2(\sin\omega t + 4\cos 2\omega t)$

$\frac{d^2y}{dt^2} \not\propto (-y)$ (Oscillatory but S.H.M. not possible)

Ex. Periodic time of a simple pendulum is 2 second and it can travel to and fro from equilibrium position upto maximum 5cm. At start the pendulum is at maximum displacement on right side of equilibrium position. Find displacement and time relation.

Sol. Displacement expression for S.H.M. , $x = A\sin(\omega t + \phi)$

Time period of simple pendulum $T = \frac{2\pi}{\omega} = 2$ s $\therefore \omega = \pi$ rad/s

Amplitude of pendulum A = 5 cm $\therefore x = 5\sin(\pi t + \phi)$

Now at t = 0, displacement x = 5 cm $\therefore 5 = 5\sin(\pi \times 0 + \phi) \Rightarrow \sin\phi = 1 \Rightarrow \phi = \frac{\pi}{2}$

Therefore, $x = 5\sin\pi(t + \frac{1}{2}) \Rightarrow x = 5\sin(\pi t + \frac{\pi}{2}) \Rightarrow x = 5\cos\pi t$

Ex. A particle executing S.H.M. having amplitude 0.01 m and frequency 60 Hz. Determine maximum acceleration of particle.

Sol. Maximum acceleration $a_{max.} = \omega^2 A = 4\pi^2 n^2 A = 4\pi^2 (60)^2 \times (0.01) = 144\,\pi^2\,m/s^2$

Ex. The velocity of a particle in S.H.M. at positions x_1 and x_2 are v_1 and v_2 respectively. Determine value of time period and amplitude.

Sol. $v = \omega\sqrt{A^2 - x^2} \Rightarrow v^2 = \omega^2 (A^2 - x^2)$

At position x_1, $v_1^2 = \omega^2 (A^2 - x_1^2)$ ...(i) At position x_2, $v_2^2 = \omega^2 (A^2 - x_2^2)$ ...(ii)

Subtracting (ii) from (i) $v_1^2 - v_2^2 = \omega^2 (x_2^2 - x_1^2) \Rightarrow \omega = \sqrt{\dfrac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}$

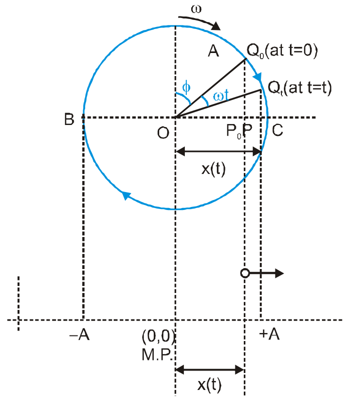
Time period $T = \dfrac{2\pi}{\omega} \Rightarrow T = 2\pi\sqrt{\dfrac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$

Dividing (i) by (ii) $\dfrac{v_1^2}{v_2^2} = \dfrac{A^2 - x_1^2}{A^2 - x_2^2} \Rightarrow v_1^2 A^2 - v_1^2 x_2^2 = v_2^2 A^2 - v_2^2 x_1^2$

So $A^2 (v_1^2 - v_2^2) = v_1^2 x_2^2 - v_2^2 x_1^2 \Rightarrow A = \sqrt{\dfrac{v_1^2 x_2^2 - v_2^2 x_1^2}{v_1^2 - v_2^2}}$

SHM As a projection of uniform circular Motion

Consider a particle Q, moving on a circle of radius A with constant angular velocity ω. The projection of Q on a diameter BC is P. It is clear from the figure that as Q moves around the circle the projection P executes a simple harmonic motion on the x-axis between B and C. The angle that the radius OQ makes with the +ve vertical in clockwise direction in at t = 0 is equal to phase constant (ϕ).



Let the radius OQ_0 makes an angle ωt with the OQ_t at time t. Then $x(t) = A\sin(\omega t + \phi)$

In the above discussion the foot of projection is x-axis so it is called horizontal phasor. Similarly the foot of perpendicular on y-axis will also executes SHM of amplitude A and angular frequency ω[$y(t) = A\cos\omega t$]. This is called vertical phasor. The phasor of the two SHM differ by π/2.

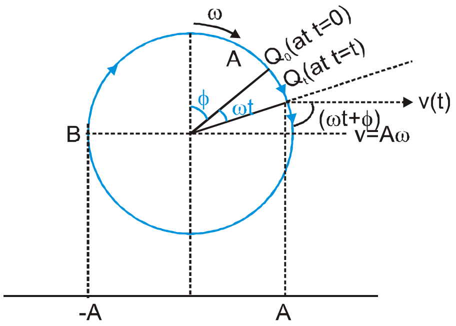
Problem solving strategy in horizontal phasor.

1. First assume circle of radius equal to amplitude of SHM.

2. Assume a particle rotating in a circular path moving with cosntant ω same as that of SHM in clockwise direction.

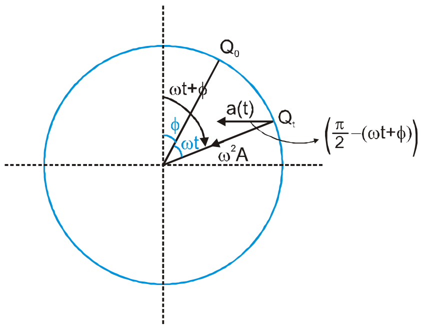
3. Angle made by the particle at t=0 with the upper vertical is equal to phase constant.

4. Horizontal component of velocity of particle gives you the velocity of particle performing SHM for example



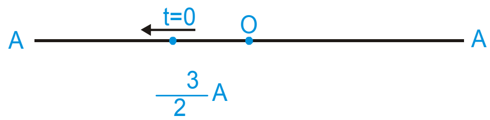
from figure v(t) =Aωcos(ωt+ϕ)

5. Component of acceleration of particle in horizontal direction is equals to the acceleration of particle performing SHM. The acceleration of a particle in uniform circular motion is only centripetal and has a magnitude a $=\omega^2 A$



from figure a(t) $=-\omega^2 A\sin(\omega t+\phi)$

Ex. A particle starts from point $x = \frac{-\sqrt{3}}{2}A$ and move towards negative extreme as shown following

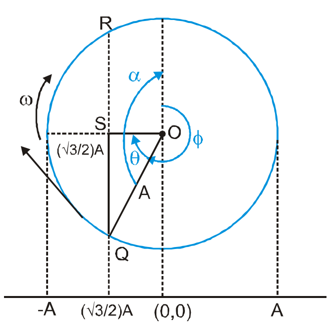


(a) Find the equation of the SHM

(b) Find the time taken by the particle to go directly from its initial position to negative extreme.

(c) Find the time taken by the particle to reach at mean position.

Sol. Figure shows the solution of the problem with the help of phasor. Horizontal component of velocity at Q gives the required direction of velocity at t = 0 so we will choose it.



In ΔOSQ :

Now $\phi = \frac{3\pi}{2} - \frac{\pi}{6} = \frac{8\pi}{6} = \frac{4\pi}{3}$

So equation of SHM is $x = A\sin\left(\omega t + \frac{4\pi}{3}\right)$

(b) Now to reach the particle at left extreme point it will travel angle θ along the circle. So time taken.

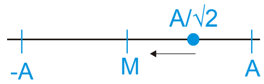
$$t = \frac{\pi}{\omega} = \frac{\pi}{6\omega} \Rightarrow t = \frac{T}{12}\text{sec}$$

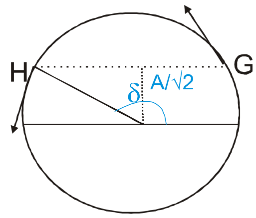
(c) To reach the particle at mean position it will travel angle $\alpha = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$

So, time taken $= \frac{\alpha}{\omega} = \frac{T}{3}\text{sec}$

Ex. Write equation of S.H.M. of ω angular frequency and A amplitude if the particle is situated at $\frac{A}{\sqrt{2}}$ at t = 0 and is going towards mean position.

Sol. At t = 0 particle was at $\frac{A}{\sqrt{2}}$ and was going towards mean position as shown in the figure below.





The same situation is described by making reference point on a circle of radius A as shown in figure .

The reference point can be located at any of the two positions G and H as shown in figure for having displacement $\frac{A}{\sqrt{2}}$, but to move towards mean position it should be H. Position H corresponds to angle $3\pi/4$.

Thus $\phi = \frac{3\pi}{4}$ and $x = A\sin\left(\omega t + \frac{3\pi}{4}\right)$

Alternative Mechanical solution

Putting t = 0 in equation x = A sin (ωt + ϕ)

We get $\frac{A}{\sqrt{2}} = A\sin\phi$

Thus, $\phi = \frac{\pi}{4}, \frac{3\pi}{4}$

but at t = 0 ; V < 0 so $\frac{\pi}{4}$ is rejected

$$x = A\sin\left(\omega t + \frac{3\pi}{4}\right)$$

Note : If mean position is not at the origin, then we can replace $x - x_0$ and the equation becomes $x - x_0 = A\sin(\omega t + \delta)$, where x_0 is the position co-ordinate of the mean position.

ENERGY CONSERVATION IN SHM

Simple harmonic motion is defined by the equation

F = –kx

The work done by the force F during a displacement from x + dx is

dW = F dx = –kx dx

The work done in a displacement from x = 0 to x is

$$W = \int_0^x (-kx)dx = -\frac{1}{2}kx^2$$

Let U**(x)** be the potential energy of the system when the displacement is x. At the change is potential energy corresponding to a force is negative of the work done by the force.

$$U(x) - U(0) = -W = \frac{1}{2}kx^2$$

Let us choose the potential energy to be zero when the particle is at the centre of oscillation x = 0.

Then U(0) = 0 and $U(x) = \frac{1}{2}kx^2$

This expression for potential energy is same as that for a spring has been used so far in this chapter

As $\omega = \sqrt{\frac{k}{m}}, k = m\omega^2$

We can write $U(x) = \frac{1}{2}m\omega^2 x^2$ .... **(i)**

The displacement and the velocity of a particle executing a simple harmonic motion are given by

x = A sin (ωt + δ)

and v = A w cos (ωt + δ)

The potential energy at time t is therefore,

$$U = \frac{1}{2}m\omega^2 x^2$$

$$= \frac{1}{2}m\omega^2 x^2 A^2 \sin^2(\omega t + \delta)$$

and the kinetic energy at time t is

$$= \frac{1}{2}mA^2\omega^2 \cos^2(\omega t + \delta)$$

The total mechanical energy at time t is

E = U + K

$$= \frac{1}{2}m\omega^2 A^2 \left[\sin^2(\omega t + \delta) + \cos^2(\omega t + \delta)\right]$$

$$= \frac{1}{2}m\omega^2 A^2 \qquad \text{........ (ii)}$$

We see that the total mechanical energy at time t is independent of t. Thus, the mechanical energy remains constant.

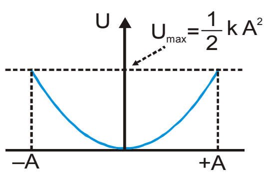
At the mean position x = 0, the potential energy is zero. The kinetic energy is $=\frac{1}{2}mv_o^2=\frac{1}{2}\omega^2 A^2$. All the mechanical energy is in the form of kinetic energy here. At the particle is displaced away from the mean position, the kinetic energy decreases and the potential energy increase. At the extreme position x = ± A, the speed v is zero and the kinetic energy decreases to zero. The potential energy is increased to its maximum value $=\frac{1}{2}kA^2=\frac{1}{2}m\omega^2 A^2$. All the mechanical energy is in the form of potential energy here.

ENERGY OF PARTICLE IN S.H.M.

Potential Energy (U or P.E.)

(i) In terms of displacement

The potential energy is related to force by the relation $F=-\frac{dU}{dx} \Rightarrow \int dU=-\int F dx$

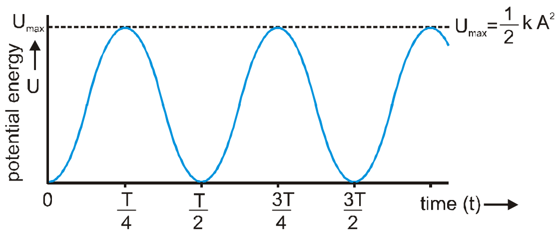


For S.H.M. $F=-kx$ so $\int dU=-\int(-kx)dx=\int kx\,dx \Rightarrow U=\frac{1}{2}kx^2+C$

At $x=0, U=U_0 \Rightarrow C=U_0$ So $U=\frac{1}{2}kx^2+U_0$

Where the potenital energy at equilibrium position $=U_0$ when $U_0=0$ then $U=\frac{1}{2}kx^2$

(ii) In terms of time



Since $x=A\sin(\omega t+\phi)$, $U=\frac{1}{2}kA^2\sin^2(\omega t+\phi)$

If initial phase (ϕ) is zero then $U=\frac{1}{2}kA^2\sin^2\omega t=\frac{1}{2}m\omega^2A^2\sin^2\omega t$

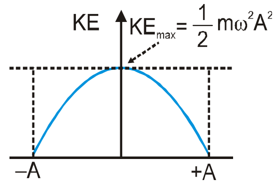
Note :

(i) In S.H.M. the potential energy is a parabolic function of displacement, the potential energy is minimum at the mean position ($x=0$) and maximum at extreme position ($x=\pm A$)

(ii) The potential energy is the periodic function of time.

It is minimum at $t=0, \frac{T}{2}, T, \frac{3T}{2}\ldots$ and maximum at $t=\frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}\ldots$

Kinetic Energy (K)



(i) In terms of displacement

If mass of the particle executing S.H.M. is m and Its velocity is v then kinetic energy at any instant.

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}k(A^2 - x^2)$$

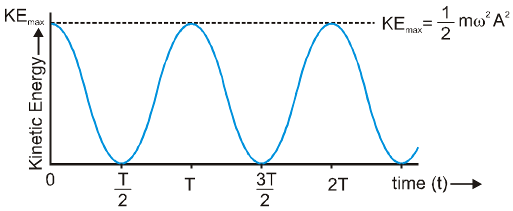
(ii) In terms of time

$$v = A\omega\cos(\omega t + \phi)$$

$$K = \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \phi)$$

If initial phase ϕ is zero

$$K = \frac{1}{2}m\omega^2A^2\cos^2\omega t$$



Note :

(i) In S.H.M. the kinetic energy is a inverted parabolic function of displacement. The kinetic energy is maximum ($\frac{1}{2}kA^2$) at mean position ($x = 0$) and minimum (zero) at extreme position ($x = \pm A$)

(ii) The kinetic energy is the periodic function of time. It is maximum at t = 0, T, 2T, 3T.............and minimum at t = $\frac{T}{2}, \frac{3T}{2}, \frac{5T}{2}$ ...

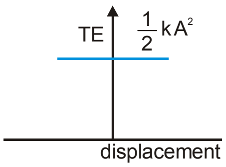
Total energy (E)

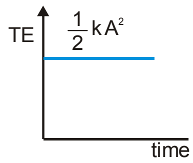
Total energy in S.H.M. is given by ; E = potential energy + kinetic energy = U + K

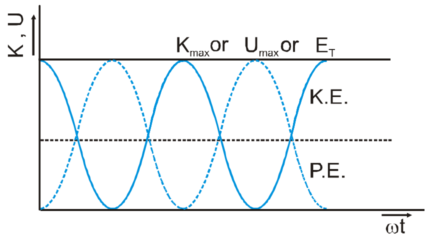
(i) w.r.t. position $E = \frac{1}{2}kx^2 + \frac{1}{2}k(A^2 - x^2) \Rightarrow E = \frac{1}{2}kA^2 = \text{constant}$

(ii) w.r.t. time

$$E = \frac{1}{2}m\omega^2A^2\sin^2\omega t + \frac{1}{2}m\omega^2A^2\cos^2\omega t = \frac{1}{2}m\omega^2A^2\left(\sin^2\omega t + \cos^2\omega t\right) = \frac{1}{2}m\omega^2A^2 = \frac{1}{2}kA^2 = \text{constant}$$







Note :

(i) Total energy of a particle in S.H.M. is same at all instant and at all displacement.

(ii) Total energy depends upon mass, amplitude and frequency of vibration of the particle executing S.H.M.

Average Energy in S.H.M.

(i) The time average of P.E. and K.E. over one cycle is

$$\text{(a)} <KE>_t = <\frac{1}{2}m\omega^2A^2\cos^2\omega t> = \frac{1}{2}m\omega^2A^2 <\cos^2\omega t> = \frac{1}{2}m\omega^2A^2\left(\frac{1}{2}\right) = \frac{1}{4}m\omega^2A^2 = \frac{1}{4}kA^2$$

$$\text{(b)} <PE>_t = <\frac{1}{2}m\omega^2A^2\cos^2\omega t> = \frac{1}{2}m\omega^2A^2 <\sin^2\omega t> = \frac{1}{2}m\omega^2A^2\left(\frac{1}{2}\right) = \frac{1}{4}m\omega^2A^2 = \frac{1}{4}kA^2$$

$$\text{(c)} <TE>_t = <\frac{1}{2}m\omega^2A^2 + U_0> = \frac{1}{2}m\omega^2A^2 + U_0 = \frac{1}{2}kA^2 + U_0$$

(ii) The position average of P.E. and K.E. between x = -A to x=A

$$\text{(a)} <KE>_x = \frac{\int_{-A}^{A}\frac{1}{2}m\omega^2\left(A^2-x^2\right)dx}{\int_{-A}^{A}dx} = \frac{\frac{1}{2}m\omega^2\left(A^2x-\frac{x^3}{3}\right)_{-A}^{A}}{2A} = \frac{1}{3}kA^2$$

$$\text{(b)} <PE>_x = \frac{\int_{-A}^{A}(PE)dx}{\int_{-A}^{A}dx} = \frac{\int_{-A}^{A}\left(U_0+\frac{1}{2}kA^2\right)}{\int_{-A}^{A}dx} = \frac{U_0(2A)+\frac{1}{2}k\left(\frac{x^3}{3}\right)_{-A}^{A}}{2A} = U_0 = \frac{1}{6}kA^2$$

$$\text{(c)} <TE>_x = \frac{\int_{-A}^{A}(TE)dx}{\int_{-A}^{A}dx} = \frac{\int_{-A}^{A}\left(\frac{1}{2}kA^2+U_0\right)}{\int_{-A}^{A}dx} = \frac{1}{2}kA^2 + U_0$$

(i) Both kinetic energy and potential energy varies periodically but the variation is not simple harmonic.

(ii) The frequency of oscillation of potential energy and kinetic energy is twice as that of displacement or velocity or acceleration of a particle executing S.H.M.

(iii) Frequency of total energy is zero because it remains constant.

Ex. In case of simple harmonic motion –

(a) What fraction of total energy is kinetic and what fraction is potential when displacement is one half of the amplitude.

(b) At what displacement the kinetic and potential energies are equal.

Sol. In S.H.M. : Kinetic Energy $K = \frac{1}{2}k(A^2-x^2)$, Potential Energy $U = \frac{1}{2}kx^2$, Total Energy (TE) $= \frac{1}{2}KA^2$

(a) Fraction of Kinetic Energy $f_{K.E.} = \frac{K}{T.E.} = \frac{A^2-x^2}{A^2}$, Fraction of Potential Energy $f_{P.E.} = \frac{U}{T.E} = \frac{x^2}{A^2}$

at $x = \dfrac{A}{2}$ $\quad f_K = \dfrac{A^2 - A^2/4}{A^2} = \dfrac{3}{4}$ and $f_U = \dfrac{A^2/4}{A^2} = \dfrac{1}{4}$

(b) $K = U \Rightarrow \dfrac{1}{2}k(A^2 - x^2) = \dfrac{1}{2}kx^2 \Rightarrow 2x^2 = A^2 \Rightarrow x = \pm\dfrac{A}{\sqrt{2}}$

Ex. The potential energy of a particle oscillating on x-axis is $U = 20 + (x-2)^2$. Here U is in joules and x in meters. Total mechanical energy of the particle is 36 J.

(a) State whether the motion of the particle is simple harmonic or not.

(b) Find the mean position.

(c) Find the maximum kinetic energy of the particle.

Sol. **(a)** $F = -\dfrac{dU}{dx} = -2(x-2)$ $\quad$ By assuming $x - 2 = X$, we have $F = -2X$

Since, $F \propto -X$ $\quad$ The motion of the particle is simple harmonic

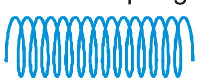
(b) The mean position of the particle is $X = 0 \Rightarrow x - 2 = 0$, which gives $x = 2m$

(c) Maximum kinetic energy of the particle is, $K_{max} = E - U_{min} = 36 - 20 = 16$ J

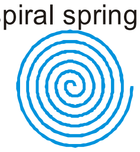
Note : U_{min} is 20 J at mean position or at x = 2m.

SPRING SYSTEM

halical spring



spiral spring

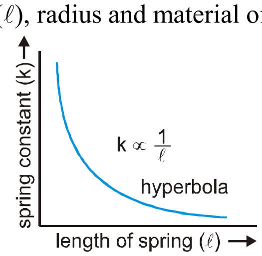


(i) When spring is given small displacement by stretching or compressing it, then restoring elastic force is developed in it because it obeys Hook's law.

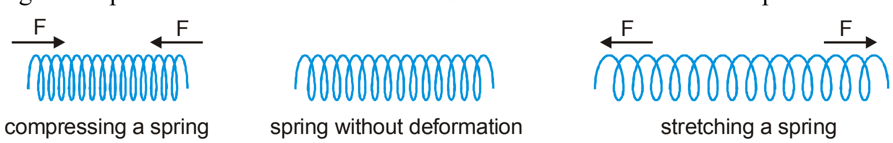
$$F \propto -x \quad \Rightarrow F = -kx \qquad \text{Here k is spring constant}$$

(ii) Spring is assumed massless, so restoring elastic force in spring is assumed same everywhere.

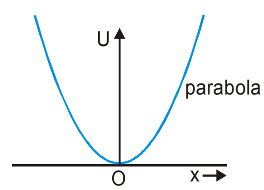
(iii) Spring constant (k) depends on length (ℓ), radius and material of wire used in spring. for spring $k\ell$ = constant



(iv) When spring is compressed or stretched then work done on it is stored as elastic potential energy.



$$W = \int F dx = \int kx\, dx \qquad \text{and} \qquad U = W = \frac{1}{2}kx^2$$



When spring is stretched from ℓ_1 to ℓ_2 then Work done $W = \dfrac{1}{2}k(\ell_2^2 - \ell_1^2)$

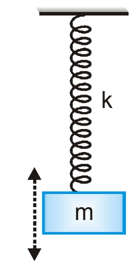
Spring Pendulum

(i) When a small mass is suspended from a mass-less spring then this arrangement is known as spring pendulum. For small linear displacement the motion of spring pendulum is simple harmonic.

(ii) For a spring pendulum

$$F = -kx \Rightarrow m\frac{d^2x}{dt^2} = -kx \quad [\because F = ma = m\frac{d^2x}{dt^2}]$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x \quad \because \frac{d^2x}{dt^2} = -\omega^2 x \Rightarrow \omega^2 = \frac{k}{m}$$

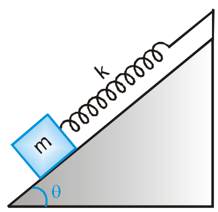


This is standard equation of linear S.H.M.

Time period $\quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$, $\quad$ Frequency $\quad n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

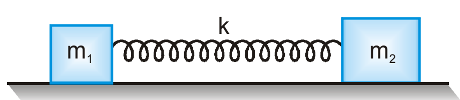
(iii) Time period of a spring pendulum is independent of acceleration due to gravity. This is why a clock based on oscillation of spring pendulum will keep proper time everywhere on a hill or moon or in a satellite or different places of earth.

(iv) If a spring pendulum oscillates in a vertical plane is made to oscillate on a horizontal surface or on an inclined plane then time period will remain unchanged.



(v) By increasing the mass, time period of spring pendulum increases $(T \propto \sqrt{m})$, but by increasing the force constant of spring (k), Its time period decreases $\left[T \propto \frac{1}{\sqrt{k}}\right]$ whereas frequency increases $(n \propto \sqrt{k})$

(vi) If two masses m_1 and m_2 are connected by a spring and made to oscillate then time period $T = 2\pi\sqrt{\frac{\mu}{k}}$

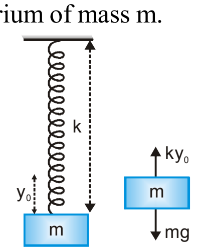


Here, $\quad \mu = \frac{m_1 m_2}{m_1 + m_2}$ = reduced mass

(vii) If the stretch in a vertically loaded spring is y_0 then for equilibrium of mass m.

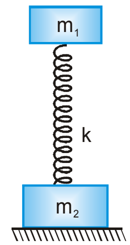
$$ky_0 = mg \text{ i.e., } \frac{m}{k} = \frac{y_0}{g}$$

So, $\quad$ time period $\quad T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{y_0}{g}}$



But remember time period of spring pendulum is independent of acceleration due to gravity.

(viii) If two particles are attached with spring in which only one is oscillating

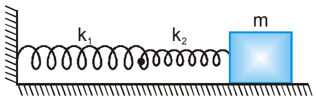


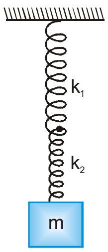
$$\text{Time period} = 2\pi\sqrt{\frac{\text{mass of oscillating particle}}{\text{force constant}}} = 2\pi\sqrt{\frac{m_1}{k}}$$

Various Spring arrangements

(i) Series combination of springs

In series combination same restoring force exerts in all springs but extension will be different.





Total displacement $x = x_1 + x_2$

Force acting on both springs $F = -k_1x_1 = -k_2x_2$

$\because\ x_1 = -\dfrac{F}{k_1}$ and $x_2 = -\dfrac{F}{k_2}$ $\quad \therefore\ x = -\left[\dfrac{F}{k_1} + \dfrac{F}{k_2}\right]$ ....**(i)**

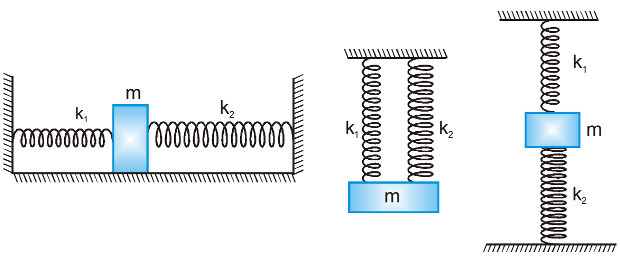
If equivalent force constant is k_s then $F = -k_s x$

So by equation **(i)** $\quad -\dfrac{F}{k_s} = -\dfrac{F}{k_1} - \dfrac{F}{k_2} \Rightarrow \dfrac{1}{k_s} = \dfrac{1}{k_1} + \dfrac{1}{k_2} \Rightarrow k_s = \dfrac{k_1 k_2}{k_1 + k_2}$

Time period $\quad T = 2\pi\sqrt{\dfrac{m}{k_s}} = 2\pi\sqrt{\dfrac{m(k_1+k_2)}{k_1 k_2}}$

Frequency $n = \dfrac{1}{2\pi}\sqrt{\dfrac{k_s}{m}}$, Angular frequency $\omega = \sqrt{\dfrac{k_s}{m}}$

(ii) Parallel Combination of springs



In parallel combination displacement on each spring is same but restoring force is different.

Force acting on the system $F = F_1 + F_2 \Rightarrow F = -k_1x - k_2x$ ...**(i)**

If equivalent force constant is k_p then, $F = -k_p x$, so by equation **(i)** $-k_p x = -k_1 x - k_2 x \Rightarrow k_p = k_1 + k_2$

Time period $\quad T = 2\pi\sqrt{\dfrac{m}{k_P}} = 2\pi\sqrt{\dfrac{m}{k_1+k_2}}$; Frequency $n = \dfrac{1}{2\pi}\sqrt{\dfrac{k_P}{m}}$; Angular frequency $\omega = \sqrt{\dfrac{k_1+k_2}{m}}$

Ex. A body of mass m attached to a spring which is oscillating with time period 4 seconds. If the mass of the body is increased by 4 kg, its timer period increases by 2 sec. Determine value of initial mass m.

Sol. In I^{st} *case* : $T = 2\pi\sqrt{\frac{m}{k}} \Rightarrow 4 = 2\pi\sqrt{\frac{m}{k}}$ ...**(i)** and in II^{nd} *case:* $6 = 2\pi\sqrt{\frac{m+4}{k}}$ ...**(ii)**

Divide **(i)** by **(ii)** $\frac{4}{6} = \sqrt{\frac{m}{m+4}} \Rightarrow \frac{16}{36} = \frac{m}{m+4} \Rightarrow m = 3.2$ kg

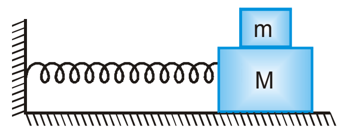
Ex. One body is suspended from a spring of length ℓ, spring constant k and has time period T. Now if spring is divided in two equal parts which are joined in parallel and the same body is suspended from this arrangement then determine new time period.

Sol. Spring constant in parallel combination $k' = 2k + 2k = 4k$

$$\therefore T' = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{4k}} = 2\pi\sqrt{\frac{m}{k}} \times \frac{1}{\sqrt{4}} = \frac{T}{\sqrt{4}} = \frac{T}{2}$$

Ex. A block is on a horizontal slab which is moving horizontally and executing S.H.M. The coefficient of static friction between block and slab is μ. If block is not separated from slab then determine angular frequency of oscillation.

Sol. If block is not separated from slab then restoring force due to S.H.M. should be less than frictional force between slab and block.

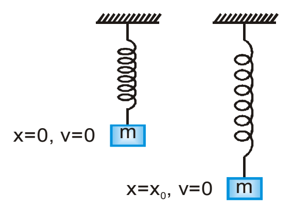


$$F_{restoring} \le F_{friction} \Rightarrow m\,a_{max.} \le \mu mg \Rightarrow a_{max.} \le \mu g \Rightarrow \omega^2 A \le \mu g \Rightarrow \omega \le \sqrt{\frac{\mu g}{A}}$$

Ex. A block of mass m is suspended from a spring of spring constant k. Find the amplitude of S.H.M.

Sol. Let amplitude of S.H.M. be x_0 then by work energy theorem $W = \Delta KE$

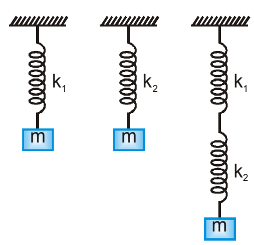
$$mgx_0 - \frac{1}{2}kx_0^2 = 0 \Rightarrow x_0 = \frac{2mg}{k}$$



Ex. Periodic time of oscillation T_1 is obtained when a mass is suspended from a spring if another spring is used with same mass then periodic time of oscillation is T_2. Now if this mass is suspended from series combination of above springs then calculate the time period.

Sol. $T_1 = 2\pi\sqrt{\frac{m}{k_1}} \Rightarrow T_1^2 = 4\pi^2\frac{m}{k_1} \Rightarrow k_1 = \frac{4\pi^2 m}{T_1^2}$

and $T_2 = 2\pi\sqrt{\frac{m}{k_2}} \Rightarrow T_2^2 = 4\pi^2\frac{m}{k_2} \Rightarrow k_2 = \frac{4\pi^2 m}{T_2^2}$



Now $T = 2\pi\sqrt{\frac{m}{k'}}$ where $\frac{1}{k'} = \frac{1}{k_1} + \frac{1}{k_2} \Rightarrow k' = \frac{k_1 k_2}{k_1 + k_2} = \frac{\left(\frac{4\pi^2 m}{T_1^2}\right)\left(\frac{4\pi^2 m}{T_2^2}\right)}{\frac{4\pi^2 m}{T_1^2} + \frac{4\pi^2 m}{T_2^2}}$; $k' = \frac{4\pi^2 m\left[\frac{4\pi^2 m}{T_1^2 T_2^2}\right]}{4\pi^2 m\left[\frac{1}{T_1^2} + \frac{1}{T_2^2}\right]} = \frac{4\pi^2 m}{T_1^2 + T_2^2}$

$$\therefore \quad T = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{\frac{4\pi^2 m}{T_1^2 + T_2^2}}} = \sqrt{T_1^2 + T_2^2}$$

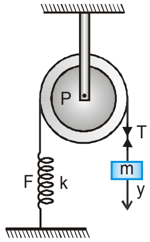
Ex. Infinite spring with force constants k, 2k, 4k, 8k, ... respectively are connected in series. Calculate the effective force constant of the spring.

Sol. $\frac{1}{k_{eff}} = \frac{1}{k} + \frac{1}{2k} + \frac{1}{4k} + \frac{1}{8k} + \dots\dots\dots\dots\infty$ (For infinite G.P. $S_\infty = \frac{a}{1-r}$ where a = First term, r = common ratio)

$$\frac{1}{k_{eff}} = \frac{1}{k}\left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\dots\dots\right] = \frac{1}{k}\left[\frac{1}{1-\frac{1}{2}}\right] = \frac{2}{k}$$ So $k_{eff} = k/2$

Ex. Figure shows a system consisting of a massless pulley, a spring of force constant k = 4000 N/m and a block of mass m = 1 kg. If the block is slightly displaced vertically down from its equilibrium position and released find the frequency of its vertical oscillation in given cases.

Sol. **Case (A) :**

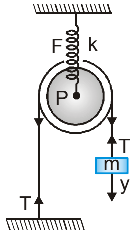


As the pully is fixed and string is inextensible, if mass m is displaced by y the spring will stretch by y, and as there is no mass between string and spring (as pully is massless)

$F = T = ky$ i.e., restoring force is linear and so motion of mass m will be linear simple harmonic with frequency

$$n_A = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4000}{1}} \approx 10 \text{ Hz}$$

Case (B) :

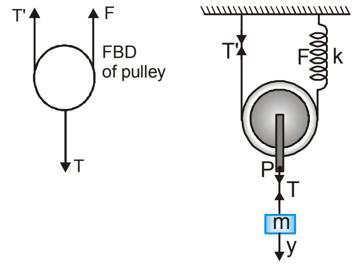


The pulley is movable and string inextensible, so if mass m moves down a distance y, the pulley will move down by (y/2). So the force in the spring F = k(y/2).

Now as pully is massless F = 2T, $\Rightarrow$ T = F/2 = (k/4)y. So the restoring force on the mass m $T = \frac{1}{4}ky = k'y \Rightarrow k' = \frac{1}{4}k$

So $$n_B = \frac{1}{2\pi}\sqrt{\frac{k'}{m}} = \frac{1}{2\pi}\sqrt{\frac{k}{4\pi}} = \frac{n_A}{2} = 5 \text{ Hz}$$

Case (C) :

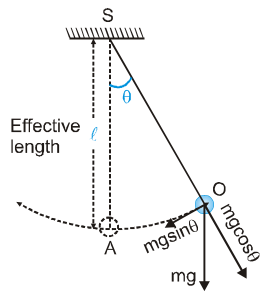


In this situation if the mass m moves by y the pully will also move by y and so the spring will stretch by 2y (as string is inextensible) and so T' = F = 2ky. Now as pulley is massless so T = F + T'= 4ky, i.e., the restoring force on the mass m

$$T = 4ky = k'y \Rightarrow k'=4k$$

So $$n_C = \frac{1}{2\pi}\sqrt{\frac{k'}{m}} = \frac{1}{2\pi}\sqrt{\frac{4k}{m}} = 2n_A = 20\text{Hz}$$

SIMPLE PENDULUM



If a heavy point mass is suspended by a weightless, inextensible and perfectly flexible string from a rigid support, then this arrangement is called a simple pendulum

Expression for time period

Restoring force acting on pendulum F = – mg sinθ

For small angle $\sin\theta \approx \frac{OA}{SA} = \frac{y}{\ell}$ $\quad \therefore ma = -mg \times \frac{y}{\ell} \Rightarrow a = -\frac{g}{\ell}y$

It proves that if displacement is small then simple pendulum performs S.H.M.

$$\because \quad |a| = \omega^2 y \Rightarrow \omega^2 = \frac{g}{\ell} \quad \Rightarrow \omega = \sqrt{\frac{g}{\ell}} \quad \therefore T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}} = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

(i) $T = 2\pi\sqrt{\frac{\ell}{g}}$ is valid when length of simple pendulum (ℓ) is negligible as compare to radius of earth ($\ell << R_e$) but if ℓ is comparable to radius of earth then time period $T = 2\pi\sqrt{\frac{R_e}{\left[1+\frac{R_e}{\ell}\right]g}}$

(ii) The time period of oscillation of simple pendulum of infinite length ($\ell \to \infty$)

$$T = 2\pi\sqrt{\frac{R}{g}} \simeq 84.6 \text{ minute} \approx 1\frac{1}{2} \text{ hour}$$ (It is maximum time period)

(iii) If angular amplitude (θ_0) is large ($\theta_0 > 15°$) then time period is given by $T = 2\pi\sqrt{\frac{\ell}{g}}\left[1+\frac{\theta_0^2}{16}\right]$ here θ_0 is in radian.

(iv) If a simple pendulum of density ρ is made to oscillate in a liquid of density σ then its time period will increase as compare to that of air and is given by $T = 2\pi\sqrt{\frac{\ell}{\left[1-\frac{\sigma}{\rho}\right]g}}$

Second's Pendulum

If the time period of a simple pendulum is 2 second then it is called second's pendulum. Second's pendulum take one second to go from one extreme position to other extreme position.

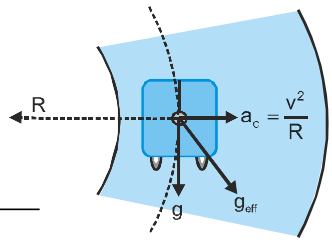
For second's pendulum, time period $T = 2 = 2\pi\sqrt{\frac{\ell}{g}}$. At the surface of earth g = 9.8 m/s^2 $\approx$ π^2 m/s^2.

So length of second pendulum at the surface of earth $\ell \approx 1$ meter

Ex. A simple pendulum of length L and mass M is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v. If the pendulum makes oscillation in a radial direction about its equilibrium position, then calculate its time period .

Sol.

Centripetal acceleration $a_c = \frac{v^2}{R}$ & Acceleration due to gravity = g



So $\quad g_{eff} = \sqrt{g^2 + \left(\frac{v^2}{R}\right)^2} \Rightarrow$ Time period $T = 2\pi\sqrt{\frac{L}{g_{eff}}} = 2\pi\sqrt{\frac{L}{\sqrt{g^2 + \frac{v^4}{R^2}}}}$

Ex. A simple pendulum is suspended from the ceiling of a lift. When the lift is at rest, its time period is T. With what acceleration should lift be accelerated upwards in order to reduce its time period to $\frac{T}{2}$.

Sol. In stationary lift $T = 2\pi\sqrt{\frac{\ell}{g}}$ ...(i) In accelerated lift $\frac{T}{2} = T' = 2\pi\sqrt{\frac{\ell}{g+a}}$ ...(ii)

Divide (i) by (ii) $2 = \sqrt{\frac{g+a}{g}} \Rightarrow g + a = 4g \Rightarrow a = 3\,g$

METHODS TO DETERMINE TIME PERIOD IN SHM

(a) Force / torque method

(b) Energy method

(a) Force / torque method.

In linear SHM the acceleration a and displacement x of the system are related by an equation of the from

$$a = -\text{(a positive constant) } x$$

which says that the acceleration is proportional to the displacement from the equilibrium position but is in the opposite direction. Once you find such an expression for an odcillating system. You can immediately compare it with equation $a = -\omega^2 x$, identify the positive constant as being equal to ω^2, and so quickly get an expression for the angular frequency of the motion.

(b) Energy Method

Let gravitational potential energy to be zero at the level of the block when spring is in its natural length. Now at a distance below that level, let speed of the block be v.

$\therefore$ $$-mgh+\frac{1}{2}kx^2+\frac{1}{2}mv^2 = \text{constant}$$

Differentiating w.r.t. time, we get

$-mgv + kxv + mva = 0$

where a is acceleration

$\therefore$ $F = ma = -kx + mg$ or $F = -k\left(x-\dfrac{mg}{k}\right)$

This shows that for the motion, force constant is k and equilibrium position is $x=\dfrac{mg}{k}$

So, the particle will perform S.H.M. and its time period would be $T=2\pi\sqrt{\dfrac{m}{k}}$

(i) If angular amplitude of simple harmonic pendulum is more, then time period

$$T=2\pi\sqrt{\frac{\ell}{g}}\left(1+\frac{\theta_0^2}{16}\right) \text{ (for other exams)}$$

Where θ_0 is in radians.

(ii) Time period of seconds pendulum is 2 sec and ℓ =0.993 m.

(iii) Simple pendulum performs angular S.H.M. but due to small angular displacement, it is considered as linear SHM.

(iv) If time period of clock based on simple pendulum increases then clock will be slow but if time period decrease then clock will be fast.

(v) If g remains constant and $\Delta\ell$ is change in length, then $\sqrt{\dfrac{\Delta T}{T}}\times 100=\dfrac{1}{2}\dfrac{\Delta\ell}{\ell}\times 100$

(vi) If ℓ remains constant and Δg is change in acceleration then, $\sqrt{\dfrac{\Delta T}{T}}\times 100=\dfrac{1}{2}\dfrac{\Delta g}{g}\times 100$

(vii) If $\Delta\ell$ is change in length and Δg is change in acceleration due to gravity then,

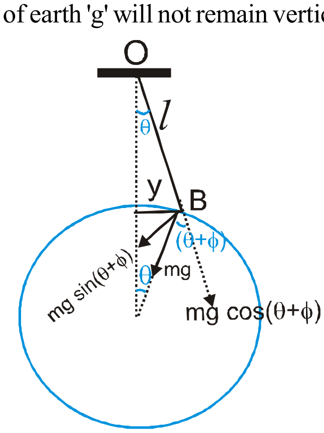
$$\frac{\Delta T}{T}\times 100=\left[\frac{1}{2}\frac{\Delta\ell}{\ell}-\frac{1}{2}\frac{\Delta g}{g}\right]\times 100$$

Time period of simple pendulum of large length

If length of the pendulum L is comparable to the radius of earth 'g' will not remain vertical but will be directed towards the centre of the earth as shown in figure.

In this case

$\tau = |mg \times OB| = mgL\sin(\theta+\phi)$



Now, $\theta = y/R$

on $\phi \approx y/R$

$\Rightarrow$ $$\tau=-mgL\theta\left(1+\frac{\phi}{\theta}\right)$$

[--ve sign indicates restoring torque]

$$= -mgL\theta\left(1+\frac{y/R}{y/L}\right) = -mgL\theta[1+(L/R)]$$

$\tau = -mgL^2\theta\,(1/L+1/R)$

$\Rightarrow \quad = -mL^2\alpha = mgL^2\theta\left(\frac{1}{L}+\frac{1}{R}\right) \quad \Rightarrow \quad \alpha = g\theta\left(\frac{1}{L}+\frac{1}{R}\right) \quad \Rightarrow \quad \alpha \propto -\theta$

i.e. here alsio oscilliations are simple harmonic in nature.

$$\left|\frac{\theta}{\alpha}\right| = \frac{1}{g\left(\frac{1}{L}+\frac{1}{R}\right)} \qquad T = 2\pi\sqrt{\frac{1}{g\left(\frac{1}{L}+\frac{1}{R}\right)}}$$

(i) General formula for time period of simple pendulum.

$$T = 2\pi\sqrt{\frac{1}{g\left(\frac{1}{R}+\frac{1}{\ell}\right)}}$$

(ii) On increasing length of simple pendulum, time period increases, but time period of simple pendulum of infinite length is 84.6 min which is maximum and is equal to $T = 2\pi\sqrt{\frac{R}{g}}$.

(Where R is radius of earth)

Time Period of Simple Pendulum in accelerating Reference Frame

$$T = 2\pi\sqrt{\frac{\ell}{g_{eff.}}}$$

$g_{eff.}$ = Effective acceleration due to gravity in reference system = $|\vec{g}-\vec{a}|$

$\vec{a}$ = acceleration of the point of suspension w.r.t. ground.

Condition for applying this formula : $|\vec{g}-\vec{a}|$ = constant.

Ex. A simple pendulum is suspended from the celling of a car accelerating uniformly on a horozontal road. If the acceleration is a_0 and the length of the pendulum is ℓ, find the time period of sma;; oscillations about the mean position.

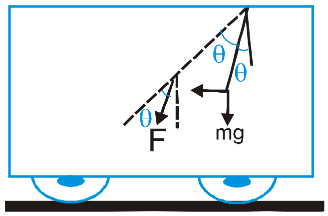
Sol. We shall work in the car frame. As it is accelerated with respect to the road, we shall have to apply a pseudo force ma_0 on the bob of mass m.

For mean position, the acceleration of the bob with respect to the car should be zero. If θ be the angle made by the string with the vertical, the tension, weight and the pseudo force will add to zero in this position.

Hence, resolution of mg and ma_θ (say $F = m\sqrt{g^2+a_0^2}$) has to be along the string.

$$\therefore \quad \tan\theta_0 = \frac{ma_\theta}{mg} = \frac{a_0}{g}$$

Now, suppose the string is further deflected by an an angle θ as shown in figure.



Now, restoring torque can be given by

$$(F\sin\theta)\ell = -m\ell\alpha$$

Substituting F and using sin θ = θ, for small θ

$$\left(m\sqrt{g^2+a_0^2}\right)\ell\theta = -m\ell^2\alpha$$

or, $\alpha = \dfrac{\sqrt{g^2+a_0^2}}{\ell}\theta$ so, $\omega^2 = \dfrac{\sqrt{g^2+a_0^2}}{\ell}$

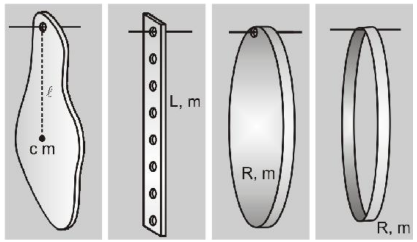
This is an equation of simple harmonic motion with time period

$$T = \frac{2\pi}{\omega} = 2\pi\frac{\sqrt{\ell}}{\left(g^2+a_0^2\right)^{1/4}}$$

COMPOUND PENDULUM

Any rigid body which is free to oscillate in a vertical plane about a horizontal axis passing through a point, is define compound pendulum

Expression for time period



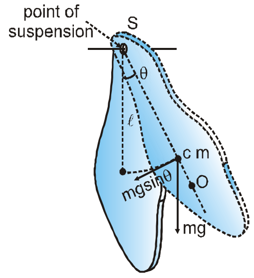
Torque acting on a body $\tau = -mg\ell\sin\theta$ if angle is very small $\sin\theta \approx \theta$ then $\tau = -mg\ell\theta$ ...**(i)** and $\tau = I_s\alpha$ ...**(ii)**

Here m = mass of the body

ℓ = distance between point of suspension and centre of mass

I_s = moment of inertia about horizontal axis passes through point of suspension

From equation **(i)** and **(ii)** $I_s\alpha = -mg\ell\theta$

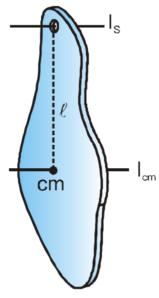


$$I_s\frac{d^2\theta}{dt^2} + mg\ell\theta = 0, \quad \frac{d^2\theta}{dt^2} + \frac{mg\ell}{I_s}\theta = 0 \qquad \text{...(iii)}$$

$$\because \frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \qquad \text{...(iv)}$$

Compare equation **(iii)** and **(iv)** $\omega^2 = \dfrac{mg\ell}{I_s} \Rightarrow \omega = \sqrt{\dfrac{mg\ell}{I_s}}$

Time period of compound pendulum $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{I_s}{mg\ell}}$



Applying parallel axis theorem $I_s = I_{CM} + m\ell^2 \Rightarrow I_s = mK^2 + m\ell^2$

$$\because T = 2\pi\sqrt{\frac{I_s}{mg\ell}} = 2\pi\sqrt{\frac{mK^2+m\ell^2}{mg\ell}} \Rightarrow T = 2\pi\sqrt{\frac{\frac{K^2}{\ell}+\ell}{g}}$$

Here S = point of suspension ; O = point of oscillation ; K = radius of gyration

$L = \frac{k^2}{\ell} + \ell$ = equivalent length of simple pendulum

ℓ = distance between point of suspension and point of oscillation

Time Period $T = 2\pi\sqrt{\frac{\frac{K^2}{\ell}+\ell}{g}}$

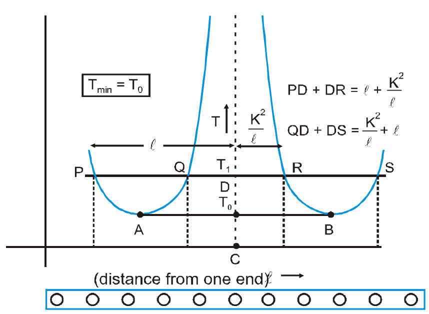
For maximum time period $\ell = 0$ Maximum time period $T_{max} = \infty$

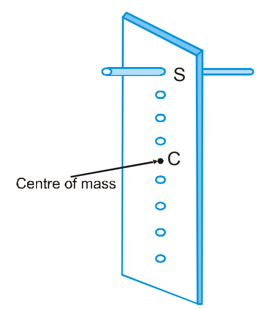
For minimum time period $\frac{dT}{d\ell} = 0$ then $K = \ell \Rightarrow T_{min} = T = 2\pi\sqrt{\frac{\frac{K^2}{K}+K}{g}} = 2\pi\sqrt{\frac{2K}{g}}$

Note : $\frac{k^2}{\ell}$ = Distance between centre of mass and point of oscillation. It has uniform cross section area and width at any point.

Bar Pendulum

A bar pendulum is a steel bar of 1 meter length with holes at regular intervals for suspension. The time period is measured for different values of ℓ (distance between S and C). The graph between T and length from one end ℓ is as shown in fig below. The time period is infinite when $\ell = 0$, i.e., when it is suspended from the centre of gravity (centre of mass).





At four points P, Q, R and S, the time period is the same T_1. The distance are such that

$$\frac{PR + QS}{2} = \ell_{eq} = \ell + K^2/\ell \qquad \text{Also } PD = \ell \qquad DR = K^2/\ell$$

The time period is minimum when $\ell = K$

In Fig. AB= 2K. The minimum period is $T_0 = 2\pi\sqrt{\frac{2K}{g}}$

Condition for T minimum

$$T^2 = \frac{4\pi^2}{g}\left(\frac{K^2}{\ell} + \ell\right)$$

diff. w.r. to ℓ

$$2T\frac{dT}{d\ell} = \left(\frac{4\pi^2}{g}\right)\left[-\frac{K^2}{\ell^2} + 1\right]$$

$\because T \neq 0$, and with $\frac{dT}{d\ell} = 0$

$-\frac{K^2}{\ell^2} + 1 = 0$ or $K^2 = \ell^2$

$K = \pm\ell$ then $T_{min} = T_0$

(i) There are maximum four points for which time period of compound pendulum is same.

(ii) Minimum time period is obtained at two points

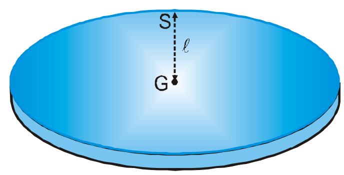
(iii) The point of suspension and point of oscillation are mutually interchangeable.

(iv) Maximum time period will obtain at centre of gravity, which is infinite means compound pendulum will not oscillate at this point.

(v) Compound pendulum executes angular S.H.M. about its mean position. Here restoring torque is provided by gravitational force.

Ex. A disc is made to oscillate about a horizontal axis passing through mid point of its radius. Determine time period.

Sol.



For disc $I = MK^2 = \dfrac{MR^2}{2} \Rightarrow K = \dfrac{R}{\sqrt{2}},\ \ell = \dfrac{R}{2}$

$$L = \ell + \frac{K^2}{\ell} = \frac{R}{2} + \frac{R^2}{2\left(\frac{R}{2}\right)} = \frac{R}{2} + R = \frac{3R}{2} \Rightarrow$$

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{3R}{2g}}$$

Ex. A rod with rectangular cross section oscillates about a horizontal axis passing through one of its ends and it behaves like a second's pendulum. Determine its length.

Sol.

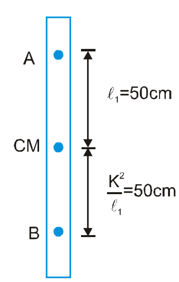
Because oscillating rod behaves as a second's pendulum so its time period will be 2 second.

$$T = 2\pi\sqrt{\frac{\ell + \frac{K^2}{\ell}}{g}} = 2s \Rightarrow \ell + \frac{K^2}{\ell} = 1 \qquad \text{...(i)} \qquad [\because \pi^2 = g]$$

Assume length of rod is L, because axis passes through one end So $\ell = \dfrac{L}{2}$ and $K^2 = \dfrac{L^2}{12}$

Putting this values in equation we get $\dfrac{L}{2} + \dfrac{L^2}{12} \times \dfrac{2}{L} = 1 \Rightarrow L = 1.5$ m

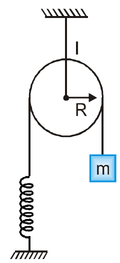
Ex. The time period of a bar pendulum when suspended at distances 30 cm and 50 cm from its centre of gravity comes out to be the same. If the mass of the body is 2kg. Find out its moment of inertia about an axis passing through first point.



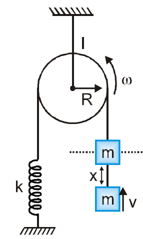
Sol. $I_A = MK^2 + M\ell_1^2$ but $\ell_1 = 50$ cm & $\dfrac{K^2}{\ell_1} = 30$ cm $\Rightarrow K^2 = 30 \times 50$ cm^2

$I_A = 2 \times (30 \times 50 \times 10^{-4}) + 2 \times (50 \times 10^{-2})^2 = 0.3 + 0.18 = 0.48$ kg-m^2

Ex. Figure shows a pulley block system in equilibrium. If the block is displaced down slightly from its equilibrium position and released. Find the time period of oscillation of the system. Assume there is sufficient friction present between pulley and string so that string will not slip over pulley surface.



Sol. If m is in equilibrium tension in string must be mg and spring is stretched by h so that mg=kh. If we displace the block downward by a distance A and released, it starts executing SHM with amplitude A. During its oscillation we consider the block at a displacement x below the equilibrium position, if it is moving at a speed v at this position, the pulley will be rotating at an angular speed ω given as $\omega = \frac{v}{r}$



Thus at this position the total energy of oscillating system is $E_T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}k(x+h)^2 - mgx$

Differentiating with respect to time,

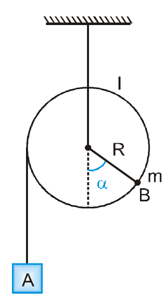
we get $$\frac{dE_T}{dt} = \frac{1}{2}m\left(2v\frac{dv}{dt}\right) + \frac{1}{2}I\left(\frac{1}{r^2}\right)\left(2v\frac{dv}{dt}\right) + \frac{1}{2}k\left[2(x+h)\frac{dx}{dt}\right] - mg\left(\frac{dx}{dt}\right) = 0$$

or $$mva + \frac{1}{r^2}va + k(x+h)v - mgv = 0$$ or $$a + \left(\frac{k}{m + \frac{1}{r^2}}\right)x = 0 \quad [\text{as } mg = kg]$$

Comparing above equation with standard differential equation of SHM we get $\omega = \sqrt{\left(\frac{k}{m + \frac{I}{1/r^2}}\right)}$

Thus time period of oscillation is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m + \frac{I}{r^2}}{k}}$

Ex. Figure shows a pulley block system in which a block A is hanging on one side of pulley and an other side small bead B of mass m is welded on pulley. The moment of inertia of pulley is I and the system is in equilibrium when bead is at an angle α from the vertical. If the system is slightly disturbed from its equilibrium position, find the time period of its oscillations.

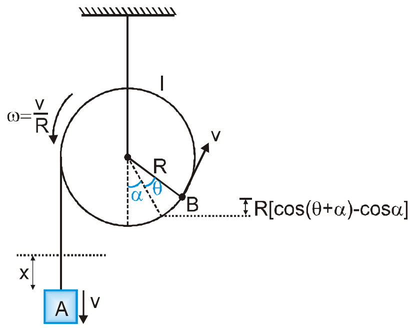


Sol. In equilibrium the net torque on pulley must be zero, thus we have

MgR = mgRsinα or M = msinα [if mass of block A is assumed to be M]

Now if block is displace down by distance A and released, it starts oscillating with amplitude A. Now consider the block at a distance x below the equilibrium position when it is going down at speed v. Figure shows the corresponding situation at this instant and the total energy of oscillating system can be written as

$$E_T = \frac{1}{2}Mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{R}\right)^2 - Mgx + mgR\left[\cos\alpha - \cos(\theta+\alpha)\right]$$



Differentiating the above equation w.r. to time, we get

$$\frac{dE_T}{dt} = \frac{1}{2}M\left(2v\frac{dv}{dt}\right) + \frac{1}{2}m\left(2v\frac{dv}{dt}\right) + \frac{1}{2}\frac{I}{R^2}\left(2v\frac{dv}{dt}\right) - Mg\left(\frac{dx}{dt}\right) + mgR\left[-\sin(\theta+\alpha)\frac{d\theta}{dt}\right] = 0$$

$$\Rightarrow Mva + mva + \frac{I}{R^2}va - Mgv + mgR\sin(\theta+\alpha)\left(\frac{v}{R}\right) = 0 \quad \left[as\ \frac{d\theta}{dt} = \omega = \frac{v}{R}\right]$$

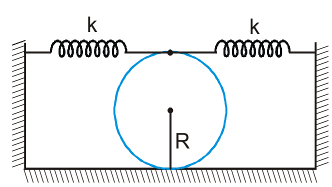
$$\Rightarrow \left(M + m + \frac{I}{R^2}\right)a - Mg - mg\,[\alpha\cos\alpha + \sin\alpha] = 0$$

$$\Rightarrow \left(M + m + \frac{I}{R^2}\right)a + mg\cos\alpha \cdot \frac{x}{R} = 0 \qquad [\text{as } M = m\sin\alpha \text{ and } \theta = \frac{x}{R}] \Rightarrow a = -\frac{mg\cos\alpha}{R\left(M + m + \frac{I}{R^2}\right)}x$$

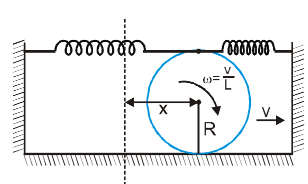
Comparing equation with basic differential equation of SHM, we get the angular frequency of SHM of system

as $\qquad \omega = \sqrt{\dfrac{mg\cos\alpha}{R\left(M + m + \dfrac{I}{R^2}\right)}}$

Ex. A solid uniform cylinder of mass M performs small oscillations in horizontal plane if slightly displaced from its mean position shown in figure. If it is given that initially springs are in natural lengths and cylinder does not slip on ground during oscillations due to friction between ground and cylinder. Force constant of each spring is k. Find time period of these oscillation.



Sol. In the situation given in problem, the cylinder is in its equilibrium position when springs are unstrained. When it slightly rolled and released. It starts executing SHM and due to friction, the cylinder is in pure rolling motion. Now during oscillations we consider the cylinder when it is at a distance x from the mean position and moving with a speed v as shown in figure. As cylinder is in pure rolling, its angular speed of rotation can be given as



As centre of cylinder is at a distance x from the initial position, the springs which are connect at a point on its rim must be compressed and stretched by a distance 2x. Thus at this intermediate position total energy of the oscillating system can be given as

$$E_T = \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2 + \frac{1}{2}k(2x)^2 \times 2$$

Differentiating with respect to time, we get

$$\frac{dE_T}{dt} = \frac{1}{2}M\left(2v\frac{dv}{dt}\right) + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\frac{1}{R^2}\left(2v\frac{dv}{dt}\right) + 4k\left(2x\frac{dx}{dt}\right) = 0$$

$$\Rightarrow Mva + \frac{1}{2}Mva + 8kxv = 0 \quad \Rightarrow \quad a = -\frac{16}{3}\frac{k}{M}x$$

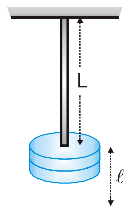
Comparing equation with basic differential equation of SHM.

We get, the angular frequency of SHM as $\omega = \sqrt{\frac{16k}{3M}}$

Thus time period of these oscillation is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{3M}{16k}} = \frac{\pi}{2}\sqrt{\frac{3M}{k}}$

Examples of Simple Harmonic Motion

1. **If a mass m is suspended from a wire of length L**, cross section A and young's modulus Y and is pulled along the length of the wire then restoring force will be developed by the elasticity of the wire.



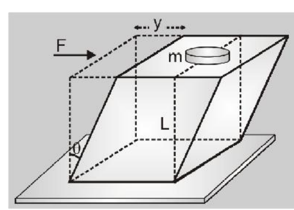
$$Y = \frac{\text{stress}}{\text{strain}} \Rightarrow \frac{F/A}{\ell/L} \Rightarrow = \frac{FL}{\ell A} \Rightarrow F = -\frac{YA}{L}\ell$$

Restoring force is linear so motion is linear simple harmonic with force constant

$$k = \frac{YA}{L} \text{ i.e., } \quad n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{YA}{mL}}$$

If the lower surface of a cube of side L and of modulus of rigidity η fixed while fixing a particle of mass m on the upper face, a force parallel to upper face is applied and withdrawn; Here restoring force will be developed due to elasticity of block.

Modulus of rigidity of the block $\eta = \frac{\text{shear stress}}{\text{shear strain}}$



$$\eta = \frac{F}{A\theta} \Rightarrow F = \eta\frac{A}{L}y \qquad [\text{as } \theta = \frac{y}{L}]$$

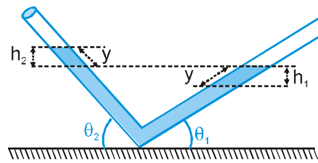
Restoring force is linear so motion will be linear S.H.M.

Force constant $(k) = \eta\frac{A}{L} = \eta L$ [as $A = L^2$] So $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{\eta L}}$

2. **Motion of a liquid in a V-shape tube when it is slightly depressed and released**

Here cross-section of the tube is uniform and the liquid is incompressible and non viscous. Initially the level of liquid in the two limbs will be at the same height. If the liquid is pressed by y in one limb, it will rise by y along the length of the tube in the other limb so the restoring force will developed by hydrostatic pressure difference, i.e.,

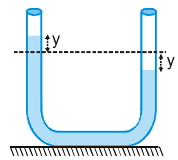
$$F = -\Delta P \times A = -(h_1 + h_2)gdA \Rightarrow \quad F = -Agd(\sin\theta_1 + \sin\theta_2)y$$



As the restoring force is linear, motion will be linear simple harmonic.

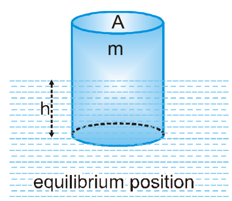
Force constant (k) = Agd ($\sin\theta_1 + \sin\theta_2$)   So $T = 2\pi\sqrt{\dfrac{m}{Adg(\sin\theta_1 + \sin\theta_2)}}$

Note : If the tube is a U-tube and liquid is filled to a height h



$\theta_1 = \theta_2 = 90°$ and m=hAd × 2   So time period $T = 2\pi\sqrt{\dfrac{h}{g}}$

3. When a partially submerged floating body is slightly pressed and released :

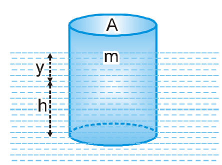


If a body of mass m and cross section A is floating in a liquid of density σ with height h inside the liquid then

$mg = \text{Thrust} = Ah\sigma g$,   i.e., $m = Ah\sigma$ ...(i)

Now from this equilibrium position if it is pressed by y, restoring force will developed due to extra thrust i.e.

$F = -A\sigma gy$



As restoring force is linear, motion will be linear simple harmonic with force constant $k = A\sigma g$,

So $T = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{m}{A\sigma g}}$

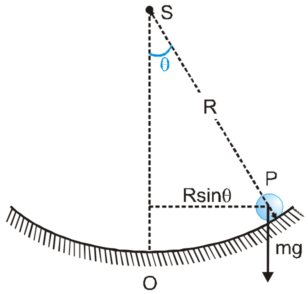
From this expression it is clear that if density of liquid decreases, time period will increase and vice-versa.

And also as from eq^n. (i) $m = Ah\sigma$, $T = 2\pi\sqrt{\dfrac{h}{g}}$ where h is the height of the body inside the liquid.

4. Motion of a ball in a bowl

If a small steel ball of mass m is placed at a small distance from O inside a smooth concave surface of radius R and released, it will oscillate about O. The restoring torque here will be due to the force of gravity mg on the ball i.e.,

$\tau = -mg\,(R\sin\theta) \quad = -mgR\theta$ [As θ is small]



Now as restoring torque is angular so motion will be angular simple harmonic. And as by definition.

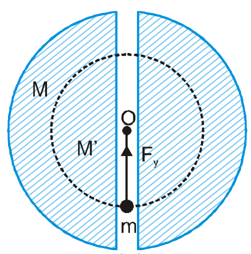
$$\tau = I\alpha = mR^2\left[\frac{d^2\theta}{dt^2}\right] \quad [\text{as } I = mR^2 \text{ and } \alpha = \frac{d^2\theta}{dt^2}]$$

So, $$mR^2\frac{d^2\theta}{dt^2} = -mgR\theta \quad \text{i.e.,} \quad \frac{d^2\theta}{dt^2} = -\omega^2\theta \Rightarrow \omega^2 = \frac{g}{R} \text{ so } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}}$$

5. Motion of a ball in a tunnel through the earth

Case I :

If the tunnel is along a diameter and a ball is released from the surface. If the ball at any time is at a distance y from the centre of earth, then the restoring force will act on the ball due to gravitation between ball and earth. But from theory of gravitation we know that force that acts on a particle inside the earth at a distance y from its centre is only due to mass M' of the earth that lies within sphere of radius y. (the portion of the earth that lies out side this sphere does not exert any net force on the particle) so $F = \frac{-GmM'}{y^2}$



(A)

But as $M = \frac{4}{3}\pi R^3\rho$ and $M' = \frac{4}{3}\pi y^3\rho$, i.e., $M' = M\left[\frac{y}{R}\right]^3$

$$F = \frac{-Gm}{y^2} \times M\left[\frac{y^3}{R^3}\right] = -\frac{GMm}{R^3}y$$

Restoring force is linear so the motion is linear SHM with force constant .

$$k = \frac{GMm}{R^3} \quad \text{so} \quad T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{R^3}{GM}}$$

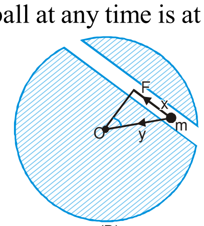
Further more as $g = \frac{GM}{R^2} \Rightarrow T = 2\pi\sqrt{\frac{R}{g}}$

Which is same as that of a simple pendulum of infinite length and is equal to 84.6 minutes.

Case II :

If the tunnel is along a chord and ball is released from the surface and if the ball at any time is at a distance x from the centre of the tunnel. The restoring force will be :

$$F' = F\sin\theta = \left[-\frac{GMm}{R^3}y\right]\left[\frac{x}{y}\right] = -\frac{GMm}{R^3}x$$



(B)

Which is again linear with same force constant $k = \frac{GMm}{R^3}$

So that motion is linear simple harmonic with same time period -

$$T = 2\pi\sqrt{\frac{m}{k}} \quad = 2\pi\sqrt{\frac{R^3}{GM}} = 2\pi\sqrt{\frac{R}{g}} = 84.6 \text{ minutes}$$

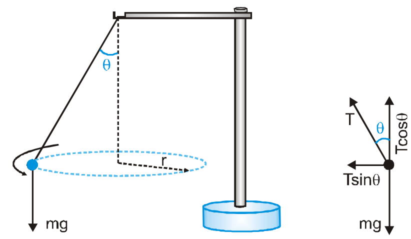
Note : In SHM $v_{max} = \omega A$

(i) In I case and II[nd] case time period will be same but v_{max} will be different.

(ii) If ball is dropped from height h it will perform oscillatory motion not SHM $\left[F \propto \frac{1}{r^2}\right.$ and not $\left.F \propto (-r)\right]$

6. Conical Pendulum

It is not example of S.H.M. but example of periodic motion.

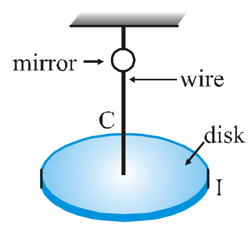


$$T = 2\pi\sqrt{\frac{h}{g}} \qquad \text{where } h = L\cos\theta$$

$$h = \sqrt{L^2 - r^2}$$

$$\omega = \sqrt{\frac{g}{L\cos\theta}}$$

7. Torsional Oscillator : (Angular SHM)



$$T = 2\pi\sqrt{\frac{I}{C}} \qquad \text{where } C = \frac{\eta\pi r^4}{2\ell}$$

η = modulus of elasticity of the wire ; $\qquad$ r = radius of the wire

L = length of the wire ; I = Moment of inertia of the disc

8. Oscillation of piston in a frictionless gas chamber piston :

$$T = 2\pi\sqrt{\frac{Vm}{A^2E}} \quad \text{where V= volume of cylinder}$$

m = mass of piston , A = area of cylinder ball , E = bulk modulus = $\frac{\Delta P}{-\Delta V / V}$

For Isothermal process : $E = P$, So $T = 2\pi\sqrt{Vm / PA^2}$

For Adiabatic process : $E = \gamma P$, So $T = 2\pi\sqrt{Vm / \gamma PA^2}$

Ex. A liquid of mass m is set into oscillations in a U-tube of cross section A. Its time period recorded is T, where $T = 2\pi\sqrt{\frac{\ell}{2g}}$, here ℓ is the length of liquid column. If the liquid of same mass is set into oscillations in U-tube of cross section $\frac{A}{16}$ then determine time period of oscillation.

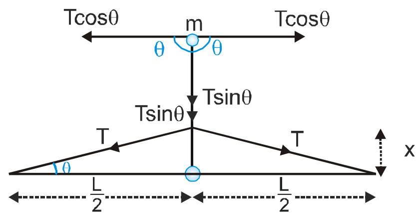
Sol. Mass is constant $\Rightarrow$ volume $\times$ density = constant $\Rightarrow$ $V_1d = V_2d$

$$(A\ell)d = \left[\frac{A}{16}\ell'\right]d \;\Rightarrow\; \ell' = 16\,\ell \qquad \because\; T = 2\pi\sqrt{\frac{\ell}{2g}} \qquad \therefore\; \frac{T'}{T} = \sqrt{\frac{\ell'}{\ell}} = \sqrt{\frac{16\ell}{\ell}} = 4 \;\Rightarrow\; T' = 4T$$

Ex. A ball of mass m kept at the centre of a string of length L is pulled from center in perpendicular direction and released. Prove that motion of ball is simple harmonic and determine time period of oscillation

Sol. Restoring force $F = -2T\sin\theta$

When θ is small $\sin\theta \approx \tan\theta \approx \theta = \dfrac{x}{L/2}$

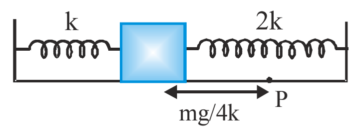


$$m\frac{d^2x}{dt^2} = -2T\sin\theta = -2T\theta = -2T\frac{x}{L/2}$$

$$\frac{d^2x}{dt^2} = -\frac{4T}{mL}x \Rightarrow \frac{d^2x}{dt^2} \propto -x$$

So motion is simple harmonic $\omega = \dfrac{2\pi}{T} = \sqrt{\dfrac{4T}{mL}} \Rightarrow T = 2\pi\sqrt{\dfrac{mL}{4T}}$

Ex. In the spring mass system shown in the figure, the spring is compressed by $x_0 = \dfrac{mg}{2k}$ from its natural length and block is released from rest. Find the speed of the block when it passes through P (mg/4k distance from mean position)

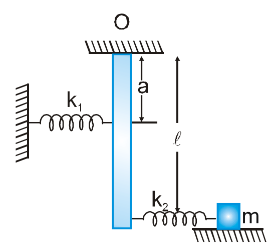


Sol. $\omega = \sqrt{\dfrac{3k}{m}}$, $x = A\sin(\omega+\phi)$, $v = A\omega\cos(\omega t+\phi)$ at $t=0$, $x=0 \Rightarrow \phi=0$

$$x = A\sin\omega t \Rightarrow \frac{mg}{4k} = \frac{mg}{2k}\sin\omega t \Rightarrow \sin\omega t = \frac{1}{2} \Rightarrow \omega t = \frac{\pi}{6} \Rightarrow t = \frac{\pi}{6\omega} = \frac{T}{12}$$

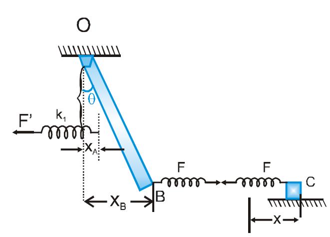
$$v = A\omega\cos\omega t\,,\ v = \frac{mg}{2k}\sqrt{\frac{3k}{m}}\cos\left[\sqrt{\frac{3k}{m}}\left(2\pi\sqrt{\frac{m}{3k}}\Big/12\right)\right] = g\sqrt{\frac{9m}{16k}}$$

Ex. A very light rod of length ℓ pivoted at O is connected with two springs of stiffness k_1 & k_2 at a distance of a & ℓ from the pivot respectively. A block of mass m attached with the spring k_2 is kept on a smooth horizontal surface. Find the angular frequency of small oscillation of the block m.



Sol. Let the block be pulled towards right through a distance x, then $x = x_B + x_{CB}$ ...**(i)**

where, x_{CB} = displacement of C (the block) relative to B



Thus $x_{CB} = \dfrac{F}{k_2}$ ...**(ii)** and $x_B = \left(\dfrac{F'}{k_1}\right)\dfrac{\ell}{a}$ ...**(iii)**

Torque acting on the rod about point O,

$$\tau_0 = F'a - F\ell \Rightarrow I_0\frac{d^2\theta}{dt^2} = F'a - F\ell$$

Since the rod is very light its moment of inertia

I_0 about O is approximately equal to zero $\Rightarrow F' = F\left(\dfrac{\ell}{a}\right)$ ...**(iv)**

Using **(iii)** & **(iv)** $\Rightarrow x_B = \frac{F}{k_1}\left(\frac{\ell}{a}\right)^2$ ....**(v)**

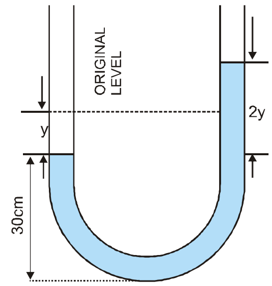
Using **(i)**, **(ii)** & **(v)** $\Rightarrow x = \frac{F}{k_1}\left(\frac{\ell}{a}\right)^2 + \frac{F}{k_2}$

As force F is opposite to displacement x, then

$$\Rightarrow F = -\frac{k_1k_2}{k_2\left(\frac{\ell}{a}\right)^2 + k_1}x \Rightarrow m\omega^2 x = \frac{k_1k_2}{k_2\left(\frac{\ell}{a}\right)^2 + k_1}x \Rightarrow \omega = \sqrt{\frac{k_1k_2a^2}{m\left(k_1a^2 + k_2\ell^2\right)}}$$

Ex. A vertical U-tube of uniform cross-section contains water upto a height of 30 cm. Show that if the water on one side is depressed and then released, its motion up and down the two sides of the tube is simple harmonic. Calculate its period.

Sol. Figure shows a U-tube of uniform cross-sectional area A. Let the liquid be depressed through the distance y in a limb, the difference of levels between two limbs will be 2 y as shown in figure.



The liquid now oscillates about the initial positions.

Excess pressure on whole liquid = (excess height of the liquid column) (density) (g)

= $2y \times 1 \times g$ (as density of water =1)

Restoring Force on the liquid = Pressure $\times$ area of cross-section = 2ygA

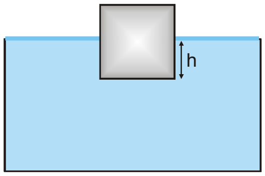
Due to this force the liquid accelerates and if its acceleration is a, we have ma = –2ygA

$$\Rightarrow (2 \times 30 \times A)\, a = -\,2ygA \Rightarrow a = -\frac{g}{30}y$$

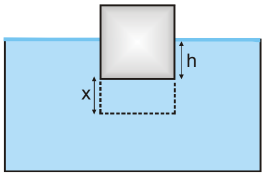
Hence acceleration is directly proportional to displacement, so the motion is simple harmonic motion. Thus the time period T is given by

$$T = \frac{2\pi}{\omega} = 2\omega\sqrt{\left(\frac{30}{g}\right)} = 2\pi\sqrt{\left(\frac{30}{980}\right)}\left[\text{as } \omega = \sqrt{\frac{g}{30}}\right] = 1.098 \text{ second}$$

Ex. Calculate the period of small oscillations of a floating box as shown in figure, which was slightly pushed down in vertical direction. The mass of box is m, area of its base is A and the density of liquid is ρ. The resistance of the liquid is assumed to be negligible.



Sol. Initially when box is floating in liquid, if its h depth is sumerged in liquid then buoyancy force on it is



F_B = weight of liquid displaced = Ahrg

As the box is in equilibrium, we have Ahρg = mg

Now if box is further pushed down by a distance x, net restoring force on it in upward (toward mean position) direction is

$F_R = -[A(h+x)\rho g - mg] = -Ax\rho g$ [as mg = Ahρg]

If a is the acceleration of box in upward direction we have $a = -\left(\frac{A\rho g}{m}\right)x$

Equation shows that the box executes SHM with angular frequency ω given as $\omega = \sqrt{\frac{A\rho g}{m}}$

Thus time period of its oscillation can be given as $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{A\rho g}}$

SUPERPOSITION OF TWO SHM's

Case - I In same direction and of same frequency

$x_1 = A_1 \sin \omega t$

$x_2 = A_2 \sin(\omega t + \theta)$, then resultant displacement

$x = x_1 + x_2 = A_2 \sin \omega t + A_2 \sin(\omega t + \theta) = A \sin(\omega t + \phi)$

where $\sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\theta}$ & $\phi = \tan^{-1}\left[\frac{A_2\sin\theta}{A_1 + A_2\cos\theta}\right]$

If θ = 0, both SHM's are in phase and $A = A_1 + A_2$

If θ = ϕ, both SHM's are out of phase and $A = [A_1 - A_2]$

The resultant amplitude due to superposition of two or more than two SHM's of this case can also be found by phasor diagram also.

Case -II In same direction but are of different frequencies

$x_1 = A_1 \sin \omega_1 t$

$x_2 = A_2 \sin \omega_2 t$

then resultant displacement $x = x_1 + x_2 = A_1 \sin \omega_1 t + A_2 \sin \omega_2 t$

the resultant motion is not SHM.

Case - III In two perpendicular directions

$x = A \sin \omega t$ ⇒ $y = B \sin(\omega t + \theta)$

then resultant displacement $x = x_1 + x_2 = A_1 \sin \omega_1 t + A_2 \sin \omega_2 t$

the resultant motion is not SHM.

If θ = 0 or π then $y = \pm (B/A)\, x$

If $\theta = \frac{\pi}{2}$ then $x = A \sin \omega t$

$y = B \sin(\omega t + \pi/2) = B \cos \omega t$

So, resultant will be $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ i.e. equation of an ellipse and if A = B, superposition will be an equation of circle.

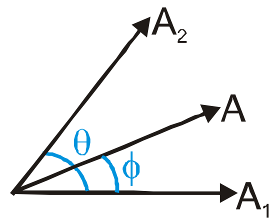
Superposition of SHM's along the same direction (using phasor diagram)

If two or more SHM's are along the same line, their resultant can be obtained by vector addition by making phasor diagram. In this method

1. Amplitude of SHM is taken as length (magnitude) of vector.
2. Please difference between the vectors is taken as the angle between these vectors. The magnitude of resultant vector gives phase constant of resultant SHM.

For example :

$x_1 = A_1 \sin \omega t \quad \Rightarrow \quad x_2 = A_2 \sin (\omega t + \theta)$



If equation of resultant SHM is taken as $x = A \sin (\omega t+\phi)$

$$A = \sqrt{A_1^2 + 2A_1A_2 \cos\theta}$$

$$\tan\phi = \frac{A_1 \sin\theta}{A_1 + A_2 \cos\theta}$$

Ex. Given : $x_1 = 3 \sin \omega t$

$x_2 = 4 \cos \omega t$

Find **(i)** amplitude of resultant SHM **(ii)** equation of the resultant SHM.

Sol. First write all SHM's in terms of sine functions with positive amplitude.

$\therefore \quad x_1 = 3 \sin \omega t$

$x_2 = 4 \sin (\omega t + \pi/2)$

$$A = \sqrt{3^2 + 4^2 + 2\times3\times4\cos\frac{\pi}{2}} = \sqrt{9+16} = \sqrt{25} = 5$$

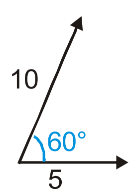
$$\tan\phi = \frac{4\sin\frac{\pi}{2}}{3+4\cos\frac{\pi}{2}} \qquad \phi = 53°$$

equation $x = 5 \sin (\omega t + 53°)$

Ex. Given $x_1 = 5 \sin (\omega t + 30°)$

$x_2 = 10 \cos (\omega t)$

Find amplitude of resultant SHM.



Sol. $x_1 = 5 \sin (\omega t + 30°)$

$$x_2 = 10\sin\left(\omega t + \frac{\pi}{2}\right)$$

Phasor diagram

$$A = \sqrt{5^2 + 10^2 + 2\times5\times10\cos 60°} = \sqrt{25+100+50} = \sqrt{175} = 5\sqrt{7}$$

1. Periodic Motion

Any motion which repeats itself after regular interval of time (i.e. time period) is called periodic motion or harmonic motion.

Ex. **(a)** Motion of planets around the sun.

(b) Motion of the pendulum of wall clock.

2. Oscillatory Motion

The motion of body is said to be oscillatory or vibratory motion if it moves back and forth (to and fro) about a fixed after regular interval of time.

The fixed point about which the body oscillates is called mean position or equilibrium position.

Ex. **(a)** Vibration of the wire of 'Sitar'

(b) Oscillation of the mass suspended from spring.

3. Simple Harmonic Motion (S.H.M.)

Simple harmonic motion is the simplest from of vibratory or oscillatory motion.

4. Some Basic Terms in SHM

(a) Mean Position

The point at which the restoring force on the particle is zero and potential energy is minimum, is known as its mean position.

(b) Restoring Force

The force acting on the particle which tends to bring the particle towards its means position, is known as its mean position.

Restoring force always acts in a direction opposite to that of displacement.

Displacement is measured from the mean position.

(c) Amplitude

The maximum (positive or negative) value of displacement of particle from mean position is defined as amplitude.

(d) Time Period (T)

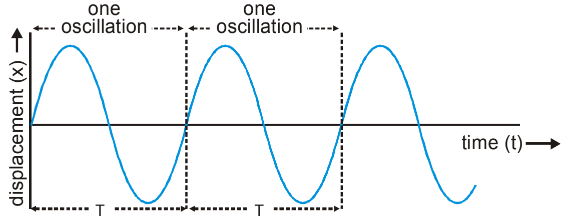
The maximum time after which the particle keeps on repeating its motion is known as time period.

The smallest time taken to complete one oscillation or vibration is also defined as time period.

It is given by $T = \frac{2\pi}{\omega} = \frac{1}{n}$ where ω is angular frequency and n is frequency.

(e) One oscillation or One vibration

When a particle goes on one side from mean position and returns back and then it goes to other side from returns back to mean position, then this process is known as one oscillation.



5. Frequency (n or f)

The number of oscillations per second is defined as frequency.

It is given by $n = \frac{1}{T} = \frac{\omega}{2\pi}$

6. Phase

Phase of a vibrating particle at any instant is the state of the vibrating particle regarding its displacement and direction of vibration at that particular instant.

In the equation $x = A\sin(\omega t + \phi)$, $(\omega t + \phi)$ is the phase of the particle.

The phase angle at time $t = 0$ is known as initial phase or epoch.

The difference of total phase angles of two particles executing SHM with respect to the mean position is known as phase difference.

Two vibrating particles are said to be in same phase if the phase difference between them is an even multiple of π, i.e. $\Delta\phi = 2n\pi$ where $n = 0, 1, 2, 3$

Two vibrating particle are said to be in opposite phase if the phase difference between them is an odd multiple of π, i.e. $\Delta\phi = (2n + 1)\pi$ where $n = 0, 1, 2, 3.....$

7. Angular frequency (ω) : The rate of change of phase angle of a particle with respect to time is defined as its

angular frequency $\omega = \sqrt{\frac{k}{m}}$

8. For linear SHM ($F \propto -x$) : $F = m\frac{d^2x}{dt^2} = -kx = -m\omega^2 x$ where $\omega^2 = \sqrt{\frac{k}{m}}$

9. For angular SHM ($\tau \propto -\theta$) : $\tau = \frac{d^2\theta}{dt^2} = I\,\alpha = -k\theta = -m\omega^2\theta$ where $\omega^2 = \sqrt{\frac{k}{m}}$

10. Displacement $x = A\sin(\omega t + \phi)$

11. Angular displacement $\theta = \theta_0 \sin(\omega t + \phi)$

12. Velocity $v = \frac{dx}{dt} = A\omega\cos(\omega t + \phi) = \omega\sqrt{A^2 - x^2}$

13. Angular velocity $\frac{d\theta}{dt} = \theta_0\,\omega\cos(\omega t + \phi)$

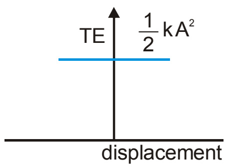
14. Acceleration $a = \frac{d^2x}{dt^2} = -A\omega^2\sin(\omega t + \phi) = -\omega^2 x$

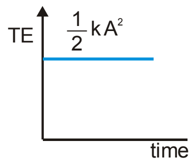
15. Angular acceleration $\frac{d^2\theta}{dt^2} = -\theta_0\omega^2\sin(\omega t + \phi) = -\omega^2\theta$

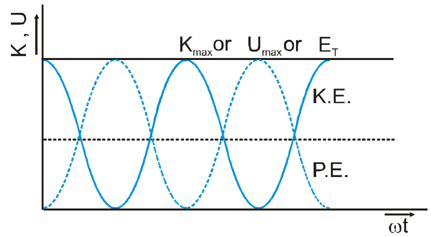
16. Kinetic energy $K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \phi)$

17. Potential energy $U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \phi)$

18. Total energy $E = K + U = \frac{1}{2}m\omega^2 A^2$ = constant







Note : **(a)** Total energy of a particle in S.H.M. is same at all instant and at all displacement.

(b) Total energy depends upon mass, amplitude and frequency of vibration of particle executing S.H.M.

19. Average energy SHM

(a) The time average of P.E. and K.E. over one cycle is

(A) $\langle K \rangle_t = \frac{1}{4}kA^2$ **(B)** $\langle PE \rangle_t = \frac{1}{4}kA^2$ **(C)** $\langle TE \rangle_t = \frac{1}{2}kA^2 + U_0$

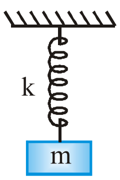
(b) The position average of P.E. and K.E. between x = – A to x = A

(A) $\langle K \rangle_x = \frac{1}{3}kA^2$ **(B)** $\langle PE \rangle_x = u_0 + \frac{1}{6}kA^2$ **(C)** $\langle TE \rangle_x = \frac{1}{2}kA^2 + U_0$

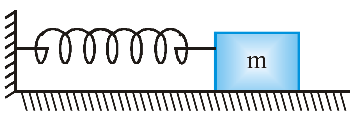
20. Differential equation of SHM

Linear SHM : $\frac{d^2x}{dt^2} + \omega^2 x = 0$ Angular SHM : $\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$

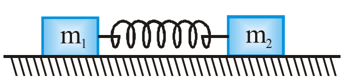
21. Spring block system



$T = 2\pi\sqrt{\frac{m}{k}}$

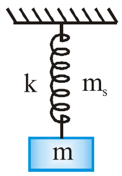


$T = 2\pi\sqrt{\frac{m}{k}}$



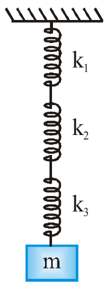
$T = 2\pi\sqrt{\frac{\mu}{k}}$ where m = reduced mass = $\frac{m_1 m_2}{m_1 + m_2}$

When spring mass is not negligible :



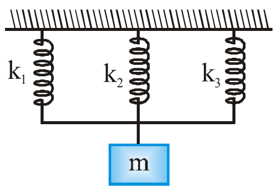
$T = 2\pi\sqrt{\frac{m + \frac{m_s}{3}}{k}}$

22. Series combination of springs



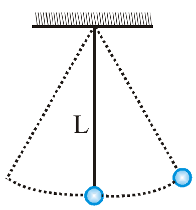
$$T = 2\pi\sqrt{\frac{m}{k_{eff}}} \qquad \text{where } \frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

23. Parallel combination of spring

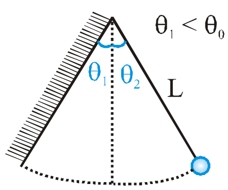


$$T = 2\pi\sqrt{\frac{m}{k_{eff}}} \quad \text{where } k_{eff} = k_1 + k_2 + k_3$$

24. Time period of simple pendulum



$$\text{Time period } T = 2\pi\sqrt{\frac{L}{g}}$$



$$\text{Time period } T = \left\{\pi - 2\sin^{-1}\left(\frac{\theta_1}{\theta_0}\right)\right\}\sqrt{\frac{L}{g}}$$

If length of simple pendulum is comparable to the radius of the earth R, then $T = 2\pi\sqrt{\dfrac{1}{g\left(\dfrac{1}{\ell} + \dfrac{1}{R}\right)}}$.

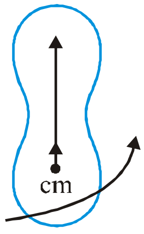
(i) If $\ell << R$ then $T = 2\pi\sqrt{\dfrac{\ell}{g}}$

(ii) If $\ell >> R$ then $T = T = 2\pi\sqrt{\dfrac{R}{g}} \approx 84$ minutes

25. Second pendulum

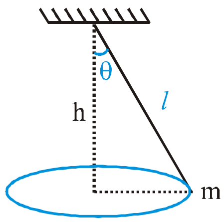
Time period = 2 second Length $\approx$ 1 meter (on earth`s surface)

26. Time period of Physical pendulum



$$T = 2\pi\sqrt{\frac{I}{mg\ell}} = 2\pi\sqrt{\frac{\frac{k^2}{\ell} + \ell}{g}} \qquad \text{where } I_{cm} = mk^2$$

27. Time period of Conical pendulum



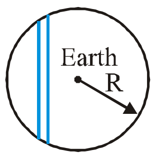
$$T = 2\pi\sqrt{\frac{\ell\cos\theta}{g}} = 2\pi\sqrt{\frac{h}{g}}$$

28. Time period of Torsional pendulum $2\pi\sqrt{\frac{I}{k}}$

where k = torsional constant of the wire

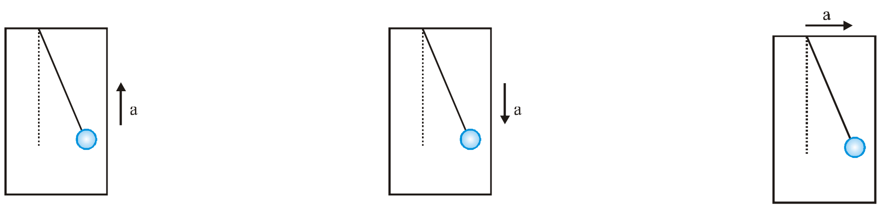
l = moment of inertia of the body about the vertical axis

29. SHM of a particle in a tunnel inside the earth



$$T = 2\pi\sqrt{\frac{R}{g}}$$

30. In accelerating cage



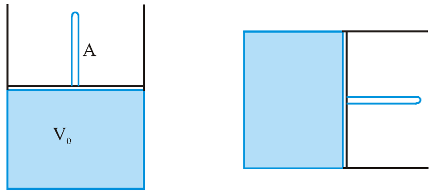
$g_{eff} = g + a$ $\qquad$ $g_{eff} = g - a$ $\qquad$ $g_{eff} = \sqrt{g^2 + a^2}$

$$T = 2\pi\sqrt{\frac{\ell}{g+a}} \qquad T = 2\pi\sqrt{\frac{\ell}{g-a}} \qquad T = 2\pi\sqrt{\frac{\ell}{\left(g^2+a^2\right)^{1/2}}}$$

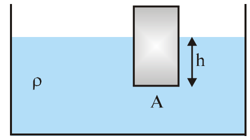
31. SHM of gas-piston system

Here elastic force is developed due to bulk elasticity of the gas

$$B = \frac{\Delta P}{\Delta V / V} \Rightarrow F = \frac{BA^2}{V_0}x \Rightarrow T = 2\pi\sqrt{\frac{m}{BA^2 / V_0}}$$



32. SHM of Floating Body



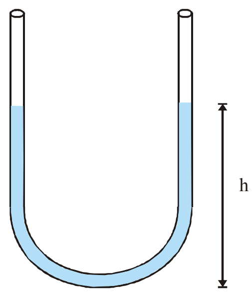
Restoring force → Thrust

$mg - \rho Agh \rightarrow$ Equilibrium

Restoring force $F = -(\rho Ag)x$

$$T = 2\pi\sqrt{\frac{h}{g}}$$

33. SHM in U-tube



$$T = 2\pi\sqrt{\frac{h}{g}}$$

34. SHM is the projection of uniform circular motion along one of the diameter of the circle.

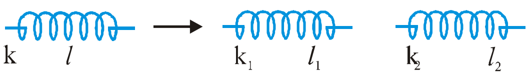
35. The periodic time of a hard spring is less as compared to that of a soft spring because the spring constant is large for hard spring.

36. For a system executing SHM, the mechanical energy remains constant.

37. Maximum kinetic energy of a particle in SHM may be greater then mechanical energy as potential energy of a system may be negative

38. The frequency of oscillation of potential energy and kinetic energy twice as that of displacement or velocity or acceleration of a particle executing S.H.M.

39. Spring cut into two parts



Here $\frac{\ell_1}{\ell_2} = \frac{m}{n}$

$$\ell_1 = \left(\frac{m}{m+n}\right)\ell, \ell_2 = \left(\frac{n}{m+n}\right)\ell$$

But $k\ell = k_1\ell_1 = k_2\ell_2$

$$k_1 = \frac{m+n}{m}k, k_2 = \frac{m+n}{n}k$$