HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

1. $I = \int_{-\alpha}^{(\pi-\alpha)} \sin|t|dt$ where $2x - \alpha = t$ $\Rightarrow$ $dx = \frac{dt}{2}$

$= \frac{1}{2}\int_{-\alpha}^{0} -\sin t\,dt + \frac{1}{2}\int_{0}^{\pi-\alpha} \sin t\,dt$

$= \frac{1}{2}\cos t\Big]_{-\alpha}^{0} - \frac{1}{2}\cos t\Big]_{0}^{\pi-\alpha}$

$= \frac{1}{2}[1 - \cos\alpha] - \frac{1}{2}[-\cos\alpha - 1]$

$= \frac{1}{2}(1 - \cos\alpha) + \frac{1}{2}(1 + \cos\alpha) = 1$

2. Note that in $\left(-\frac{1}{2}, \frac{1}{2}\right)$, $\sin^{-1}(3x - 4x^3) = 3\sin^{-1}x$

and $\cos^{-1}(4x^3 - 3x) = 2\pi - 3\cos^{-1}x$

hence $f(x) = 3\sin^{-1}x - 2\pi + 3\cos^{-1}x = -\frac{\pi}{2}$

$\therefore$ $I = -\frac{\pi}{2}\int_{-1/2}^{1/2} dx = -\frac{\pi}{2}$

5. $f'(x) = e^{g(x)}.\, g'(x)\,;\, g'(x) = \frac{x}{1 + x^4}$;

$f'(x) = e^{g(x)} \cdot \frac{x}{1+x^4}$; $e^{g(2)} = e^0 = 1$

hence $f'(2) = e^{g(2)}.\, g'(2) = e^0.\frac{2}{17} = \frac{2}{17}$

6. put $ln(ln(ln\,x)) = t$

$\int_1^e \frac{dt}{t} = \ln t|_1^e = 1$

7. $I = \int_1^e (x+1)e^x \ell nx\,dx = \int_1^e [x\ell nx + \ell nx + 1 - 1]e^x dx$

$= \int_1^e [\underbrace{x\ell nx}_{f(x)} + \underbrace{\ell nx + 1}_{f'(x)}]e^x dx - \int_1^e e^x dx$

$= [x\ell nxe^x]_1^e - [e^x]_1^e = ee^e - e^e + e$

9. $\int_2^4 \left[\frac{1}{\log_2 x} - \frac{1}{\ell n2(\log_2 x)^2}\right]dx$

$= \int_2^4 \left[\underbrace{\frac{1}{\log_2 x}}_{f(x)} + \underbrace{\frac{(-x)}{x\ell n2(\log_2 x)^2}}_{f'(x)}\right]dx = \left[\frac{x}{\log_2 x}\right]_2^4 = 0$

10. $F(x) = \int \frac{\sin x}{x}dx$

Now $I = \int_1^3 \frac{\sin 2x}{x}dx$ [put 2x = t]

$= \int_2^6 \frac{2}{2}\frac{\sin t}{t}dt = [F(x)]_2^6 = F(6) - F(2)$

11. put $e^{x^2} = t$; $e^{x^2} \cdot 2x\,dx = dt$;

$\int_1^{\pi/2} \cos t\,dt = [\sin t]_1^{\pi/2} = 1 - (\sin 1)$

13. $\int_{-3}^{3} \frac{t^2 \sin 2t}{t^2 + 1}dt = 0$ as the integrand is an odd function.

also $\int_0^1 \frac{dt}{t^2 + 2t\cos\alpha + 1} = \frac{1}{\sin\alpha}\tan^{-1}\frac{t + \cos\alpha}{\sin\alpha}\Big|_0^1 = \frac{\alpha}{2\sin\alpha}$

Thus the given equation reduces to

$x^2\frac{\alpha}{2\sin\alpha} - 2 = 0 \Rightarrow x = \pm\, 2\sqrt{\frac{\sin\alpha}{\alpha}}$

14 put $x = \tan\theta$

$I = \int_0^{\pi/2} \frac{d\theta}{1 + (\tan\theta)^a} = \int_0^{\pi/2} \frac{(\cos\theta)^a}{(\sin\theta)^a + (\cos\theta)^a}d\theta \Rightarrow I = \frac{\pi}{4}$

16. $f'(ln\,x) = \begin{cases} 1 & \text{for } 0 < x \le 1 \\ x & \text{for } x > 1 \end{cases}$

put $ln\,x = t$ $\Rightarrow$ $x = e^t$

for $x > 1$; $f'(t) = e^t$ for $t > 0$

integrating $f(t) = e^t + C$; $f(0) = e^0 + C$

$\Rightarrow$ $C = -1$ (given $f(0) = 0$)

$\therefore$ $f(t) = e^t - 1$ for $t > 0$ (corresponding to $x > 1$)

hence $f(x) = e^x - 1$ for $x > 0$ ....(1)

again $\quad$ for $0 < x \le 1$

$f'(\ln x) = 1 \qquad (x = e^t)$

$f'(t) = 1 \quad \text{for } t \le 0$

$f(t) = t + C$

$f(0) = 0 + C$

$\Rightarrow C = 0 \quad \Rightarrow f(t) = t \quad \text{for } t \le 0$

$\Rightarrow f(x) = x \quad \text{for } x \le 0$

18. $T_r = \dfrac{\pi}{6n}\sec^2\dfrac{r\pi}{6n}$

$$S = \sum T_r = \frac{\pi}{6n}\sum_{r=1}^{n}\sec^2\frac{r\pi}{6n} = \frac{\pi}{6}\int_0^1 \sec^2\frac{\pi x}{6}\,dx = \tan\frac{\pi x}{6}\Big|_0^1$$

$$= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

19. $$\lim_{\lambda\to 0}\left(\int_0^1 (1+x)^{\lambda}\,dx\right)^{1/\lambda} = \lim_{\lambda\to 0}\left(\frac{(1+x)^{\lambda+1}}{\lambda+1}\Bigg|_0^1\right)^{1/\lambda}$$

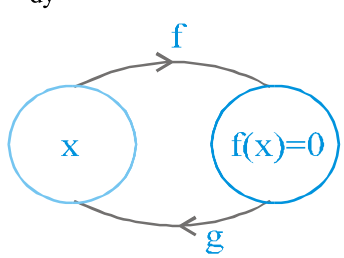
$$= \lim_{\lambda\to 0}\left(\frac{2^{\lambda+1}-1}{\lambda+1}\right)^{1/\lambda} \qquad (1^{\infty}\text{ form})$$

$$= e^{\lim\limits_{\lambda\to 0}\frac{1}{\lambda}\left(\frac{2^{\lambda+1}-1-\lambda-1}{\lambda+1}\right)} = e^{\lim\limits_{\lambda\to 0}\left(\frac{2^{\lambda+1}-2-\lambda}{\lambda(\lambda+1)}\right)}$$

$$= e^{\lim\limits_{\lambda\to 0}\left(\frac{2(2^{\lambda}-1)}{\lambda}-1\right)} = e^{2\ln 2-1} = e^{\ln\left(\frac{4}{e}\right)} = \frac{4}{e}$$

20. $f'(x) = \dfrac{1}{\sqrt{1+x^4}} = \dfrac{dy}{dx}$

now $g'(x) = \dfrac{dx}{dy} = \sqrt{1+x^4}$



when $y = 0$ i.e. $\displaystyle\int_2^x \frac{dt}{\sqrt{1+t^4}} = 0$ then $x = 2$ (think !)

hence $g'(0) = \sqrt{1+16} = \sqrt{17}$

22. $$I = \int_0^{3\pi/4}(\sin x + \cos x)\,dx + \int_0^{3\pi/4}\underset{I}{x}\,\underbrace{(\sin x - \cos x)}_{II}\,dx$$

$$= \int_0^{3\pi/4}(\sin x + \cos x)\,dx + \underbrace{x(-\cos x - \sin x)\Big|_0^{3\pi/4}}_{\text{zero}}$$

$$+ \int_0^{3\pi/4}(\sin x + \cos x)\,dx$$

$$= 2\int_0^{3\pi/4}(\sin x + \cos x)\,dx = 2\left(\sqrt{2}+1\right)$$

23. $$\int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx$$

$$= \int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx +$$

$$\int_1^2 (1+\cos^8 x)(ax^2+bx+c)\,dx$$

$$\therefore \int_1^2 (1+\cos^8 x)(ax^2+bx+c)\,dx = 0$$

since $1+\cos^8 x$ is always positive

$$= \int_a^b f(x)\,dx = 0 \quad (b > a)$$

means f(x) is positive in some portion and negative in some portion from a to b

$\therefore$ ax^2+bx+c is positive and negative in (1, 2)

$\therefore$ ax^2+bx+c has a root in (1, 2)

24. $y = f(x) \Rightarrow x = f^{-1}(y)$ and $dy = f'(x)\,dx$

now $I = \displaystyle\int_3^7 f^{-1}(x)\,dx = \int_3^7 f^{-1}(y)\,dy = \int_2^5 x\,f'(x)\,dx$;

(when $y = 3$ then $x = 2$ and $y = 7$ then $x = 5$)

hence $I = \displaystyle\int_2^5 x\,f'(x)\,dx$. Integrating by parts gives,

$$I = x\,f(x)\Big|_2^5 - \int_2^5 f(x)\,dx$$

$$I = 5\cdot 7 - 2\cdot 3 - 17 = 35 - 6 - 17 = 12$$

25. $\int_a^y \cos t^2 \ dt = \int_a^{x^2} \frac{\sin t}{t} dt$

differentiating both sides w.r.t x we get

$$\frac{d}{dx}\int_a^y \cot t^2\, dt = \frac{d}{dx}\int_a^{x^2} \frac{\sin t}{t} dt$$

RHS $= \frac{\sin[x^2]}{x^2}\,\frac{dx^2}{dx} = 2\times\frac{\sin x^2}{x^2}$

L.H.S. $= \frac{d}{dy}\left(\int_a^y \cos t^2\, dt\right)\frac{dy}{dx} = \cos y^2 \frac{dy}{dx}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\sin x^2}{x\cos y^2}$$

26. $g(x+\pi) = \int_0^{x+\pi} \cos^4 t\, dt$

$$= \int_0^x \cos^4 t\, dt + \int_x^{x+\pi} \cos^4 t\, dt$$

$$= g(x) + \int_0^{\pi} \cos^4 t\, dt = g(x) + g(\pi)$$

27. $S'(x) = \ln x^3 . 3x^2 - \ln x^2 . 2x = 9x^2 \ln x - 4x \ln x = x\, l n\, x\, (9x-4).$

Hence $\frac{S'(x)}{x} = \ln x\,(9x-4).$

Now it is obvious that $\frac{S'(x)}{x}$ is continuous and derivable in its domain.

29. $\int_a^0 3^{-x}(3^{-x}-2)\, dx \ge 0$ put $3^{-x} = t$

$\Rightarrow 3^{-x}\ln 3\, dx = -dt$

$$\ln 3\int_1^{3^{-a}} (t-2)\ dt \ \ge 0 \quad \Rightarrow \quad \left.\frac{t^2}{2} - 2t\right]_1^{3^{-a}} \ \ge \ 0$$

$$\left(\frac{3^{-2a}}{2} - 2.3^{-a}\right) - \left(\frac{1}{2} - 2\right) \ \ge \ 0$$

$3^{-2a} - 4\times 3^{-a} + 3 > 0$

$(3^{-a} - 3)(3^{-a} - 1) > 0$

$3^{-a} > 3^1 \ \Rightarrow \ a < 1$

or $3^{-a} < 3^0 \Rightarrow a > 0$

Hence $a \in (-\infty, -1) \cup [0, \infty)$

30. The given integrand is perfect differential coeff. of $\prod_{r=1}^{n}(x+r) \ \Rightarrow \ I = \left[\prod_{r=1}^{n}(x+r)\right]_0^1 = (n+1)! - n! = n\cdot n!$

31. Use $\int_0^a f(x) = \int_0^a (a-x)\,dx$ and add two integrals

34. $a_n = \int_0^{\pi/2} (1-\sin t)^n \sin 2t\, dt$

Let $1 - \sin t = u \ \Rightarrow \ -\cos t\, dt = du$

$$= 2\int_0^1 u^n(1-u)\,du$$

$$= 2\left(\int_0^1 u^n\, du - \int_0^1 u^{n+1}\, du\right)$$

$$= 2\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

hence $\frac{a_n}{n} = 2\left(\frac{1}{n(n+1)} - \frac{1}{n(n+2)}\right)$

$$\lim_{n\to\infty}\sum_{n=1}^{n}\frac{a_n}{n}$$

$$= 2\left(\sum\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{2}\sum\left(\frac{1}{n} - \frac{1}{n+2}\right)\right)$$

$$= 2\sum_{n=1}^{n}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \sum_{n=1}^{n}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$= 2(1) - \left[\left(1-\frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + ...\right]$$

$$= 2 - \frac{3}{2} = \frac{1}{2}$$

36. $I=\int_2^3 \frac{(x+2)^2}{2x^2-10x+53}dx$ ......**(i)**

$I=\int_2^3 \frac{(7-x)^2}{2(5-x)^2-10(5-x)+53}dx$

$I=\int_2^3 \frac{(7-x)^2}{2x^2-10x+53}dx$ .....**(ii)**

add**(i)** & **(ii)**

$2I=\int_2^3 \frac{(x+2)^2+(7-x)^2}{2x^2-10x+53}dx$

$=\int_2^3 dx=1$ $\quad \therefore I=1/2$

38. Differentiate both sides w.r.t. x

$f'(x)=\cos x+f'(x)(2\sin x-\sin^2 x)$

$\therefore \quad (1+\sin^2 x-2\sin x)f'(x)=\cos x$

$f'(x)=\frac{\cos x}{1+\sin^2 x-2\sin x}=\frac{\cos x}{(1-\sin x)^2}$

Integrating $\quad f(x)=\int\frac{\cos x\,dx}{(1-\sin x)^2}$ (Put $1-\sin x=t$);

$f(x)=-\int\frac{dt}{t^2}=\frac{1}{t}=\frac{1}{1-\sin x}+C$

also $f(0)=0$, hence $C=-1$

$f(x)=\frac{1}{1-\sin x}-1=\frac{1-1+\sin x}{1-\sin x}$

$=\frac{\sin x}{1-\sin x}$

39. Put $\pi x=t \Rightarrow dx=\frac{dt}{\pi}$

$I=\frac{1}{\pi}\frac{\pi}{\pi}\int_0^{2008\pi} t|\sin t|dt=\frac{1}{\pi}\int_0^{2008\pi} t|\sin t|dt$ ....**(1)**

$I=\frac{1}{\pi}\int_0^{2008\pi}(2008\pi-t)|\sin t|dt$ ....**(2)**

$(1)+(2) \Rightarrow 2I=\frac{2008\pi}{\pi}\int_0^{2008\pi}|\sin t|dt=(2008)^2.$

$\int_0^{\pi}|\sin t|dt$

$I=(2008)^2$; hence Here $\sqrt{I}=2008$

41. $I=\int_0^{\pi/2}\sqrt{\tan x}\,dx$ ....**(1)**

$I=\int_0^{\pi/2}\sqrt{\cot x}\,dx$ ....**(2)**

adding (1) and (2), we get

$2I=\int_0^{\pi/2}\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx=\sqrt{2}\int_0^{\pi/2}\frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx$

$=\sqrt{2}\int_0^{\pi/2}\frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}}dx$

$=\sqrt{2}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}=2\sqrt{2}\int_0^1\frac{dt}{\sqrt{1-t^2}}=\sqrt{2}\pi$

(where $\sin x-\cos x=t$)

$\therefore \quad I=\frac{\pi}{\sqrt{2}}$

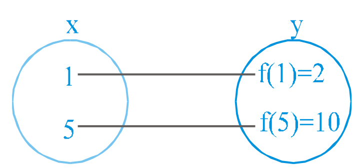
42. $I=\int_{\ell n\lambda}^{-\ell n\lambda}\frac{f\left(\frac{x^2}{4}\right)[f9x)-f(-x)]}{g\left(\frac{x^2}{4}\right)[g(x)+g(-x)]}dx=0.$

Since $\frac{f\left(\frac{x^2}{4}\right)[f(x)-f(-x)]}{g\left(\frac{x^2}{4}\right)\left[g(x)+\int(-x)\right]}$ is an odd function

43. $y=f(x) \quad \Rightarrow \quad x=f^{-1}(y)=g(y)$

$dy=f'(x)\,dx$

$\therefore \quad I=\int_1^5 f(x)dx+\int_1^5 x f'(x)dx$



where y is 2 then x = 1

y is 10 then x = 5

$\therefore \quad I=\int_1^5\left(f(x)+x f'(x)\right)dx$

$=x f(x)\Big|_1^5=5f(5)-f(1)=5\cdot 10-2=48$

45. $I=\int_{-\pi/2}^{\pi/2}\frac{\cos\theta\,d\theta}{(2-\sin\theta)\cos\theta}$ (putting $x=\sin\theta$)

$=\int_0^{\pi/2}\left(\frac{1}{2-\sin\theta}+\frac{1}{2+\sin\theta}\right)d\theta$

$\left[\text{using }\int_{-a}^{a}f(x)dx=\int_0^a[f(x)+f(-x)\,dx\right]$

$=4\int_0^{\pi/2}\frac{d\theta}{4-\sin^2\theta}=\frac{4}{3}\int_0^{\pi/2}\frac{\sec^2\theta\,d\theta}{\frac{4}{3}+\tan^2\theta}=\frac{4}{3}\int_0^{\infty}\frac{d\theta}{t^2+\frac{4}{3}}$

$=\frac{4}{\sqrt{3}}.\frac{\sqrt{3}}{2}.\tan^{-1}\frac{\sqrt{3}\,t}{2}\Big|_0^{\infty}=\frac{2}{\sqrt{3}}.\frac{\pi}{2}=\frac{\pi}{\sqrt{3}}$

46. $I=\lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{\pi}{n}\sin\frac{r\pi}{n}=\pi\int_0^1\sin\pi x\,dx=\pi\left[-\frac{\cos\pi x}{\pi}\right]_0^1$

$=[-\cos\pi+1]=2$

47. $C_n=\int_{\frac{1}{n+1}}^{\frac{1}{n}}\frac{\tan^{-1}(nx)}{\sin^{-1}(nx)}dx$ (put $nx=t$) $\Rightarrow$ C_n

$=\frac{1}{n}\int_{\frac{n}{n+1}}^{1}\frac{\tan^{-1}(t)}{\sin^{-1}(t)}dt$

$L=\lim_{n\to\infty}n^2\cdot C_n=\lim_{n\to\infty}n\cdot\int_{\frac{n}{n+1}}^{1}\frac{\tan^{-1}t}{\sin^{-1}t}dt\quad(\infty\times 0);$

$L==\frac{\int_{\frac{n}{n+1}}^{1}\frac{\tan^{-1}t}{\sin^{-1}t}dt}{\frac{1}{n}}\quad\left(\frac{0}{0}\right)$

applying Leibnitz rule

$L=\lim_{n\to\infty}\frac{0-\frac{\tan^{-1}\frac{n}{n+1}}{\sin^{-1}\frac{n}{n+1}}\left(\frac{1}{(n+1)^2}\right)}{-\frac{1}{n^2}}=\frac{\pi}{4}\cdot\frac{2}{\pi}=\frac{1}{2}$

49. Let $f(a)=\int_a^{a^2}\frac{dx}{x+\sqrt{x}}$

$f'(a)=\frac{2a}{a+a^2}-\frac{1}{a+\sqrt{a}}=0$

$\Rightarrow\ 2a^2+2a\sqrt{a}=a^2+a\ \Rightarrow\ a^2+2a\sqrt{a}-a=0$

$a+2\sqrt{a}-1=0\ \Rightarrow\ (\sqrt{a}+1)^2=2$

$\Rightarrow\ \sqrt{a}=\sqrt{2}-1\ \Rightarrow\ \tan\frac{\pi}{8}$

$a=(\sqrt{2}-1)^2=\tan^2\left(\frac{\pi}{8}\right)$

50. $I=\int_{2-\ell n3}^{3+\ell n}\frac{\ell n(4+x)}{\ell n(4+x)+\ell n(9-x)}dx$

$=\int\frac{\ell n(9-x)}{\ell n(9-x)+\ell n(4-x)}dx$

$2I=\int_{2-\ell n3}^{3+\ell n3}1.dx=3+\ell n3-(2-\ell n3)=1+2\ell n3$

51. We have $\int_{-1}^{1}(px+q)(x^{2n+1}+a_nx+b_n)dx=0$

equating the odd component to be zero and integrating we get

$\frac{2p}{2n+3}+\frac{2a_np}{3}+2b_nq=0$ for all p, q

hence $b_n=0$ and $a_n=-\frac{3}{2n+3}$

52. $\int_0^{n\pi+V}\sqrt{\frac{2\cos^2x}{2}}dx=\int_0^{n\pi+V}|\cos x|\,dx$

$=\int_0^{n\pi}|\cos x|\,dx+\int_{n\pi}^{n\pi+V}|\cos x|\,dx$

$=n\int_0^{\pi}|\cos x|\,dx+\int_0^{V}|\cos x|\,dx$

$=2n+\int_0^{\pi/2}\cos x\,dx-\int_{\pi/2}^{V}\cos x\,dx=2n+2-\sin V$

53. Given $f(x)=\int_{-1}^{x} e^{t^2}dt$;

$h(x)=f(1+g(x))$; $g(x)<0$ for $x>0$

now $h(x)=\int_{-1}^{1+g(x)} e^{t^2}dt = f(1+g(x))$ (given)

differentiating $h'(x)=e^{(1+g(x))^2}\cdot g'(x)$

$h'(1)=e$ (given)

$e^{(1+g(1))^2}\cdot g'(1) = e$

$\therefore$ $(1+g(1))^2 = 1$

$1+g(1)=\pm 1$

$\Rightarrow$ $g(1)=0$ (not possible) or $g(1)=-2$

$\Rightarrow$ **(C)**

54. put $-x/2=t$ $\Rightarrow$ $dx=-2dt$

$$I=-2\int_{-\pi/4}^{-\pi/3}\frac{e^t\sqrt{1+\sin 2t}}{1+\cos 2t}dt$$

$$=2\int_{-\pi/3}^{-\pi/4}\frac{e^t|\sin t+\cos t|}{2\cos^2 t}dt$$

$$=-\int_{-\pi/3}^{-\pi/4}\frac{e^t(\sin t+\cos t)}{\cos^2 t}dt$$

$$=-\int_{-\pi/3}^{-\pi/4}e^t(\sec t\tan t+\sec t)\,dt$$

$$=-\left[e^t\sec t\right]_{-\pi/3}^{-\pi/4} = -[e^{-\pi/4}\sqrt{2}-e^{-\pi/3}2]$$

55. $$I=\underset{h\to 0}{\text{Limit}}\frac{\int_a^x \ell n^2 t\,dt+\int_x^{x+h}\ell n^2 t\,dx-\int_a^x \ell n^2 t\,dt}{h}$$

$$\Rightarrow I=\underset{h\to 0}{\text{Limit}}\frac{\int_x^{x+h}\ell n^2 t\,dx}{h}$$

Using L hospital we get

$$I=\underset{h\to 0}{\text{Limit}}\ \ell n^2(x+h)=\ell n^2 x$$

56. Consider $I=\int_{-1/x}^{1/x}\frac{\ln(1+t^2)}{1+e^t}dt$ ....**(1)**

$$=\int_{-1/x}^{1/x}\frac{\ln(1+t^2)}{1+e^{-t}}dt \quad \text{(Using King)}$$

$$I=\int_{-1/x}^{1/x}\frac{\ln(1+t^2)e^t}{1+e^t}dt \quad ....\textbf{(2)}$$

(1) + (2)

$$2I=\int_{-1/x}^{1/x}\ln(1+t^2)dt = 2\int_0^{1/x}\ln(1+t^2)dt$$

$$\Rightarrow I=\int_0^{1/x}\ln(1+t^2)dt$$

hence $l=\lim_{x\to\infty}x^3\int_0^{1/x}\ln(1+t^2)dt$

$$=\lim_{x\to\infty}\frac{\int_0^{1/x}\ln(1+t^2)dt}{x^{-3}}\quad\left(\frac{0}{0}\text{ form}\right)$$

Using L'Hospital's Rule

$$l=\lim_{x\to\infty}\frac{x^4\ln\left(1+\frac{1}{x^2}\right)\cdot\left(-\frac{1}{x^2}\right)}{-3}$$

$$=\frac{1}{3}\lim_{x\to\infty}x^2\ln\left(1+\frac{1}{x^2}\right)=\frac{1}{3}\lim_{x\to\infty}\ln\left(1+\frac{1}{x^2}\right)^{x^2}\quad(1^\infty\text{ form})$$

$$=\lim_{x\to\infty}\frac{1}{3}x^2\left(1+\frac{1}{x^2}-1\right)=\frac{1}{3}$$

57. $$f'(x)=\frac{1}{\sqrt{1+g^2(x)}}\cdot g'(x);$$

$$f'\left(\frac{\pi}{2}\right)=\frac{g'(\pi/2)}{\sqrt{1+g^2(\pi/2)}};\quad g\left(\frac{\pi}{2}\right)=0$$

$$=g'\left(\frac{\pi}{2}\right)$$

but $g'(x)=[1+\sin(\cos^2 x)](-\sin x)$

$$g'\left(\frac{\pi}{2}\right)=1(-1)=-1$$

hence $f'\left(\frac{\pi}{2}\right)=-1$ as $h'(0^+)=-1$ $\Rightarrow$ **(C)**

59. put $x = 19 + y \Rightarrow dx = dy,$

$$I = \int_0^{18} \left(\{19+y\}^2 + \sin 2y\right) dy = \int_0^{18} (\{y\}^2 + \underbrace{\sin 2\pi y}_{\text{zero}}) dy$$

$$= 18\int_0^1 y^2\, dy = 6$$

60. $$\sqrt{5x-6-x^2} + \frac{\pi x}{2} > x\frac{\pi}{2} \Rightarrow 5x-6-x^2>0$$

61. $$T_r = \frac{1}{\sqrt{\frac{r}{n}}\cdot n\left(3\sqrt{\frac{r}{n}}+4\right)^2}$$

$$S = \frac{1}{n}\sum_1^{4n} \frac{1}{\left(3\sqrt{\frac{r}{n}}+4\right)^2\cdot\sqrt{\frac{r}{n}}} = \int_0^4 \frac{dx}{\sqrt{x}(3\sqrt{x}+4)^2}$$

put $3\sqrt{x}+4 = t \Rightarrow \frac{3}{2}\frac{1}{\sqrt{x}}dx = dt$

$$= \frac{2}{3}\int_4^{10}\frac{dt}{t^2} = \frac{2}{3}\left[\frac{1}{t}\right]_{10}^4 = \frac{2}{3}\left[\frac{1}{4}-\frac{1}{10}\right] = \frac{2}{3}\cdot\frac{6}{40} = \frac{1}{10}$$

62. $$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\frac{2x}{1+x^2}dx \quad ....(1)$$

$$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\left(\frac{-2x}{1-x^4}\right)dx \quad \text{(using King)}$$

$$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\left(\pi - \cos^{-1}\frac{2x}{1-x^4}\right)dx \quad ....(2)$$

add (1) and (2)

$$\therefore\ 2I = \pi\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}dx$$

$$2I = 2\pi\int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4}dx$$

$\therefore\ k = \pi$

63. Given $U_n = \int_0^1 x^n.(2-x)^n\, dx$;

$$V_n = \int_0^1 x^n.(1-x)^n\, dx$$

in U_n put $x = 2t \Rightarrow dx = 2dt$

$$\therefore\ U_n = 2\int_0^{1/2} 2^n.t^n\, 2^n\,(1-t)^n\, dt \quad ....(1)$$

Now $V_n = 2\int_0^{1/2} x^n\,(1-x)^n\, dx$

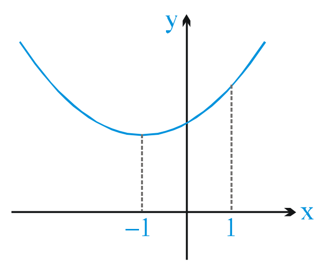
(Using Queen) .....(2)

From (1) and (2)

$U_n = 2^{2n}.V_n \Rightarrow$ **(C)**

65. $$A = \int_{-1}^1 (ax^2+bx+c)dx = 2\int_0^1 (ax^2+c)dx$$

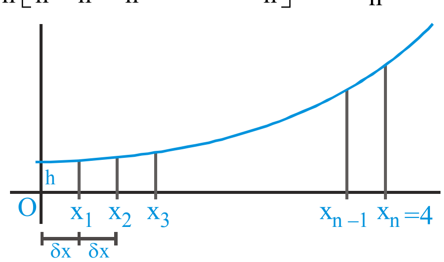
$$= 2\left[\frac{a}{3}+c\right] = \frac{1}{3}[2a+6c]$$



$$\therefore\ A = \frac{1}{3}[f(-1)+4f(0)+f(1)]$$

67. $$L = \lim_{\delta x\to 0}\delta x(x_1+x_2+x_3+.....+x_n)\ 77$$

$$= \lim_{n\to\infty}\frac{4}{n}\left[\frac{4}{n}+\frac{8}{n}+\frac{12}{n}+.......+4\cdot\frac{n}{n}\right] \quad (\delta x = \frac{4}{n})$$



$$= \lim_{n\to\infty}\frac{16}{n^2}(1+2+3+......+n) = \lim_{n\to\infty}\frac{16}{n^2}\cdot\frac{n(n+1)}{2} = 8$$

68. $S = \lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)...\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$

$\ell n\, S =$

$\lim_{n\to\infty}\frac{1}{n}\left[\ell n\left(1+\frac{1}{n^2}\right)+\ell n\left(1+\frac{2^2}{n^2}\right)+........+\ell n\left(1+\frac{n^2}{n^2}\right)\right]$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ell n\left(1+\frac{r^2}{n^2}\right) = \int_0^1 1.\ell n(1+x^2)\,dx$

$\Rightarrow \ \ell n\, S = \ell n[2.e^{\frac{\pi}{2}-2}]$

$\Rightarrow \ S = 2e^{\frac{\pi}{2}-2} = 2e^{\frac{\pi}{2}}\cdot e^{-2} = \frac{2}{e^2}e^{\frac{\pi}{2}}$

69. $I = \int_0^1 x\ \ln\left(\frac{x+2}{2}\right)dx$

$= \int_0^1 x\,(\ln(x+2) - \ln 2)\ \ dx$

$\therefore \ \ I = \int_0^1 x\ \ln(x+2)dx\ - \ln 2\int_0^1 x\ dx\ ;$

hence $\ \ I = \left.\ln(x+2).\frac{x^2}{2}\right]_0^1 - \int_0^1 \frac{x^2}{x+2}\ dx - \frac{\ln 2}{2}$

$= \frac{1}{2}\ln 3 - \int_0^1 \frac{x^2-4+4}{x+2}\ dx - \frac{\ln 2}{2}$

$\Rightarrow \frac{1}{2}\ln\frac{3}{2} - \int_0^1\left((x-2) + \frac{4}{x+2}\right)dx$ now proceed

70. $I(a) = \int_0^{\pi}\left(\frac{x^2}{a^2} + a^2\sin^2 x + 2x\sin x\right)dx$

$\left(\int_0^{\pi} x\sin x\,dx = \pi\right)$

$\therefore \ \ I(a) = \frac{\pi^3}{3a^2} + \frac{\pi a^2}{2} + 2\pi = \pi\left[\frac{\pi^2}{3a^2} + \frac{a^2}{2}\right] + 2\pi$

$= \pi\left[\left(\frac{\pi}{\sqrt{3}a} - \frac{a}{\sqrt{2}}\right)^2 + \frac{2\pi}{\sqrt{6}}\right] + 2\pi$

$I(a)$ is minimum when $\frac{\pi}{\sqrt{3}a} = \frac{a}{\sqrt{2}} \ \Rightarrow \ a^2 = \pi\sqrt{\frac{2}{3}}$

$\Rightarrow \ a = \sqrt{\pi\sqrt{\frac{2}{3}}}$

Also $\ \ I(a)|_{min} = 2\pi + \pi^2\sqrt{\frac{2}{3}}$

71. $\left|\int_{10}^{19}\frac{\sin x}{1+x^8}\right| \le \int_{10}^{19}\frac{|\sin x|}{1+x^8}dx \le \int_{10}^{19}\frac{dx}{1+x^8} < \int_{10}^{19}\frac{dx}{x^8}$

$= \left[\frac{x^{-7}}{-7}\right]_{10}^{19}$

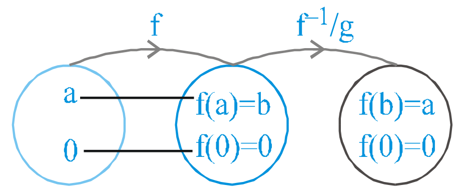
$= -\frac{1}{7}[19^{-7} - 10^{-7}] = \frac{1}{7}[10^{-7} - 19^{-7}]$

Ans : $\ < 10^{-7}$

73. $y = f(x) \ \Rightarrow \ \ x = g(y)$ and $dy = f'(x)\,dx$

$I = \int_0^a f(x)dx + \int_0^b g(y)dy\ ;\ \ y = f(x) \Rightarrow x = f^{-1}(y) = g(y)$

$= \int_0^a f(x)dx + \int_0^a x f'(x)dx$



$= \int_0^a \left(f(x) + x f'(x)\right)dx = \left[x f(x)\right]_0^a = a f(a) = ab$

75. $L = \lim_{n\to\infty}\frac{n}{(n!)^{1/n}} = \lim_{n\to\infty}\frac{n}{(1\cdot 2\cdot 3\cdot ....n)^{1/n}}$

$= \lim_{n\to\infty}\frac{1}{\left(\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}........\frac{n}{n}\right)^{1/n}}$

$\ln L = -\frac{1}{n}\left[\ln\left(\frac{1}{n}\right) + \ln\left(\frac{2}{n}\right) + \ln\left(\frac{3}{n}\right) + ... + + \ln\left(\frac{n}{n}\right)\right]$

general term of $\ln L$ is

$T_r = -\frac{1}{n}\ \ln\ \frac{r}{n}$

$\therefore \ \ S = \lim_{n\to\infty} -\frac{1}{n}\sum_{r=1}^{n}\ln\frac{r}{n} = -\int_0^1 \ln x\,dx = -[x\ln x - x]_0^1$

$= -[(0-1)-(0)] = 1$

$\therefore \ \ $ Hence $\ln L = 1 \ \ \Rightarrow \ \ L = e$

EXERCISE - 2

Part # I : Multiple Choice

1. $I=\int_0^1(10x^4-3x^2-1)dx = 2x^5-x^3-x\Big]_0^1 = 0 \Rightarrow A.$

Since f(x) is even hence must have a root in (–1, 0) also

$\Rightarrow$ A and B $\Rightarrow$ C

3. Given $f(f(x)) = -x+1$

replacing $x \to f(x)$

$f(f(f(x))) = -f(x)+1$

$f(1-x) = -f(x)+1$

$f(x)+f(1-x)=1$ ....(1) $\Rightarrow$ **(A)**

now $J=\int_0^1 f(x)dx = \int_0^1 f(1-x)dx$

$2J=\int_0^1 (f(x)+f(1-x))dx\ ;\ 2J=\int_0^1 dx = 1 \Rightarrow J=\frac{1}{2}$

4. $v=\int_0^\infty \frac{x^2dx}{x^4+7x^2+1}$

Put $x=\frac{1}{t} \Rightarrow dx=\frac{-1}{t^2}dt$

$v=-\int_\infty^0 \frac{\frac{1}{t^2}\cdot\frac{1}{t^2}dt}{\frac{1}{t^4}+\frac{7}{t^2}+1} = \int_0^\infty \frac{dx}{x^4+7x^2+1}$

v = u

Hence $2u=\int_0^\infty\left(\frac{x^2+1}{x^4+7x^2+1}\right)dx$

$=\int_0^\infty\left(\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+7}\right)dx = \int_0^\infty \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+3^2} = \int_0^\infty \frac{dt}{t^2+9}$

$=\frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^\infty$

$2u=\pi/3$

5. $f(x+\pi)=\int_0^{x+\pi}(\cos^4t+\sin^4t)\,dt$

$=\int_0^x(\cos^4t+\sin^4t)\,dt + \int_x^{x+\pi}(\sin^4t+\cos^4t)\,dt$

$=f(x)+\int_x^{x+\pi}(\sin^4t+\cos^4t)\,dt$

put $t=x+y=f(x)+\int_0^\pi(\sin^4y+\cos^4y)\ dy=f(x)+f(\pi)$

6. Given $(f'(x))^2+(g(x))^2=1$

$f(x)+\int_0^x g(t)dt = \sin x(\cos x-\sin x)$

differentiating both sides

$f'(x)+g(x)=\cos 2x-\sin 2x$ .....**(1)**

squaring (1)

$(f'(x))^2+(g(x))^2+2f'(x)\cdot g(x)=1-\sin 4x$

$1+2f'(x)\cdot g(x)=1-\sin 4x$

$\therefore\ 2f'(x)g(x)=-\sin 4x$

now, substituting $g(x)=-\frac{\sin 4x}{2f'(x)}$ in equation (1)

$f'(x)-\frac{\sin 4x}{2f'(x)}=\cos 2x-\sin 2x$

put $f'(x)=t$

$2t^2-2(\cos 2x-\sin 2x)t-\sin 4x=0$

$\Rightarrow\ t=\frac{2(\cos 2x-\sin 2x)\pm\sqrt{4(1-\sin 4x)+8\sin 4x}}{4}$

$\therefore\ 4t=2(\cos 2x-\sin 2x)\pm\sqrt{4(1-\sin 4x)+8\sin 4x} \Rightarrow$

$2t=(\cos 2x-\sin 2x)\pm\sqrt{1+\sin 4x}$

taking (+)ve sign, $2t=\cos 2x-\sin 2x+\cos 2x+\sin 2x$

$\Rightarrow\ t=\cos 2x$

taking (–) ve sign, $t=-\sin 2x$

hence $f'(x)=\cos 2x$ or $f'(x)=-\sin 2x$

Integrating

$f(x)=\frac{1}{2}\sin 2x+C_1$ or $f(x)=\frac{\cos 2x}{2}+C_2$

$f(0)=0 \Rightarrow C_1=0$ and $C_2=-1/2$

$\therefore\ f(x)=\frac{1}{2}\sin 2x$ or $f(x)=\frac{\cos 2x-1}{2}$

if $f'(x)=\cos 2x$ then $g(x)=-\sin 2x$

if $f'(x)=-\sin 2x$ then $g(x)=\cos 2x$

i.e. $f(x)=\frac{1}{2}\sin 2x$ and $g(x)=-\sin 2x \Rightarrow$ **(C)**

$f(x)=\frac{\cos 2x-1}{2}$ and $g(x)=\cos 2x \Rightarrow$ **(D)**

8. **(A)** $I = \int_0^{\pi/2} \ln(\cot x)\,dx \Rightarrow I = \int_0^{\pi/2} \ln(\tan x)\,dx$

$I = -\int_0^{\pi/2} \ln(\cot x)\,dx \Rightarrow I = -I \Rightarrow I = 0$

(B) $I = \int_0^{2\pi} \sin^3 x\,dx = -\int_0^{2\pi} \sin^3 x\,dx \Rightarrow I = 0$

(C) at $x = 1/t$, $I = \int_e^{1/e} \frac{-(1/t^2)dt}{-1/t(\ln t)^{1/3}} = -\int_{1/e}^{e} \frac{dt}{t(\ln t)^{1/3}}$

$I = -I \Rightarrow I = 0$

(D) $\sqrt{\frac{1+\cos 2x}{2}} > 0 \Rightarrow \int_0^{\pi} \sqrt{\frac{1+\cos 2x}{2}}\,dx > 0$

9. Numerator $= 2(x^2+2x+2)-(x+1)$

10. $f(x) = 2\int_0^1 \underbrace{(1-t)}_{I}\underbrace{\cos(xt)}_{II}\,dt$

$= 2\left[(1-t)\frac{\sin xt}{x}\Big|_0^1 + \frac{1}{x}\int_0^1 \sin xt\,dt\right] = 2\left[0 - \frac{1}{x^2}\cos xt\Big|_0^1\right]$

$f(x) = 2\left[\frac{1-\cos x}{x^2}\right] \quad (x \neq 0)$

if $x = 0$ then $f(x) = \int_{-1}^{1} (1-|t|)\,dt = 2\int_0^1 (1-t)\,dt = 1$

$\Rightarrow$ **(C)** is correct

hence $f(x) = \begin{cases} 2\left(\frac{1-\cos x}{x^2}\right) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

$\therefore$ f is continuous at $x = 0 \Rightarrow$ **(D)** is correct]

12. We have $f(x) = x^2 + ax^2 + bx^3$

where $a = \int_{-1}^{1} t\cdot f(t)\,dt$ and $b = \int_{-1}^{1} f(t)\,dt$

How $a = \int_{-1}^{1} t[(a+1)t^2 + bt^3]\,dt$

$a = 2b\int_0^1 t^4\,dt = \frac{2b}{5}$ .... **(1)**

Again $b = \int_{-1}^{1} f(t)\,dt = \int_{-1}^{1} ((a+1)t^2 + bt^3)\,dt$

$= 2\int_0^1 (a+1)t^2\,dt$

$b = \frac{2(a+1)}{3}$ ....**(2)**

From (1) and (2)

$\frac{5a}{2} = \frac{2(a+1)}{3} \quad \left(\frac{5}{2} - \frac{2}{3}\right)a = \frac{2}{3} \quad \Rightarrow \frac{11}{6}a = \frac{2}{3}$

$a = \frac{4}{11}$ and $b = \frac{10}{11}$

Hence $\int_{-1}^{1} t\cdot f(t)\,dt = \frac{4}{11}$ and $\int_{-1}^{1} f(t)\,dt = \frac{10}{11}$

$\therefore \quad f(x) = (a+1)x^2 + bx^3$

$\left.\begin{aligned} f(1) &= (a+1)+b \\ f(-1) &= (a+1)-b \end{aligned}\right] \Rightarrow f(1) + f(-1) = 2(a+1) = \frac{30}{11}$

and $f(1) - f(-1) = 2b = \frac{20}{11} \quad \Rightarrow$ **B, D correct.**

13. $f(-x) = -f(x)$ ... (1)

$f(x+2) = f(x)$ ... (2)

$g(2n) = \int_0^{2n} f(t)\,dt = n\int_0^2 f(t)\,dt$

$\Rightarrow g(2n) = n\,g(2)$ ... **(3)**

Now $g(-x) = \int_0^{-x} f(t)\,dt$

put $t = -z \Rightarrow dt = -dz$

$= \int_0^x f(-z)(-dz) = -\int_0^x f(-z)\,dz$ (from (1))

$= \int_0^x f(t)\,dt = g(x) \quad \therefore g(-x) = g(x)$

Again $g(x+2) = \int_0^{x+2} f(t)\,dt + \int_x^{x+2} f(t)\,dt$

$\therefore \; g(x+2) = \int_0^x f(t)\,dt + \int_0^2 f(t)\,dt \quad (\because f \to \text{period})$

$\Rightarrow g(x+2) = g(x) + g(2)$ .... **(4)**

Putting $x = 0, 2, ...$

$g(2) = g(0 + g(2) \Rightarrow g(0) = 0$

$g(4) = g(2) + g(2) \Rightarrow g(4) = 2g(2)$

putting $x \to -x$ we get

$g(2-x) = g(-x) + g(2) = g(x) + g(2)$

at $x = 2$

$g(0) = 2g(2) \Rightarrow g(2) = 0$

$\therefore g(0) = g(\pm 2) = g(\pm 4) = ... = 0$

from (3) $g(2n) = 0$

& from (4) $g(x+2) = g(x) \Rightarrow$ prd. of $g(x)$ is 2

14. Let $f(x) = \frac{ax^5}{5} + \frac{bx^3}{3} + cx$

It is continuous & differentiable everywhere

Now $f(0) = 0, f(1) = \frac{3a + 5b + 15}{15} = 0$

and $f(-1) = 0$

so $f'(x) = 0$ will have at least one root in $(-1, 0)$ atleast one root in $(0, 1)$, so it will have atleast two roots in $(-1, 1)$

16. $f(x) = \frac{\ell n^2 x}{2} = 2 \Rightarrow$ C, D

17. $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \frac{n}{(n+r)(n+2r)}$

$= \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \frac{n}{n^2\left(1+\frac{r}{n}\right)\left(1+\frac{2r}{n}\right)}$

$= \int_0^1 \frac{dx}{(1+x)(1+2x)} = \int_0^1 \left(\frac{2}{1+2x} - \frac{1}{1+x}\right) dx$

$= \ell n(1+2x) - \ell n(1+x)\Big|_0^1 = \ell n3 - \ell n2 = \ell n\frac{3}{2}$

18. $I_1 = \int_0^{\pi/2} \cos(\pi \sin^2 x)\, dx$

Use king

$I_1 = \int_0^{\pi/2} \cos(\pi \cos^2 x)\, dx$

On adding

$2I_1 = \int_0^{\pi/2} \cos(\pi \sin^2 x) + \cos(\pi\cos^2 x)\, dx$

$= \int_0^{\pi/2} 2\cos\left(\frac{\pi}{2}\right)\cdot\cos\left(\frac{\pi}{2}\cos 2x\right) dx = 0$

$\Rightarrow I_1 = 0$ ...**(1)**

$I_2 = \int_0^{\pi/2} \cos(\pi(1-\cos 2x)\, dx = -\int_0^{\pi/2} \cos(\pi\cos 2x)\, dx$

$= -\frac{1}{2}\int_0^{\pi} \cos(\pi\cos t)\, dt$ [Put $2x = t$]

$= -\frac{2}{2}\int_0^{\pi/2} \cos(\pi \cos t)\, dt = I_3 \Rightarrow I_2 + I_3 = 0$

$I_2 = -\int_0^{\pi/2} \cos(\pi\sin t)\, dt$

$\therefore I_2 + I_3 = 0$ ...**(2)**

Hence, $I_1 + I_2 + I_3 = 0$ ...**(3)**

29. $I = \int_{\infty}^{0} \frac{ze^{-z}}{\sqrt{1 - e^{-2z}}} dz$

put $e^{-z} = \sin\theta$

$I = -\int_0^{\pi/2} \frac{\ell n(\sin\theta)(-\cos\theta)d\theta}{\sqrt{1-\sin^2\theta}} = \int_0^{\pi/2} \ell n \sin\theta\, d\theta$

$= \frac{-\pi}{2}\ell n2$

21. I.B.P. taking 1 as the 2[nd] and $\frac{1}{(1+x^2)^n}$ as the 1[st] function

23. Consider $f(x) = \int_{-x}^{x} (\underbrace{t\sin at}_{even} + \underbrace{bt}_{odd} + \underbrace{c}_{even})\, dt$

$= 2\int_0^x (t\sin at + c)\, dt$

$= 2\left[-t\frac{\cos at}{a}\Big|_0^x + \int_0^x \frac{\cos at}{a} dt + ct\Big|_0^x\right]$ (using I.B.P.)

$= 2\left[\frac{-x\cos ax}{a} + \frac{1}{a^2}\sin ax + cx\right]$

$\therefore \lim\limits_{x\to 0} \frac{f(x)}{x} = \lim\limits_{x\to 0} 2\left[-\frac{\cos ax}{a} + \frac{\sin ax}{a\cdot ax} + c\right]$

$= 2\left[-\frac{1}{a} + \frac{1}{a} + c\right] = 2c$

24. Consider $I = \int_a^{\infty} \frac{n\, dx}{n^2\left(x^2 + \frac{1}{n^2}\right)} = \frac{1}{n}\cdot n\left(\tan^{-1} nx\right)_a^{\infty}$

$$= \left(\frac{\pi}{2} - \tan^{-1} an\right)$$

$$\therefore \quad L = \lim_{n\to\infty}\left(\frac{\pi}{2} - \tan^{-1} an\right) = \begin{cases} \pi & \text{if } a < 0 \\ \pi/2 & \text{if } a = 0 \\ 0 & \text{if } a > 0 \end{cases}$$

Part # II : Assertion & Reason

2. $I = \int_{-\pi/4}^{\pi/4} \frac{dx}{1+\sin x}$

$2I = \int_{-\pi/4}^{\pi/4} \frac{2\,dx}{1-\sin^2 x} \quad \Rightarrow \quad I$

$= \int_{-\pi/4}^{\pi/4} \frac{dx}{\cos^2 x}$

$I = 2\int_0^{\pi/4} \sec^2 x\,dx \neq 0 \quad \Rightarrow \quad$ Statement-1 is false

3. $\int_0^t \{x\}\,dx = \int_0^{[t]} \{x\}\,dx + \int_{[t]}^t \{x\}\,dx = [t]\int_0^1 x\,dx + \int_0^{\{t\}} x\,dx$

$= \frac{[t]}{2} + \frac{\{t\}^2}{2}$

$\therefore$ statement-2 is true.

$\int_0^{5.5} \{x\}\,dx = \frac{5}{2} + \frac{(.5)^2}{2} = \frac{21}{8}$

$\therefore$ statement-1 is true and is explained by statement-2.

5. Statement-1 :

Put $x = \frac{1}{t} \quad \Rightarrow \quad dx = -\frac{1}{t^2}\,dt$

$I = -\int_3^{1/3} t\,\text{cosec}^{99}\left(\frac{1}{t} - t\right)\frac{1}{t^2}\,dt$

$= -\int_{1/3}^{3} \frac{1}{t}\,\text{cosec}^{99}\left(t - \frac{1}{t}\right)dt$

$I = -I \quad \Rightarrow \quad 2I = 0 \quad \Rightarrow \quad I = 0$

6. Let $\int_0^1 f(t)\,dt = k$, so

$f(x) = xk + 1$, now

$\int_0^1 (kt+1)\,dt = k$

$\Rightarrow \quad \frac{k}{2} + 1 = k$, so $k = 2$

$\therefore \quad f(x) = 2x + 1$,

Also $\quad \int_0^3 f(x)\,dx = 12$

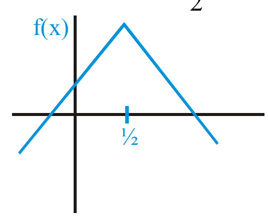
$\Rightarrow$ option **(C)** is correct.

7. $f(x) = -x^2 + x + 1$

$f'(x) = 1 - 2x$

$f'(x) > 0 \Rightarrow 1 - 2x > 0 \Rightarrow x < \frac{1}{2}$

$f'(x) < 0 \Rightarrow 1 - 2x < 0 \Rightarrow x > \frac{1}{2}$



$\Rightarrow$ f(x) is increasing in $(0, \frac{1}{2})$ and decreasing in $(\frac{1}{2}, 1)$

Now $g(x) = \max\{f(t)\,;\, 0 \le t \le x\}$

$$= \begin{cases} x - x^2 + 1 & 0 \le x \le \frac{1}{2} \\ \frac{5}{4} & \frac{1}{2} \le x \le 1 \end{cases}$$

$\int_0^1 g(x)\,dx = \int_0^{1/2} (x - x^2 + 1)\,dx + \int_{1/2}^1 5/4\,dx = \frac{29}{24}$

8. Statement-1 :

$I = \int_0^{\pi} x\tan x\cos^3 x\,dx$ ....**(i)**

$I = \int_0^{\pi} (\pi - x)\tan x\cos^3 x\,dx$ ....**(ii)**

(i) + **(ii)**

$2I = \pi\int_0^{\pi} \tan x.\cos^3 x\,dx$

$I = \frac{\pi}{2}\int_0^{\pi} \tan x \cos^3 x dx$ (true)

Statement-2 :

$I = \int_a^b x f(x)dx$ ....(i)

$I = \int_a^b (a+b-x) f(a+b-x)dx$ ....(ii)

(i) + (ii)

$2I = (a+b)\int_a^b f(x)dx$

{If $f(a+b-x) = f(x)$

$I = \frac{a+b}{2}\int_a^b f(x)dx$

Hence Statement-2 false

but if $f(a+b-x) \neq f(x)$, then $I \neq \frac{a+b}{2}\int_a^b f(x)$

9. If f(x) is odd $\Rightarrow$ f '(x) is even but converse is not true

e.g. if $f'(x) = x \sin x$

then $f(x) = \sin x - x\cos x + C$

$f(-x) = -\sin x + x\cos x + C$

$f(x) + f(-x) =$ constant which need not to be zero

for S-1: $f(x) = \int_0^x \sqrt{1+t^2}\, dt$; $g(x) = \sqrt{1+x^2}$

$f(-x) = \int_0^{-x} \sqrt{1+t^2}\, dt$; $t = -y$

$f(-x) = -\int_0^x \sqrt{1+y^2}\, dy$

$\therefore$ $f(x) + f(-x) = 0 \Rightarrow$ f is odd and g is obviously even

12. $P(x) = ax^2 + bx + c$; $f(x) = \sin^3 x$

$I = \int_0^{2\pi} \underbrace{P(x)}_{I} \cdot \underbrace{f''(x)}_{II} dx$

Using I.B.P.

$= \underbrace{P(x)\cdot f'(x)\Big|_0^{2\pi}}_{\text{zero}} - \int_0^{2\pi} \underbrace{P'(x)}_{I} \cdot \underbrace{f'(x)}_{II} dx$

$= -\left[P'(x)\cdot f(x)\Big|_0^{2\pi} - \int_0^{2\pi} P''(x)\cdot f(x)dx\right]$

$= \int_0^{2\pi} P''(x)\cdot f(x)dx = 2\int_0^{2\pi} \sin^3 x\, dx = 0$]

13. $\int_{-\pi}^{\pi} (\sin mx.\sin nx)dx = 0$ if $m \neq n$

and $\int_{-\pi}^{\pi} (\sin mx.\sin nx)dx = \pi$ if $m = n$

$\therefore$ $a = \cos 0 = 1$ and $b = \cos\pi = -1$

$\therefore$ $a + b = 0$

EXERCISE - 3

Part # I : Matrix Match Type

1. **(A)** Apply L'Hospital's rule twice or use expansion of $e^{x \cos x}$

(B) $x = u^6 \Rightarrow dx = 6u^5\,du$

$$I = \int_0^1 \frac{6u^5 du}{u^3+u^2} = 6\int_0^1 \frac{u^3+1-1}{u+1}du = 5 - 6\ln 2$$

$\Rightarrow a + b = 5 - 6 = -1$ Ans.

(C) $e^n \int_0^n e^{-\theta}(\sec^2\theta - \tan\theta)\,d\theta = 1$

put $-\theta = t$; $d\theta = -dt$

$$-e^n \int_0^{-n} e^t(\sec^2 t + \tan t)\,dt = 1$$

[use $\int e^x(f(x) + f'(x)) = e^x f(x)$]

$$-e^n\left[e^t \tan t\right]_0^{-n} = 1 \Rightarrow -e^n[-e^{-n}\tan n] = 1$$

$\Rightarrow \tan n = +1$ Ans.

(D) $$\lim_{n\to\infty}\frac{\int_{\frac{1}{n+1}}^{\frac{1}{n}} \tan^{-1}(nx)\,dx}{\int_{\frac{1}{n+1}}^{\frac{1}{n}} \sin^{-1}(nx)\,dx} \quad (\text{put } nx = t);$$

$$\therefore \lim_{n\to\infty}\frac{\frac{1}{n}\int_{\frac{n}{n+1}}^{1} \tan^{-1}(t)\,dt}{\frac{1}{n}\int_{\frac{n}{n+1}}^{1} \sin^{-1}(t)\,dt} \quad \left(\frac{0}{0}\right)$$

use L'Hospital's rule

$$\lim_{n\to\infty}\frac{\tan^{-1}\left(\frac{n}{n+1}\right)}{\sin^{-1}\left(\frac{n}{n+1}\right)} = \frac{\frac{\pi}{4}}{\frac{\pi}{2}} = \frac{1}{2}$$

2. **(A)** Let $a = 2008$

$$I = \int_0^1 \left(1 + ax^a\right)e^{x^a}dx\ ; \quad I = \int_0^1\left(e^{x^a} + ax^a e^{x^a}\right)dx$$

(Note: $ax^a = ax \cdot x^{a-1}$)

$$\therefore \quad I = \int_0^1\left(e^{x^a} + e^{x^a}\cdot x \cdot ax^{a-1}\right)dx = \int_0^1 (f(x) + x f'(x))\,dx$$

where $f(x) = e^{x^a}$

hence $I = \left[xe^{x^a}\right]_0^1 = e$ **Ans.** $\Rightarrow$ **(S)**

(B) $I = I_1 + I_2$

Consider $I_2 = \int_1^{1/e}\sqrt{-\ln x}\,dx$

Put $\sqrt{-\ln x} = t \Rightarrow -\ln x = t^2 \Rightarrow x = e^{-t^2}$

$\Rightarrow dx = -2t\,e^{-t^2}\,dt$

$$\therefore \quad I_2 = \int_0^1 \underset{\text{I}}{t}\cdot\underset{\text{II}}{(-2t\,e^{-t^2})}\,dt = t\,e^{-t^2}\Big]_0^1 - \int_0^1 e^{-t^2}dt$$

$$= \frac{1}{e} - \int_0^1 e^{-t^2}dt$$

hence $I = \int_0^1 e^{-x}dx + \frac{1}{e} - \int_0^1 e^{-t^2}dt = \frac{1}{e} = e^{-1}$ **Ans.**

$\Rightarrow$ **(P)**

note that if $f(x) = e^{-x^2}$ then $f^{-1}(x) = \sqrt{-\ln x}$

(C) $$L = \lim_{n\to\infty}\left(\left(\frac{1}{n}\right)^1\cdot\left(\frac{2}{n}\right)^2\cdot\left(\frac{3}{n}\right)^3 \ldots\ldots\left(\frac{n}{n}\right)^n\right)^{\frac{1}{n^2}}$$

$\ln L$

$$= \lim_{n\to\infty}\frac{1}{n^2}\left(1\cdot\ln\left(\frac{1}{n}\right) + 2\cdot\ln\left(\frac{2}{n}\right) + 3\cdot\ln\left(\frac{3}{n}\right) + \ldots\ldots + n\cdot\ln\left(\frac{n}{n}\right)\right)$$

general term of $\ln L = \frac{r}{n^2}\ln\frac{r}{n}$

$$\text{Sum} = \frac{1}{n}\cdot\sum_{r=1}^{n}\frac{r}{n}\ln\left(\frac{r}{n}\right)$$

$$\ln L = \int_0^1 x\ln x\,dx = \frac{x^2\cdot\ln x}{2}\Bigg]_0^1 - \frac{1}{2}\int_0^1 x^2\frac{1}{x}dx = 0$$

$$-\frac{1}{2}\frac{x^2}{2}\Bigg]_0^1 = -\frac{1}{4}$$

$L = e^{-1/4}$ **Ans.** $\Rightarrow$ **(Q)**

4. $f(\theta) = \left.\frac{(x+\sin\theta)^3}{3}\right]_0^1 = \frac{(1+\sin\theta)^3 - \sin^3\theta}{3}$

and $g(\theta) = \left.\frac{(x+\cos\theta)^3}{3}\right]_0^1 = \frac{(1+\cos\theta)^3 - \cos^3\theta}{3}$

$f(\theta) = \frac{1+3\sin\theta+3\sin^2\theta}{3}$; $g(\theta)$

$= \frac{1+3\cos\theta+3\cos^2\theta}{3}$

$f(\theta) - g(\theta) = (\sin\theta - \cos\theta) + (\sin^2\theta - \cos^2\theta)$

$f(\theta) - g(\theta) = (\sin\theta - \cos\theta)(1 + \sin\theta + \cos\theta)$

now verify all matching.

5. (A) $f(n) = \frac{\log 3}{\log 2}\cdot\frac{\log 4}{\log 3}.......\frac{\log n}{n-1}$

$f(n) = \frac{\log n}{\log 2} = \log_2(n)$

$\therefore\ f(2^k) = \log_2(2^k) = k$

$\therefore\ \sum_{k=2}^{100} f(2^k) = \sum_{k=2}^{100} k = 2+3+4+......+100 = 5050 - 1$

$= 5049$ **Ans.**

(B) $f(x) = \sqrt{1+x\sqrt{1+(x+1)\sqrt{1+(x+2)(x+4)}}}$

$= \sqrt{1+x\sqrt{1+(x+1)\sqrt{x^2+6x+9}}}$

$= \sqrt{1+x\sqrt{1+(x+1)(x+3)}}$

$= \sqrt{1+x\sqrt{x^2+4x+4}} = \sqrt{1+x(x+2)}$

$= \sqrt{x^2+2x+1} = (x+1)$

$\therefore\ I = \int_0^{100}(x+1)\,dx = \left.\frac{(x+1)^2}{2}\right|_0^{100} = \frac{(101)^2 - 1^2}{2}$

$= \frac{100\times 102}{2} = 5100$ **Ans.**

(C) A.P. is a, (a + d), (a + 2d) .... (a + 98d)

sum of odd terms = 2550

$\underbrace{a+(a+2d)+(a+4d)+.....+(a+98d)}_{50\text{ terms}} = 2550$

$\frac{50}{2}[2a+98d] = 2550$ or $50[a+49d] = 2550$ or $a+49d = 51$

This is the 50^{th} term of the A.P. Hence $S_{99} = 51\times 99 = 5049$

6. (A) $\int_0^{\pi/2} \ell n\,(\tan x + \cot x)\ dx$

$= \int_0^{\pi/2} -\ell n\,(\sin x\cdot\cos x)\ dx$

$= -\int_0^{\pi/2} \ell n \sin x\, dx - \int_0^{\pi/2} \ell n\ \cos x\, dx$

$= -2\left(-\frac{\pi}{2}\ell n 2\right) = \pi\ \ell n\,2$

(B) $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{(\sin x + \cos x)^2}\ dx$

$= \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{\left(\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)\right)^2}\ dx$

$= \int_0^{\pi/2} \frac{\cos x - \sin x}{(\sin x + \cos x)^2}\ dx = -I \qquad \therefore\ I = 0$

(C) $I = \int_0^{2\pi} x\ \sin^2 x\cos^2 x\ dx$

$= \int_0^{2\pi} (2\pi - x)\ \sin^2 x\cos^2 x\ dx$

$I = \pi\int_0^{2\pi} \sin^2 x\cos^2 x\ dx$

$= \frac{\pi}{4}\int_0^{2\pi} 4\sin^2 x\cos^2 x\ dx = \frac{\pi}{4}\int_0^{2\pi}\sin^2 2x\ dx$

$= \frac{\pi}{8}\int_0^{2\pi}(1-\cos 4x)\ dx = \frac{\pi}{8}\left.\left(x - \frac{\sin 4x}{4}\right)\right|_0^{2\pi}$

$= \frac{\pi}{8}(2\pi) = \frac{\pi^2}{4}$

(D) $\int_0^{\pi/2} (2\ \ell n \sin x - \ell n\ 2 - \ell n \sin x - \ell n \cos x)\ dx$

$= -\int_0^{\pi/2} \ell n\ 2\ dx = -\frac{\pi}{2}\ \ell n\ 2$

8. (A) $f'(x) = \dfrac{g'(x)}{\sqrt{1+g^3(x)}}$

and $g'(x) = [1 + \sin(\cos^2 x)]\,(-\sin x)$

hence $f'(x) = \dfrac{[1+\sin(\cos^2 x)](-\sin x)}{\sqrt{1+g^3(x)}}$

$f'\left(\dfrac{\pi}{2}\right) = \dfrac{1+0}{\sqrt{1+g^3(\pi/2)}} = \dfrac{-1}{1+0} = -1$

as $g\left(\dfrac{\pi}{2}\right) = 0$

$\therefore$ $f'\left(\dfrac{\pi}{2}\right) = -1$ Ans.

(C) Maximum when $a = -1;\ b = 2$

$\Rightarrow$ $a + b = 1$

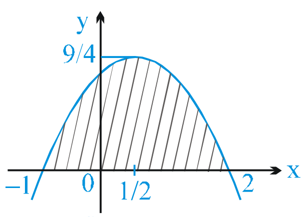


figure (C)

(D) If $\displaystyle\lim_{x\to 0} \frac{\sin 2x}{x^3} + a + \frac{b}{x^2} = 0$

$\Rightarrow$ $\displaystyle\lim_{x\to 0} \frac{\sin 2x + ax^3 + bx}{x^3} = 0$

for limit to exist $2 + b = 0$ $\Rightarrow$ $\boxed{b = -2}$

$\therefore$ $\displaystyle\lim_{x\to 0} \frac{\sin 2x + ax^3 - 2x}{x^3} = 0$

apply LHS rule, $\displaystyle\lim_{x\to 0} \frac{2\cos 2x + 3ax^2 - 2}{3x^2} = 0$

$\therefore$ $a = \displaystyle\lim_{x\to 0} \frac{2(1-\cos 2x)}{3x^2}$

$\Rightarrow$ $\boxed{a = \dfrac{4\sin^2 x}{3x^2} = \dfrac{4}{3}}$

$\therefore$ $3a + b = 3 \cdot \dfrac{4}{3} - 2 = 2$ Ans.

9. (A) $I = \displaystyle\int_0^{\pi} x\left(\sin^2(\sin x) + \cos^2(\cos x)\right)dx$;

$I = \displaystyle\int_0^{\pi} (\pi - x)\left(\sin^2(\sin x) + \cos^2(\cos x)\right)dx$

$2I = \pi\displaystyle\int_0^{\pi} \left(\sin^2(\sin x) + \cos^2(\cos x)\right)dx$;

$2I = 2\pi\displaystyle\int_0^{\pi/2} \left(\sin^2(\sin x) + \cos^2(\cos x)\right)dx$ (Using Queen)

also $I = \pi\displaystyle\int_0^{\pi/2} \left(\sin^2(\cos x) + \cos^2(\sin x)\right)dx$;

use king and add,

$2I = \pi\displaystyle\int_0^{\pi/2} 2\,dx \Rightarrow I = \pi\int_0^{\pi/2} dx = \frac{\pi^2}{2}$ **Ans.** $\Rightarrow$ **(Q)**

(B) $I = \displaystyle\int_0^{\pi} \frac{\pi - x}{1+\sin^2 x}dx$;

$\therefore$ $2I = \pi\displaystyle\int_0^{\pi} \frac{dx}{1+\sin^2 x}$

$2I = 2\pi\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sin^2 x} \Rightarrow I = \pi\int_0^{\pi/2} \frac{\sec^2 x\,dx}{1+2\tan^2 x}$

$= \dfrac{\pi}{2}\displaystyle\int_0^{\pi/2} \frac{\sec^2 x\,dx}{\tan^2 x + (1/\sqrt{2})^2} = \frac{\pi\sqrt{2}}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{2\sqrt{2}}$ **Ans.**

$\Rightarrow$ **(S)**

(C) $f(x) = 2\sin\sqrt{x}$

$f'(x) = \dfrac{2\cos\sqrt{x}}{2\sqrt{x}} = \dfrac{\cos\sqrt{x}}{\sqrt{x}}$

$x f'(x) = \sqrt{x}\cos\sqrt{x}$

$\therefore$ $I = \displaystyle\int_0^{\pi^2/4} \left(f(x) + xf'(x)\right)dx$

$= x f(x)\Big]_0^{\pi^2/2} = 2x\sin\sqrt{x}\Big]_0^{\pi^2/2} = \dfrac{2\pi^2}{4} = \dfrac{\pi^2}{2}$ **Ans.**

$\Rightarrow$ **(Q)**

10. (A) $f(x) = \int x^{\sin x}\left(1 + x\cos x \cdot \ln x + \sin x\right)dx$

if $F(x) = x^{\sin x} = e^{\sin x \ln x}$

$\therefore \quad x F'(x) = x^{\sin x}(x\cos x \ln x + \sin x)$

$\therefore \quad f(x) = \int\left(F(x) + x F'(x)\right) = x F(x) + C$

$f(x) = x \cdot x^{\sin x} + C$

$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cdot\frac{\pi}{2} + C \quad \Rightarrow \quad C = 0$

$\therefore \quad f(x) = x\,(x)^{\sin x}\ ; \quad f(\pi) = \pi(\pi)^0 = \pi$ (irrational)

$\Rightarrow$ **(Q)**

(B) $g(x) = \int \frac{\cos x(\cos x + 2) + \sin^2 x}{(\cos x + 2)^2}dx$

$= \int \underbrace{\cos x}_{\text{II}} \cdot \underbrace{\frac{1}{(\cos x + 2)}}_{\text{I}}dx + \int \frac{\sin^2 x}{\cos x + 2}dx$

$= \frac{1}{\cos x + 2}\cdot \sin x - \int \frac{\sin^2 x}{(\cos x + 2)^2}dx + \int \frac{\sin^2 x}{(\cos x + 2)^2}dx$

$g(x) = \frac{\sin x}{\cos x + 2} + C$

11. (A) $I = \int_4^{10} \frac{[x^2]dx}{[(14-x)^2] + [x^2]}$ --------- **(i)**

$I = \int_4^{10} \frac{[(14-x)^2]}{[x^2] + [(14-x)^2]}dx$ --------- **(ii)**

add **(i)** & **(ii)**

$2I = \int_4^{10} dx$

$\Rightarrow 2I = 6 \Rightarrow I = 3$

(B) $\int_{-1}^{2} \frac{|x|}{x}dx = \int_{-1}^{0}(-1)dx + \int_0^2 (1)dx = 1$

(C) $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^{99}}{n^{100}} = \int_0^1 x^{99}dx = \left[\frac{x^{100}}{100}\right]_0^1 = \frac{1}{100}$

(D) $5050\int_{-1}^{1}\sqrt{x^{200}}\,dx = 5050 \times 2\int_0^1 |x^{100}|\,dx$

$= 5050 \times 2\int_0^1 x^{100}\,dx$

$= 10100 \times \left[\frac{x^{101}}{101}\right]_0^1 = 100 = \frac{1}{\alpha}$

$\Rightarrow \alpha = \frac{1}{100}$

12. (A) for $a = 0, I = \int_0^T \sin^2 x\,dx = \int_0^T \frac{1 - \cos 2x}{2}dx$

$= \left.\frac{x}{2} - \frac{1}{4}\sin 2x\right|_0^T = \frac{T}{2} - \frac{1}{4}\sin 2T$

$\therefore \quad L = \frac{1}{2} - \lim_{T\to\infty}\frac{1}{4}\frac{\sin 2T}{T} = \frac{1}{2} \quad \Rightarrow \quad$ **(A)** $\to$ q

(B) for $a = 1, \int_0^T 4\sin^2 x\,dx$

$\Rightarrow L = 2,$ hence **(B)** $\to$ s

(C) $a = -1, \quad \int_0^T 0\,dx = 0 \quad \Rightarrow \quad L = 0,$

hence **(C)** $\to$ p

(D) for $a \neq 0, -1, 1,$

$I = \int_0^T (\sin^2 x + \sin^2 ax + 2\sin x \cdot \sin ax)dx$

$= \int_0^T \left(\frac{1 - \cos 2x}{2} + \frac{1 - \cos 2ax}{2} + \cos(a-1)x - \cos(a+1)x\right)dx$

$= \left[x - \frac{1}{4}\sin 2x - \frac{1}{4a}\sin 2ax + \frac{\sin(a-1)x}{a-1} - \frac{\sin(a+1)x}{a+1}\right]_0^T$

$\therefore \quad L = \lim_{T\to\infty}\frac{T}{T}$

$- \lim_{T\to\infty}\frac{1}{T}$

$\left[\frac{1}{4}\sin 2x - \frac{1}{4a}\sin 2ax + \frac{\sin(a-1)x}{a-1} - \frac{\sin(a+1)x}{a+1}\right]_0^T$

$L = 1$ hence **(D)** $\to$ r

13. (A) $L = \lim_{x\to 0} \dfrac{\int_0^{\ln(1+x)} (1-\tan 2y)^{1/y}\,dy}{\frac{\sin x}{x}\cdot x}$

$= \lim_{x\to 0} \dfrac{\int_0^{\ln(1+x)} (1-\tan 2y)^{1/y}\,dy}{x}$ Using L'Hospital's Rule

$L = \lim_{x\to 0} \dfrac{[1-\tan 2(\ln(1+x))]^{\frac{1}{\ln(1+x)}}}{(1+x)}$ (1^∞ form)

$L = \lim_{x\to 0} e^{l}$

where $l = \lim_{x\to 0} \dfrac{-1}{\ln(1+x)} \tan\left(2(\ln(1+x)\right)$

$= -\lim_{x\to 0} \dfrac{2\cdot\tan\left(2(\ln(1+x)\right)}{2\ln(1+x)} = -2$

hence $L = e^{-2} \Rightarrow$ **(s)**

(B) $\ln l = \lim_{x\to\infty} \dfrac{\ln(e^{2x}+e^x+x)}{x} = \lim_{x\to\infty} \dfrac{2e^{2x}+e^x+1}{e^{2x}+e^x+1}$

$= \lim_{x\to\infty} \dfrac{2+e^{-x}+e^{-2x}}{1+e^{-x}+e^{-2x}} = 2$

$\Rightarrow l = e^2 \Rightarrow$ **(r)**

(C) $f(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \dfrac{n}{n^2+k^2x^2} = \lim_{n\to\infty} \dfrac{1}{n} \sum_{k=1}^{n} \dfrac{1}{1+\left(\frac{kx}{n}\right)^2}$

$= \int_0^1 \dfrac{dy}{1+x^2y^2} = \dfrac{1}{x^2}\cdot\int_0^1 \dfrac{dy}{\frac{1}{x^2}+y^2}$

$= \dfrac{1}{x^2}\cdot x\cdot \tan^{-1} yx \Big]_0^1 = \dfrac{\tan^{-1}x}{x}$

$\Rightarrow \lim_{x\to 0} \dfrac{\tan^{-1}x}{x} = 1 \Rightarrow$ **(p)**

Part # II : Comprehension

Comprehension # 1

1. $\lim_{x\to 0} \dfrac{\int_0^x \frac{t^2}{(a+t^r)^{1/p}}}{bx - \sin x}$

using L'Hospital's Rule

$\lim_{x\to 0} \dfrac{\frac{x^2}{(a+x^r)^{1/p}}}{b-\cos x}$

for existence of limit $\lim_{x\to 0} D^r \to 0$

$\therefore\ b-1=0 \Rightarrow b=1$ **Ans.**

2. $\therefore\ l = \lim_{x\to 0} \dfrac{x^2}{(a+x^r)^{1/p}} \dfrac{x^2}{(1-\cos x)}\cdot\dfrac{1}{x^2}$

$= 2\lim_{x\to 0} \dfrac{1}{(a+x^r)^{1/p}} = \dfrac{2}{a^{1/p}}$

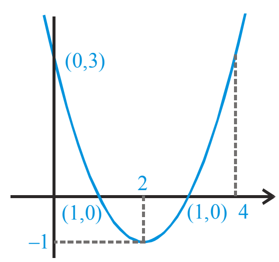
if $p = 3$ and $l = 1$

$\therefore\ 1 = \dfrac{2}{a^{1/3}} \Rightarrow a = 8$ **Ans.**

3. again $p = 2$ and $a = 9$ then

$l = \dfrac{2}{9^{1/3}} = \dfrac{2}{3}$ **Ans.**

Comprehension # 2



$f(x) = x^2 - 4x + 3$

$f(x)\big|_{max} = 3 \quad x \in [0,4]$

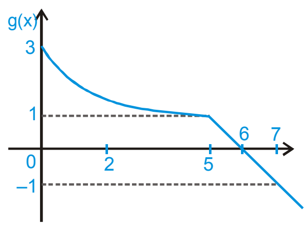
$f(x)\big|_{min} = \begin{cases} x^2-4x+3 & x \in [0,2) \\ -1 & x \in [2,4] \end{cases}$

Now, $g(x) = \begin{cases} \dfrac{x^2-4x+6}{2} & 0 \le x < 2 \\ \dfrac{3-1}{2} = 1 & 2 \le x \le 4 \\ -x+5+x-4 = 1 & 4 < x < 5 \\ \tan\left(\tan^{-1}\left(\dfrac{6-x}{1}\right)\right) = 6-x & x \ge 5 \end{cases}$

$$g(x) = \begin{cases} \dfrac{x^2-4x+6}{2} & 0 \le x < 2 \\ 1 & 2 \le x < 5 \\ 6-x & x \ge 5 \end{cases}$$

1. $\displaystyle\int_2^5 g(x)dx = 5-2 = 3$

2. $h(x) = \displaystyle\int_0^{x^2} g(t)dt$



$h'(x) = g(x^2) . 2x$

$g(x^2) = 0$ at $x = \sqrt{6}$

$\therefore$ $h'(x) < 0$ in $(\sqrt{6}, 7]$

and hence h(x) is decreasing

3. $\displaystyle\lim_{x\to4} \frac{g(x)-g(2)}{\ell n(\cos(4-x))}$ $\left(\dfrac{0}{0} \text{ from}\right)$

$$\lim_{x\to4} \frac{g'(x)}{\dfrac{1}{\cos(4-x)}(\sin(4-x))}$$

$$= \lim_{x\to4} \frac{g'(x)}{\tan(4-x)} \quad \left(\frac{0}{0} \text{ from}\right)$$

$$\Rightarrow \lim_{x\to4} \frac{-g''(x)}{\sec^2(4-x)} = 0 \quad (\because\ g''(4) = 0)$$

Comprehension # 4

1. $\displaystyle\int_0^1 \frac{\sin x}{x^2}dx - \int_0^1 \frac{\cos x}{x}dx = \sin x\left(-\frac{1}{x}\right)\Bigg|_0^1 + \int_0^1 \cos x\left(\frac{1}{x}\right)dx$

$$- \int_0^1 \frac{\cos x}{x}dx$$

$= -\left[\dfrac{\sin x}{x}\right]_0^1 = (1) - \sin(1)$ **Ans.**

2. $\displaystyle\lim_{t\to0} \frac{\int_0^t f(x)dx}{t^2}$

$$= \lim_{t\to0} \frac{\int_0^t \frac{\sin x - x\cos x}{x^2}dx}{t^2}$$

using L'Hospital's rule

$$l = \lim_{t\to0} \frac{\sin t - t\cos t}{t^2 \cdot 2t} = \lim_{t\to0} \frac{\cos t(\tan t - t)}{2t^3}$$

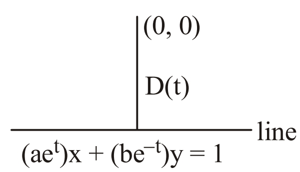
$$= \frac{1}{2}\lim_{t\to0} \frac{\sec^2 t - 1}{3t^2} = \frac{1}{6}$$

Comprehension # 5

1. $D(t) = \left|\dfrac{1}{\sqrt{(ae^t)^2 + (be^{-t})^2}}\right|$

$$= \frac{1}{\sqrt{a^2e^{2t} + b^2e^{-2t}}}$$

$$\frac{1}{(D(t))^2} = (a^2e^{2t} + b^2e^{-2t})$$



$$\therefore \quad I = \int_0^1 \left(a^2e^{2t} + b^2e^{-2t}\right)dt = \left[\frac{a^2e^{2t}}{2} - \frac{b^2e^{-2t}}{2}\right]_0^1$$

$$= \left(\frac{a^2e^2 - b^2e^{-2}}{2}\right) - \left(\frac{a^2 - b^2}{2}\right)$$

$$= \frac{a^2(e^2-1) - b^2(e^{-2}-1)}{2} = \frac{a^2(e^2-1) + \frac{b^2}{e^2}(e^2-1)}{2}$$

$$= \frac{e^2-1}{2}\left(a^2 + \frac{b^2}{e^2}\right)$$

2. now put $b = \frac{1}{a}$

$$I = \frac{e^2-1}{2}\left(a^2 + \frac{1}{a^2e^2}\right)$$

$$= \frac{e^2-1}{2}\left(\left(a - \frac{1}{ae}\right)^2 + \frac{2}{e}\right)$$

I is minimum if $a = \frac{1}{ae} \Rightarrow a^2 = \frac{1}{e} \Rightarrow a = \frac{1}{\sqrt{e}}$

$\Rightarrow b = \sqrt{e}$

3. and $I_{min} = \frac{e^2-1}{2}\cdot\frac{2}{e} = e - \frac{1}{e}$

Comprehension # 7

1. $g(x) = \int_0^x f(t)\,dt$

$g'(x) = f(x)$

From the graph it is clear that

$f(x) > 0$ in $x \in [0, 3)$ and

$f(x) < 0$ in $x \in (3, 7)$

$\therefore$ g(x) is increasing in [0, 3] and

g(x) is decreasing in [3, 7]

$\therefore$ maximum value of g(x) occurs at x = 3

$\therefore$ $g(3) = \int_0^3 f(t)dt$

$$= \int_0^1 1.dt + \int_1^2 (2t-1)dt + \int_2^3 (3t+9)dt$$

$$= 1 + (t^2 - t)_1^2 + \left(9t - 3\frac{t^2}{2}\right)_2^3$$

$$= 1 + (4 - 2 - 0) + \left(27 - \frac{27}{2} - 18 + 6\right) = \frac{9}{2}$$

2. g(x) start decreasing from x = 3

$$g(4) = \int_0^4 f(t)\,dt = \int_0^3 f(t)\,dt + \int_3^4 f(t)\,dt$$

$$= \frac{9}{2} + \int_3^4 (-3t+9)dt = \frac{9}{2} + \left(9t + \frac{3t^2}{2}\right)_3^4$$

$$= \frac{9}{2} + \left(36 - 24 - 27 + \frac{27}{2}\right) = 3$$

Now, $g(x) = \int_0^x f(t)dt$

$$= \int_0^4 f(t)\,dt + \int_4^x f(t)\,dt \qquad 0 \le x \le 6$$

$$= 3 + \int_4^x (-3)dt = 3 - 3(x-4) = 15 - 3x$$

$g(x) = 0 \Rightarrow 15 - 3x = 0 \Rightarrow x = 5$

wich lies in [0, 6]

3. g(x) becomes zero at x = 5

$\therefore$ g(x) will be negative in (5, 7)

Comprehension # 8

1. Given $f(x) \cdot f'(-x) = f(-x) \cdot f'(x)$

$$\frac{f'(x)}{f(x)} = \frac{f'(-x)}{f(-x)}$$

Integrating

$\ln f(x) = -\ln f(-x) + C$

$\ln\left(f(x)\cdot f(-x)\right) = C \qquad f(x)\cdot f(-x) = C$

but $f(0) = 3$

$\therefore$ $f^2(0) = C \Rightarrow C = 9$

$\therefore$ $f(x)\cdot f(-x) = 9 \Rightarrow$ **(B)**

2. Let $I = \int_{-51}^{51} \frac{dx}{3 + f(x)}$

$$= \int_{-51}^{51} \frac{dx}{3 + f(-x)}$$

$$2I = \int_{-51}^{51} \frac{6 + f(x) + f(-x)}{(3 + f(x))(3 + f(-x))}dx$$

$$= \int_{-51}^{51} \frac{6 + f(x) + f(-x)}{9 + 3(f(x) + f(-x)) + f(x).f(-x)}dx$$

$$2I = \int_{-51}^{51} \frac{6 + f(x) + f(-x)}{18 + 3(f(x) + f(-x))}dx = \frac{1}{3}\int_{-51}^{51} dx = \frac{2\cdot 51}{3}$$

$\Rightarrow I = \frac{51}{3} = 17$ **Ans.**

3. Let $x = \alpha$ be the root of $f(x) = 0$

$\therefore$ $f(\alpha) = 0$

$f(x) \cdot f(-x) = 9$

put $x = \alpha$ $\Rightarrow$ $0 = 9$ impossible

hence $f(x)$ has no root

but $f(0) = 3$

hence $f(x) > 0 \ \forall \ x \in R$ as f is continuous

possible function $f(x) = 3e^{-x}$ **Ans.**

Comprehension # 11

1. $f(x) = e^x \underbrace{\int_0^1 e^t \cdot f(t)\,dt}_{A \text{ say}}$ $\qquad f(x) = Ae^x$ **....(1)**

$\Rightarrow$ $f(t) = Ae^t$

where, $A = \int_0^1 e^t \cdot f(t)\,dt$

$\Rightarrow$ $A = \int_0^1 e^t \cdot Ae^t\,dt$; $A = A\int_0^1 e^{2t}\,dt$

now $A\left[\int_0^1 e^{2t}\,dt - 1\right] = 0$ $\Rightarrow$ $A = 0$ as $\int_0^1 e^{2t}\,dt \neq 0$

hence $f(x) = 0$ $\Rightarrow$ $f(1) = 0$ **Ans.**

2. again $g(x) = e^x \int_0^1 e^t g(t)\,dt + x$

$g(x) = Be^x + x$ **....(2)**

$\Rightarrow$ $g(t) = Be^t + t$

where $B = \int_0^1 e^t g(t)\,dt$; $B = \int_0^1 e^t (Be^t + t)\,dt$;

$B = B\int_0^1 e^{2t}\,dt + \int_0^1 e^t \cdot t\,dt$

but $\int_0^1 e^{2t}\,dt = \frac{1}{2}(e^2 - 1)$ and $\int_0^1 te^t\,dt = 1$

$\therefore$ $B = \frac{B}{2}(e^2 - 1) + 1$ $\Rightarrow$ $2B = B(e^2 - 1) + 2$

$\Rightarrow$ $3B = Be^2 + 2$ $\Rightarrow$ $B = \frac{2}{3 - e^2}$

$\therefore$ from (2)

$g(x) = \left(\frac{2}{3 - e^2}\right)e^x + x$; $g(0) = \frac{2}{3 - e^2}$ **Ans.**

3. $g(2) = \frac{2e^2}{3 - e^2} + 2 = \frac{6}{3 - e^2}$;

$\frac{g(0)}{g(2)} = \frac{2}{3 - e^2} \cdot \frac{3 - e^2}{6} = \frac{1}{3}$ **Ans.**

EXERCISE - 4
Subjective Type

1. **(I)** $-5\left(\sqrt[5]{16}-1\right)$ **(II)** $0.2\,(e-1)^5$

(III) $\dfrac{\pi(9-4\sqrt{3})}{36}+\dfrac{1}{2}\ln\dfrac{3}{2}$ **(IV)** $2-\dfrac{\pi}{2}$

(V) $\sqrt{2}-\dfrac{2}{\sqrt{3}}+\ln\dfrac{2+\sqrt{3}}{1+\sqrt{2}}$

(VI) $\dfrac{1}{32}\left(\pi+\dfrac{7\sqrt{3}}{2}-8\right)$ **(VII)** $\tan^{-1}\dfrac{1}{2}$

(VIII) $\dfrac{11}{6}$ **(IX)** 2

(X) $\int_0^2 [x^2]dx=\int_0^1 0.dx+\int_1^{\sqrt{2}} dx+\int_{\sqrt{2}}^{\sqrt{3}} 2.dx+3\int_{\sqrt{3}}^{2} dx$

$=5-\sqrt{2}-\sqrt{3}$

(XI) $\int_{-1}^{1}[\cos^{-1}x]dx=3\int_{-1}^{\cos 3}dx+2\int_{\cos 3}^{\cos 2}dx+\int_{\cos 2}^{\cos 1}dx+\int_{\cos 1}^{0}0.dx$

$=\cos 1+\cos 2+\cos 3+3$

(XII) $I=\int_{-\infty}^{\infty}\dfrac{dx}{(x+1)^2+1}=\left[\tan^{-1}(x+1)\right]_{-\infty}^{\infty}$

$=\dfrac{\pi}{2}-\left(-\dfrac{\pi}{2}\right)=\pi$

(XIII) $x=\sec\theta$

$\Rightarrow\ dx=\sec\theta\tan\theta\, d\theta$

$\Rightarrow\ I=\int_{\pi/4}^{\pi/2}\dfrac{\sec\theta\tan\theta\, d\theta}{\sec\theta.\tan\theta}dx=\dfrac{\pi}{2}-\dfrac{\pi}{4}=\dfrac{\pi}{4}$

(XIV) $I=\int_0^4\dfrac{x^2+1-1}{1+x}dx=\int_0^4(x-1)\,dx+\int_0^4\dfrac{1}{1+x}dx$

$=\left[\dfrac{x^2}{2}-x\right]_0^4+\left[\ell n(1+x)\right]_0^4=4+\ell n5$

(XV) Let $\cos\theta=t-\sin\theta\, d\theta=dt$

$I=-\int_1^0\sqrt{t}(1-t^2).dt$

$I=-\left[\dfrac{t^{3/2}}{3/2}-\dfrac{t^{7/2}}{7/2}\right]_1^0$

$I=-\left(-\dfrac{2}{3}+\dfrac{2}{3}\right)=-\left(\dfrac{-14+6}{21}\right)$

$I=+\dfrac{8}{2}$

(XVI) $\dfrac{\pi-2}{2}$ **(XVII)** $\dfrac{1}{2}\ell n\left(\dfrac{e}{2}\right)$

(XVIII) 1 **(XIX)** $\dfrac{\pi}{6}-\dfrac{2}{9}$

(XX) 0 **(XXI)** 0

2. **(I)** $-\pi\log 2$ **(II)** t

(III) $\dfrac{\pi}{8}\log 2$ **(IV)** $\dfrac{\pi^2}{8}$

(V) $\int_0^1\dfrac{x^4(1-x)^4}{1+x^2}dx=\int_0^1\dfrac{x^4[(1+x^2)-2x]^2}{1+x^2}dx$

$=\int_0^1 x^4(1+x^2)dx-4\int_0^1 x^5dx+4\int_0^1\dfrac{x^6}{1+x^2}dx$

$=\left[\dfrac{x^5}{5}+\dfrac{x^7}{7}\right]_0^1-4\left[\dfrac{x^6}{6}\right]_0^1+4\int_0^1\dfrac{-dx}{1+x^2}+$

$4\int_0^1\dfrac{(x^2+1)(x^4+1-x^2)}{1+x^2}dx$

$=\left(\dfrac{1}{5}+\dfrac{1}{7}\right)-4\left(\dfrac{1}{6}\right)-4\left[\tan^{-1}x\right]_0^1+4\left(\dfrac{1}{5}+1-\dfrac{1}{3}\right)$

$=\dfrac{22}{7}-\pi$

(VI) $I=\int_0^{\pi/2}\dfrac{a\sin x+b\cos x}{\sin\left(\dfrac{\pi}{4}+x\right)}dx$

$I=\sqrt{2}\int_0^{\pi/2}\dfrac{a\sin x+b\cos x}{\sin x+\cos x}dx$ .....**(1)**

$I=\sqrt{2}\int_0^{\pi/2}\dfrac{a\cos x+b\sin x}{\cos x+\sin x}$ .....**(2)**

add.(1) & (2)

$2I=\sqrt{2}(a+b)\int_0^{\pi/2}dx\ \Rightarrow\ I=\dfrac{(a+b)\pi}{2\sqrt{2}}$

(VIII) $\frac{\pi}{2\sqrt{2}}-\frac{16\sqrt{2}}{5}$

(IX) $I=\int_0^1 \frac{1-x}{1+x}.\frac{dx}{\sqrt{x+x^2+x^3}}$

$=\int_0^1 \frac{(1-x^2)}{(x^2+2x+1)\sqrt{x+x^2+x^3}}$

$=\int_0^1 \frac{\left(\frac{1}{x^2}-1\right)dx}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}$

Put $x+\frac{1}{x}+1=t^2$

$\Rightarrow \left(1-\frac{1}{x^2}\right)dx=2t\,dt$

$I=\int_{\infty}^{\sqrt{3}} \frac{-2t\,dt}{(t^2+1)t}=2\int_{\sqrt{3}}^{\infty}\frac{dt}{t^2+1} \quad =2[\tan^{-1}t]_{\sqrt{3}}^{\infty}=\frac{\pi}{3}$

(X) $\frac{\pi}{8}\ell n2$

(XI) $I=\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Let $x=\tan\theta \Rightarrow dx=\sec^2\theta\,d\theta$

$\Rightarrow I=\int_0^{\pi/4} 2\theta\sec^2\theta\,d\theta$

$=[\theta\tan\theta]_0^{\pi/4}-2\int_0^{\pi/4}\tan\theta\,d\theta \quad =2\left(\frac{\pi}{4}\right)-2[\ell n\sec\theta]_0^{\pi/4}$

$=\frac{\pi}{2}-2\ell n\sqrt{2}$

$=\frac{\pi}{2}-\ell n2$

(XII) Let $\tan^{-1}x=t \Rightarrow \frac{dx}{1+x^2}=dt$

$\therefore I=\int_0^{\pi/4}\frac{t\tan t\cdot dt}{\sqrt{1+\tan^2 t}}=\int_0^{\pi/4} t\cdot\sin t\,dt$

$=\frac{\pi}{4\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{4-\pi}{4\sqrt{2}}$

(XIII) $I=\int_a^b \sqrt{(x-a)(b-x)}\,dx$

put $x=a\sin^2\theta+b\cos^2\theta$

$I=2\int_0^{\pi/2} -(b-a)^2\sin^2\theta\cos^2\theta\,d\theta$

$I=-2(b-a)^2\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$

$I=-\frac{(b-a)^2}{2}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta$

$I=-\frac{(b-a)^2}{4}\left[\theta-\frac{\sin 4\theta}{4}\right]_0^{\pi/2} \quad =-\frac{(b-a)^2\pi}{8}$

(XIV) $I=\int_0^{\sqrt{3}}\tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$

put $x=\tan\theta$

$I=\int_0^{\pi/3}\tan^{-1}(\tan 2\theta)\sec^2\theta d\theta+\int_{\pi/4}^{\pi/3}(\pi-2\theta)\sec^2\theta d\theta$

$I=\pi\left(1-\frac{1}{\sqrt{3}}\right)-\ell n4$

(XV) $\frac{\pi}{4}$ **(XVI)** $2\left(\frac{5}{6}-\ell n2\right)$

(XVII) $\ell n\left(\frac{9}{8}\right)$ **(XVIII)** $\frac{\pi}{2}$

(XIX) $\frac{1}{20}\ \ell n3$ **(XX)** 0 **(XXI)** 0

(XXII) $I=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

$I=\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\cos x}\cdot\sqrt{\sin x}}$

$\Rightarrow 2I=\int_0^{\pi/2}dx \Rightarrow I=\frac{\pi}{4}$

(XXIII)

$$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos x\left(\frac{\pi}{2} - x\right)} dx$$

$$I = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx$$

$$2I = \int_0^{\pi/2} 1.dx = \frac{\pi}{2} \quad \Rightarrow \quad I = \frac{\pi}{4}$$

(XXIV) $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx = \int_0^a \frac{\sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} dx$

$$2I = \int_0^a 1.dx = a \quad \Rightarrow \quad I = \frac{a}{2}$$

(XXV) $I = \int_0^{\pi/2} \frac{a \sin x + 6\cos x}{\sin x + \cos x} dx$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{(a+b)\sin x + (a+b)\cos x}{\sin x + \cos x} dx$$

$$\Rightarrow I = (a+b)\frac{\pi}{4}$$

3. **(I)** $\frac{\pi}{2}$

(II) $I = \int_0^1 \frac{\sin^{-1}\sqrt{x}}{x^2 - x + 1} dx$ ...**(i)**

$$I = \int_0^1 \frac{\sin^{-1}\sqrt{1-x}}{(1-x)^2 - (1-x) + 1} dx = \int_0^1 \frac{\sin^{-1}\sqrt{1-x}}{x^2 - x + 1} dx \quad \text{...(ii)}$$

Add. **(i)** and **(ii)**

$$2I = \int_0^1 \frac{\sin^{-1}\sqrt{x} + \sin^{-1}\sqrt{1-x}}{x^2 - x + 1} dx$$

$$\Rightarrow 2I = \frac{\pi}{2}\int_0^1 \frac{dx}{(x - 1/2)^2 + (\sqrt{3}/2)^2}$$

$$\Rightarrow I = \frac{\pi}{4}\frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{2x-1}{\sqrt{3}}\right]_0^1 = \frac{\pi^2}{6\sqrt{3}}$$

(III) $\frac{1}{3}\left(\tan^{-1}\frac{\sqrt{2}}{3} - \tan^{-1}\frac{1}{3}\right)$

(IV) $I = \int_0^\pi \frac{x \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi} dx$ ....**(i)**

then,

$$I = \int_0^\pi \frac{(\pi - x)\sin 2(\pi - x)\sin\left(\frac{\pi}{2}\cos(\pi - x)\right)}{2(\pi - x) - \pi} dx \quad \text{....(ii)}$$

$$= \int_0^\pi \frac{(\pi - x)\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{\pi - 2x} dx$$

$$= \int_0^\pi \frac{(x - \pi)\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x - \pi} dx$$

add equation **(i)** & **(ii)**

$$2I = \int_0^\pi \sin 2x \sin\left(\frac{\pi}{2}\cos x\right) dx$$

$$\therefore \quad I = \int_0^\pi \sin x \cos x \sin\left(\frac{\pi}{2}\cos x\right) dx$$

Put $\frac{\pi}{2}\cos x = t \quad \Rightarrow \quad \sin x dx = -\frac{2}{\pi} dt$

$$\therefore \quad I = -\frac{2}{\pi}\int_{\pi/2}^{-\pi/2} \frac{2t}{\pi}\sin t dt = \frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} t \sin t dt$$

$$\Rightarrow \quad I = \frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} t \sin t dt = \frac{4}{\pi^2}\left[-t\cos t + \sin t\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{4}{\pi^2} \times 2 = \frac{8}{\pi^2}$$

(V) $-\frac{3\sqrt{2}}{5}\left(e^{2\pi} + 1\right)$

(VI) $\frac{\pi^2}{16} - \frac{\pi}{4}\ell n2$

4. **(V)** $\int_a^b \frac{x^{n-1}\{nx^2 - 2x^2 + n(a+b)x - (a+b)x + nab\}}{(x+a)^2(x+b)^2} dx$

$$= \int_a^b \frac{x^{n-1}\{n(x+a)(x+b) - x(2x+a+b)\}}{(x+a)^2(x+b)^2} dx$$

$$= \int_a^b \frac{nx^{n-1}}{(x+a)(x+b)} dx - \int_a^b \frac{x^n(x+a+x+b)}{(x+a)^2(x+b)^2} dx$$

$$= \int_a^b \left(\frac{d}{dx}\frac{x^n}{(x+a)(x+b)}\right) dx$$

$$= \left[\frac{x^n}{(x+a)(x+b)}\right]_a^b = \frac{b^{n-1} - a^{n-1}}{2(a+b)}$$

(VII) $\int_0^1 \underset{\text{I}}{x^m} \underset{\text{II}}{(1-x)^n}\,dx$

$$= \left[-x^m \frac{(1-x)^{n+1}}{n+1}\right]_0^1 + \frac{m}{n+1}\int_0^1 x^{m-1}(1-x)^{n+1}\,dx$$

$$= 0 + \frac{m}{n+1}\int_0^1 \underset{\text{I}}{x^{m-1}} \underset{\text{II}}{(1-x)^{n+1}}\,dx$$

$$= \frac{m(m-1)}{(n+1)(n+2)}\int_0^1 x^{m-2}(1-x)^{n+2}\,dx$$

$$= \frac{m(m-1)\ldots\ldots..1}{(n+1)(n+2)\ldots\ldots.(n+m+1)} = \frac{\lfloor m \lfloor n}{\lfloor m+n+1}$$

7. 5250

9. $\dfrac{t^{n+1}-1}{(t-1)(n+1)}$

10. $u_n = \{x(1-x)\}^n$

$$\frac{du_n}{dx} = n\{x(1-x)\}^{n-1}\{1-2x\}$$

$$\frac{du_n}{dx} = n.u_{n-1} - 2nxu_{n-1}$$

$$\frac{d^2u_n}{dx^2} = n(n-1)u_{n-2}\{1-2x\}$$

$$-2n\{u_{n-1} + x.(n-1)u_{n-2}\{1-2x\}\}$$

$= n(n-1)u_{n-2} - 2xn(n-1)u_{n-2}$

$-2n.u_{n-1} - x2n(n-1)(1-2x)u_{n-2}$

$= n(n-1)u_{n-2} - 2nx(n-1)u_{n-2}\{1+1-2x\} - 2n\,u_{n-1}$

$= n(n-1)u_{n-2} - 4nx(1-x)u_{n-2}(n-1) - 2n\,u_{n-1}$

$= n(n-1)u_{n-2} - 2nu_{n-1}\{2n-1\}$

$$v_n = \int_0^1 e^x.u_n\,dx$$

II I

& apply by parts twice

11.

$$F(x) = \begin{cases} \int_0^x (1-t)\,dt & ; \quad 0 \le x \le 1 \\ \int_0^1 (1-t)\,dt + \int_1^x 0.\,dx & ; \quad 1 < x \le 2 \\ \int_0^1 (1-t)\,dt + \int_1^2 0.\,dx + \int_2^x (2-t)^2\,dt & ; \quad 2 < x \le 3 \end{cases}$$

$$F(x) = \begin{cases} x - \frac{x^2}{2} & ; \quad 0 \le x \le 1 \\ \frac{1}{2} & ; \quad 1 < x \le 2 \\ \frac{(x-2)3}{3} + \frac{1}{2} & ; \quad 2 < x \le 3 \end{cases}$$

13. $I = \int_0^\pi f(x)\,dx = \int_0^\pi \frac{\sin x}{x}\,dx$ ... **(i)**

$$I = \int_0^\pi f(\pi - x)\,dx = \int_0^\pi \frac{\sin(\pi - x)}{\pi - x}\,dx = \int_0^\pi \frac{\sin x}{\pi - x}\,dx$$

... **(ii)**

(i) + (ii)

$$\Rightarrow \quad 2I = \int_0^\pi \left\{\frac{\sin x}{x} + \frac{\sin x}{\pi - x}\right\}dx \quad \Rightarrow \quad I$$

$$= \frac{\pi}{2}\int_0^\pi \frac{\sin x}{x(\pi - x)}\,dx \qquad \ldots \textbf{(iii)}$$

Now $\quad \dfrac{\pi}{2}\displaystyle\int_0^{\pi/2} f(x)f\left(\frac{\pi}{2} - x\right)dx$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x}{x} \times \frac{\sin\left(\frac{\pi}{2} - x\right)}{\frac{\pi}{2} - x}\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x}{x} \cdot \frac{\cos x}{\frac{\pi}{2} - x}\,dx = \frac{\pi}{4} \cdot \int_0^{\pi/2} \frac{\sin 2x}{x\left(\frac{\pi}{2} - x\right)}\,dx$$

$$= \frac{\pi}{8}\int_0^\pi \frac{\sin t}{\frac{t}{2}\left(\frac{\pi}{2} - \frac{t}{2}\right)}\,dt, \text{ where } \quad t = 2 \qquad \ldots \textbf{(iv)}$$

(iii) + (iv)

$$\Rightarrow \quad \frac{\pi}{2}\int_0^\pi f(x)f\left(\frac{\pi}{2} - x\right)dx = \int_0^\pi f(x)\,dx$$

14. $(1-x)^n = C_0 - C_1 x + C_2 x^2 \text{ ....} + (-1)^n C_n x^n$

$x^{n-1}(1-x)^{n+1} = (C_0 x^{n-1} - C_1 x^n + C_2 x^{n+1} + \ldots (-1)^n C_n x^{2n-1})(1-x)$

$= (C_0 x^{n-1} - C_1 x^n + C_2 x^{n+1} + \ldots (-1)^n C_n x^{2n-1})$

$-(C_0 x^n - C_1 x^{n+1} + C_2 x^{n+2} + \ldots (-1)^n C_n x^{2n})$

$$\int_0^1 x^{n-1}(1-x)^{n+1}\,dx$$

$$= \left[\frac{C_0 x^n}{n} - \frac{C_1 x^{n+1}}{n+1} + \frac{C_2 x^{n+2}}{n+2} - \ldots\ldots \frac{(-1)^n C_n x^{2n}}{2n}\right]_0^1$$

$$-\left[\frac{C_0 x^{n+1}}{n+1} - \frac{C_1 x^{n+2}}{n+2} + \frac{C_2 x^{n+3}}{n+3} - \ldots + \frac{(-1)^n C_n x^{2n+1}}{2n+1}\right]_0^1$$

$$= \left[\frac{C_0}{n} - \frac{C_1}{n+1} + \frac{C_2}{n+2} + \ldots \frac{(-1)^n C_n}{2n}\right]$$

$$-\left(\frac{C_0}{n+1} - \frac{C_1}{n+2} + \frac{C_2}{n+3} + \ldots (-1)^n \frac{C_n}{2n+1}\right)$$

$$= \frac{C_0}{n(n+1)} - \frac{C_1}{(n+1)(n+2)} + \frac{C_2}{(n+2)(n+3)} + \ldots$$

upto (n + 1) terms

$$\int_0^1 x^{n-1}(1-x)^{n+1}\,dx$$

put $x = \sin^2\theta \quad \Rightarrow \quad dx = 2\sin\theta\cos\theta\, d\theta$

$$\int_0^1 x^{n-1}(1-x)^{n+1}\,dx = \int_0^{\pi/2} \sin^{2n-2}\theta\cos^{2n+2}\theta(2\sin\theta\cos\theta)d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n-1}\theta\cos^{2n+3}\theta\, d\theta$$

$$= \frac{2\Gamma\left(\frac{2n-1+1}{2}\right)\Gamma\left(\frac{2n+3+1}{2}\right)}{2\Gamma\left(\frac{2n-1+2n+3+2}{2}\right)}$$

$$= \frac{\Gamma(n)\Gamma(n+1)}{\Gamma(2n+2)} = \frac{\lfloor n-1 \lfloor n+1}{\lfloor 2n+1}$$

15. $$\int_1^2 \frac{(x^2-1)dx}{x^3\sqrt{2x^4-2x^2+1}} = \int_1^2 \frac{x(x^2-1)dx}{x^4\sqrt{2x^4-2x^2+1}}$$

Let $x^2 = t \quad \Rightarrow \quad xdx = dt/2$

$$= \frac{1}{2}\int_1^4 \frac{(t-1)dt}{t^2\sqrt{2t^2-2t+1}}$$

$$= \frac{1}{2}\int_1^4 \frac{t-1}{t^3\sqrt{2-\frac{2}{t}+\frac{1}{t^2}}}dt = \frac{1}{2}\int_1^4 \frac{\frac{1}{t^2}-\frac{1}{t^3}}{\sqrt{2-\frac{2}{t}+\frac{1}{t^2}}}dt$$

Let $2-\frac{2}{t}+\frac{1}{t^2} = z^2 \Rightarrow \left(\frac{2}{t^2}-\frac{2}{t^3}\right)dt = 2zdz$

$$= \frac{1}{2}\int_1^{5/4} \frac{zdz}{\sqrt{z^2}} = \frac{1}{2}\int_1^{5/4} dz = \frac{1}{8} = \frac{U}{V}$$

$\Rightarrow (1000)\frac{U}{V} = \frac{1000}{8} = 125$

16. – 1

17. $\frac{2\pi}{\sqrt{3}}$

18. $$I = \int_{-4}^{-5} e^{(x+5)^2}dx + 3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2}dx$$

Let $I_1 = \int_{-4}^{-5} e^{(x+5)^2}dx$

$$= (-5+4)\int_0^1 e^{((-5+4)x-4+5)^2}dx$$

$\{$using property $\int_a^b f(x)dx = (b-a)\int_0^1 f((b-a)x+a)dx\}$

$$= -\int_0^1 e^{(x-1)^2}dx$$

$$I_2 = \int_{1/3}^{2/3} e^{9(x-2/3)^2}dx$$

$$= \left(\frac{2}{3}-\frac{1}{3}\right)\int_0^1 e^{9\left[\left(\frac{2}{3}-\frac{1}{3}\right)x+\frac{1}{3}-\frac{2}{3}\right]^2}dx$$

$$= \frac{1}{3}\int_0^1 e^{(x-1)^2}dx = \frac{-1}{3}I_1$$

where $I = I_1 + 3I_2$

$= I_1 + 3(-I_1/3) = 0$

$\therefore \ I = 0$

19. $x^2 + 2x + 1 = k + 1 + \int_0^1 |t + k|\, dt$

$(x + 1)^2 = (k + 1) + \int_0^1 |t + k|\, dt$

If $k \geq -1$ R.H.S. ≥ 0

so there will be two real and distinct roots for $k \geq -1$

If $k < -1$

$(x + 1)^2 = k + 1 - \int_0^1 (t + k)dt$

$(x + 1)^2 = 1/2$

so there will have two real and distinct roots for $k < -1$

$\Rightarrow$ The equation will have two real and distinct roots for $k \in R$,

20. **(II)** $\left(0, \frac{5}{2}\right]$ **(III)** 1, 3

21. **(III)** $I = \int_0^1 \frac{dx}{2 + x^2} + \int_1^2 \frac{dx}{2 + x^2}$

$\frac{1}{3} \leq \int_0^1 \frac{dx}{2 + x^2} \leq \frac{1}{2}$ ......**(1)**

$\frac{1}{6} \leq \int_1^2 \frac{dx}{2 + x^2} \leq \frac{1}{3}$ ......**(2)**

add (1) & (2)

$\frac{1}{2} \leq I \leq \frac{5}{6}$

(IV) $\because$ $\frac{\sin x}{x}$ is monotonic decreasing

$\therefore$ $\frac{\sqrt{3}}{8} < \int_{\pi/4}^{\pi/3} \frac{\sin x}{x}\, dx < \frac{\sqrt{2}}{6}$

(V) Let $f(x) = \sqrt{3 + x^3}$

$f'(x) = \frac{3x^2}{2\sqrt{3 + x^3}} > 0 \; \forall \; x \in (1, 3)$ $\Rightarrow$ f is strictly increasing is (1, 3)

$\because$ $m = \text{least value} = f(1) = \sqrt{3+1} = 2$

$M = \text{greatest value} = f(3) = \sqrt{30}$

$\because$ $m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$

$\Rightarrow$ $2(2) \leq \int_1^3 \sqrt{3 + x^3}\, dx \leq \sqrt{30} \times 2$

$\Rightarrow$ $4 \leq \int_1^3 \sqrt{3 + x^3}\, dx \leq 2\sqrt{30}$

22. 0

25. $\sec(1) - 1$

26. $I = \int_0^\infty f\left(\frac{a}{x} + \frac{x}{a}\right) \frac{\ell nx}{x}\, dx$

Let $x = \frac{a}{t}$ $\Rightarrow$ $dx = -\frac{a}{t^2}dt$

$\Rightarrow$ $I = \int_0^\infty f\left(t + \frac{1}{t}\right) \frac{\ell n\left(\frac{a}{t}\right)}{a/t} \cdot \left(\frac{a}{t^2}\right)dt$

Again put

$t = \frac{z}{a}$ $\Rightarrow$ $dt = \frac{dz}{a}$

$\Rightarrow$ $I = \int_0^\infty f\left(\frac{z}{a} + \frac{a}{z}\right) \frac{\ell n\left(\frac{a^2}{z}\right)}{\frac{z}{a}} \frac{dz}{a}$

$= \int_0^\infty f\left(\frac{z}{a} + \frac{a}{z}\right) \frac{[2\ell na - \ell nz]}{z} dz$

$= \int_0^\infty f\left(\frac{z}{a} + \frac{a}{z}\right) \frac{2\ell na}{z} dz - \int_0^\infty f\left(\frac{z}{a} + \frac{a}{z}\right) \frac{\ell nz}{z} dz$

$= 2\, \ell na \int_0^\infty f\left(\frac{z}{a} + \frac{a}{z}\right) \frac{dz}{z} - I$

$\Rightarrow$ $2I = 2\ell na \int_0^\infty f\left(\frac{a}{z} + \frac{z}{a}\right) \cdot \frac{dz}{z}$ $\Rightarrow$ I

$= \ell na \int_0^\infty f\left(\frac{a}{z} + \frac{z}{a}\right) \cdot \frac{dz}{z} = \ell na \int_0^\infty f\left(\frac{a}{x} + \frac{x}{a}\right) \cdot \frac{dx}{x}$

27. $I_n = \int_0^1 e^x.(x-1)^n\,dx$

$= e^x.(x-1)^n\Big|_0^1 - n\int_0^1 e^x(x-1)^{n-1}\,dx$

$I_n = -(-1)^n - n\,I_{n-1} = (-1)^{n+1} - n\,I_{n-1}$

$n = 1,\; I = \int_0^1 e^x(x-1)dx$

$= (x-1)e^x\Big|_0^1 - \int_0^1 e^x dx \;= 2 - e$

$I_2 = -1 - 2(2-e) = 2e - 5$

$I_3 = 1 - 3.(2e-5) = 16 - 6e$

so $n = 3$

28. (A) $f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}dt$

Put $t = \sin^2\theta$ in I[st] integral and $t = \cos^2\phi$ in the second integral

then $f(x) = \int_0^x \theta\sin 2\theta d\theta - \int_{\pi/2}^x \phi\sin 2\phi d\phi$

$= \int_0^x \theta\sin 2\theta d\theta + \int_x^{\pi/2} \theta\sin 2\theta d\theta$

$= \int_0^{\pi/2} \theta\sin 2\theta d\theta \;= \frac{\pi}{4}$

29. (i) $4\sqrt{2}$

31. $J_m = \int_1^e \ell n^m x\,dx = [x\ell n^m x]_1^e - m\int_1^e \ell n^{m-1} x.\frac{1}{x}dx$

$= e - mJ_{m-1}$

34. $b\beta - a\alpha$

35. (I) $2\,e^{(1/2)(\pi-4)}$ **(II)** 11 **(V)** Let $P = \lim_{n\to\infty}\left(\frac{\lfloor n}{n^n}\right)^{1/n}$

$P = \lim_{n\to\infty}\left(\frac{1.2.3.4..............n}{n.nnn...............n}\right)^{1/n}$

$P = \lim_{n\to\infty}\left(\left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\left(\frac{3}{n}\right).........\left(\frac{n}{n}\right)\right)^{1/n}$

$\Rightarrow \ell n P = \lim_{n\to\infty}\frac{1}{n}\left(\log\left(\frac{1}{n}\right) + \log\left(\frac{2}{n}\right) + ...\log\left(\frac{n}{n}\right)\right)$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\frac{r}{n} = \int_0^1 \ell nx dx \quad = [x\ell nx - x]_0^1$

$= (0-1) - \lim_{x\to 0}(x\ell nx) + 0$

$= -1 - \lim_{x\to 0}\frac{\ell nx}{1/x} = -1 - \lim_{x\to 0}\frac{1/x}{(-1/x^2)}$

$= -1 - \lim_{x\to 0} x = -1 + 0 = -1$

$\Rightarrow \ell np = -1$

$P = e^{-1} = 1/e$

(VI) $\frac{2}{3}$ **(VII)** 0

(VIII) $\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}} = \lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n}-\frac{1}{\sqrt{1-\left(\frac{n}{r}\right)^2}}$

$= \int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \left[\sin^{-1}x\right]_0^1 = \frac{\pi}{2}$

(IX) $\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{3}{n}\sqrt{\frac{n}{n+3r}} = \lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{3}{n}\sqrt{\frac{n/r}{3+n/r}}$

$= \int_0^1 3\sqrt{\frac{x}{x+3}}dx$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

3. $= \int_0^{\pi} |\sin x|\,dx + \int_{\pi}^{10\pi} |\sin x|\,dx - \int_0^{\pi} |\sin x|\,dx$

$= \int_0^{10\pi} |\sin x|\,dx - \int_0^{\pi} |\sin x|\,dx$

$= 10\int_0^{\pi} |\sin x|\,dx - \int_0^{\pi} |\sin x|\,dx = 9\int_0^{\pi} |\sin x|\,dx$

$= 9 \times 2 = 18$

4. $I = \int_0^{\sqrt{2}} [x^2]dx = \int_0^{1} [x^2]dx + \int_1^{\sqrt{2}} [x^2]dx$

$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} dx = [x]_1^{\sqrt{2}} = \sqrt{2} - 1$

11. $f(y) = e^y,\ g(y) = y\ ;\ y > 0$

and $F(t) = \int_0^t f(t-y)g(y)dy$.

$= \int_0^t e^{t-y} y\,dy = e^t \int_0^t e^{-y} y\,dy = e^t[-ye^{-y} - e^{-y}]_0^t$

$= -e^t[te^{-t} + e^{-t} - 0 - 1] = e^t - (1 + t)$

17. $f(x) = \frac{e^x}{1+e^x}$ $\qquad I_1 = \int_{f(-a)}^{f(a)} xg[x(1-x)]dx$

$I_2 = \int_{f(-a)}^{f(a)} g[x(1-x)]dx$

$f(a) = \frac{e^a}{1+e^a},\ f(-a) = \frac{e^{-a}}{1+e^{-a}}$

$\therefore f(a) + f(-a) = 1$

$2I_1 = \int_{f(-a)}^{f(a)} xg\{x(1-x)\}dx + \int_{f(-a)}^{f(a)} \{f(a) + f(-a) - x\}g(1-x)(x)dx$

$2I_1 = \int_{f(-a)}^{f(a)} g\{x(1-x)\}dx = I_2$

$\because\ f(a) + f(-a) = 1$

$2I_1 = I_2$

$\frac{I_2}{I_1} = 2$

18. $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{r}{n^2}\sec^2\frac{r^2}{n^2}$

$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n}.\frac{r}{n}\sec^2\frac{r^2}{n^2}$ $\qquad$ Put $\frac{1}{n} = dx\ ;\ \frac{r}{n} = x$

lower limit $x = \frac{r}{n}$

$r = 1$	$x = 1/n$
$n \to \infty$	$x = 0$
$r = n$	$x = 1$

$= \int_0^1 x\sec^2 x^2 dx$

Put $x^2 = t\ ;\ 2xdx = dt\ ;\ xdx = \frac{dt}{2}$

$x = 0, t = 0$

$x = 1, t = 1$

$= \frac{1}{2}\int_0^1 \sec^2 t\,dt$

$= \frac{1}{2}(\tan t)_0^1 \qquad = \frac{1}{2}\tan 1$

19. for $0 < x < 1,\quad x^2 > x^3$ and

for $1 < x < 2,\quad x^3 > x^2$

for, $0 < x < 1,\quad 2^{x^2} > 2^{x^3}$ and

for $1 < x < 2,\quad 2^{x^2} < 2^{x^3}$

$\therefore \int_0^1 2^{x^2}dx > \int_0^1 2^{x^3}dx$ and $\int_1^2 2^{x^2}dx < \int_1^2 2^{x^3}dx$

$\therefore I_1 > I_2$ and $I_3 < I_4$

21. Putting $-x$ for x

$$I = \int_{\pi}^{-\pi} \frac{\cos^2 x}{1+a^{-x}}(-dx) = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^{-x}} dx$$

$$I + I = \int_{-\pi}^{\pi} \cos^2 x\left(\frac{1}{1+a^x} + \frac{1}{1+a^{-x}}\right) dx$$

$$= \int_{-\pi}^{\pi} \cos^2 x dx \Rightarrow 2I = 2\int_0^{\pi} \cos^2 x dx$$

$$= \int_0^{\pi} (1+\cos 2x) dx$$

$$2I = \left[x + \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$2I = \pi \quad \Rightarrow \quad I = \frac{\pi}{2}$$

25. $= \int_1^2 1.f'(x)dx + \int_2^3 2.f'(x)dx + ... + \int_{[a]}^{a} [a]f'(x)dx$

$= [f(2) - f(1)] + 2[f(3) - f(2)] + ... + [a]\,[f(a)]$

$= [a]\, f(a) - \{f(1) + f(2) + ... + f[a]\}$

26. $F(x) = f(x) + f(1/x)$ put $x = e$

$$F(e) = \int_1^e \frac{\log t}{1+t} dt + \int_1^{1/e} \frac{\log t}{1+t} dt$$

let $t = \frac{1}{z} \Rightarrow \frac{dt}{dz} = \left(\frac{-1}{z^2}\right)$

$$= \int_1^e \frac{\ln t}{(1+t)} dt + \int_1^e \frac{\ln 1/z}{(1+1/z)}\left(\frac{-1}{z^2}\right) dz$$

by property $\int_a^b f(x)dx = \int_a^b f(t)dt$

$$\int_1^e \frac{\ln t}{(1+t)} dt + \int_1^e \frac{\ln t}{t(1+t)} dt = \int_1^e \frac{\ln t}{t} dt = \frac{1}{2}$$

28. Now

$$\sin x < x \quad \Rightarrow \quad \frac{\sin x}{\sqrt{x}} < \sqrt{x}$$

$$\int_0^1 \frac{\sin x}{\sqrt{x}} dx < \int_0^1 \sqrt{x}\, dx$$

$$I < \left[\frac{2}{3} x^{3/2}\right]_0^1$$

$$I < \frac{2}{3}$$

$\because \quad \cos x < 1$

$$\frac{\cos x}{\sqrt{x}} < \frac{1}{\sqrt{x}} \Rightarrow \int_0^1 \frac{\cos x}{\sqrt{x}} dx < \int_0^1 \frac{1}{\sqrt{x}} dx < \left[2\sqrt{x}\right]_0^1 < 2$$

$J < 2$

29. $I = \int_0^{\pi} [\cot x] dx$ .... **(1)**

$I = \int_0^{\pi} [\cot(\pi - x)] dx = \int_0^{\pi} [-\cot x] dx$ ...... **(2)**

add (1) & (2)

$2I = \int_0^{\pi} [\cot x] + [-\cot x] dx \quad \because [x] + [-x] = -1$

$$= \int_0^{\pi} -1\, dx = -[x]_0^{\pi} \Rightarrow I = -\frac{\pi}{2}$$

32. $\int_0^{1.5} x[x^2] dx$

$$\int_0^1 0\, dx + \int_1^{\sqrt{2}} x\, dx + \int_{\sqrt{2}}^{1.5} 2x\, dx$$

$$\left[\frac{x^2}{2}\right]_1^{\sqrt{2}} + \left[x^2\right]_{\sqrt{2}}^{1.5}$$

$$\left(\frac{2}{2} - \frac{1}{2}\right) + (2.25 - 2)$$

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

33. $g(x) = \int_0^x \cos 4t\, dt$

$g(x+\pi) = \int_0^{x+\pi} \cos 4t\, dt$

$= \int_0^x \cos 4t\, dt + \int_x^{x+\pi} \cos 4t\, dt$

$= \int_0^x \cos 4t\, dt + \int_0^{\pi} \cos 4t\, dt = g(x) + g(\pi)$

Because $g(\pi) = 0$ so $g(x) - g(\pi)$ is also correct Ans.

34. Statement-I : $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

$I = \int_{\pi/6}^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$ ......**(1)**

use $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}dx}{\sqrt{\cos x}+\sqrt{\sin x}}$ .....**(2)**

$(1)+(2)$

$2I = \int_{\pi/6}^{\pi/3} dx$

$2I = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

$I = \frac{\pi}{12}$

So Statement-I is false.

and statement-II is true as it is property.

Part # II : IIT-JEE ADVANCED

6. Given that f(x) is an even function, then to prove

$\int_0^{\pi/2} f(\cos 2x)\cos x dx = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\, dx$

Let $I = \int_0^{\pi/2} f(\cos 2x)\cos x\, dx$ ....**(1)**

$= \int_0^{\pi/2} f\left[\cos 2\left(\frac{\pi}{2}-x\right)\right]\cos\left(\frac{\pi}{2}-x\right) dx$

$\left[\text{Using } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$= \int_0^{\pi/2} f(-\cos 2x)\sin x dx$

$I = \int_0^{\pi/2} f(\cos 2x)\sin x dx$ .....**(2)**

[As $f(x)$ is an even function]

adding two values of I in (1) and (2) we get

$2I = \int_0^{\pi/2} f(\cos 2x)(\sin x + \cos x)dx$

$\Rightarrow I = \frac{\sqrt{2}}{2}\int_0^{\pi/2} f(\cos 2x)\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right]dx$

$I = \frac{\sqrt{2}}{2}\int_0^{\pi/2} f(\cos 2x)\cos(x-\pi/4)dx$

Let $x - \pi/4 = t \Rightarrow dx = dt$

$\therefore I = \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} f[\cos 2(t+\pi/4)]\cos t\, dx$

$= \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} f[-\sin 2t]\cos t\, dt$

$= \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} f(\sin 2t)\cos t\, dt$

[$\because$ f is an even function]

$= \frac{2}{\sqrt{2}}\int_0^{\pi/4} f(\sin 2t)\cos t\, dt$

[$\because$ f is an even function]

$= \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\cos x\, dx = \text{R.H.S.}$

8. **(B)** $I = \int_{-2}^{0} [x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)]dx$

$= \left[\frac{x^4}{4} + x^3 + \frac{3x^2}{2} + 3x + (x+1)\sin(x+1) + \cos(x+1)\right]_{-2}^{0}$

$= 4$

9. Let $I=\int_0^{\pi} e^{|\cos x|}\left[2\sin\left(\frac{1}{2}\cos x\right)+3\cos\left(\frac{1}{2}\cos x\right)\right]\sin x\,dx$

$$=\int_0^{\pi} e^{|\cos x|}\,2\sin\left(\frac{1}{2}\cos x\right)\sin x\,dx$$

$$+\int_0^{\pi} e^{|\cos x|}\,3\cos\left(\frac{1}{2}\cos x\right)\sin x\,dx$$

$= I_1 + I_2$

Now using the property that

$$\int_0^{2a} f(x)dx=0 \qquad \text{if } f(2a-x)=-f(x)$$

$$=2\int_0^{a} f(x)dx \qquad \text{if } f(2a-x)=f(x)$$

We get, $I_1 = 0$

and $I_2=2\int_0^{\pi/2} e^{|\cos x|}\,3\cos\left(\frac{1}{2}\cos x\right)\sin x\,dx$

$$=6\int_0^{\pi/2} e^{\cos x}\cos\left(\frac{1}{2}\cos x\right)\sin x\,dx$$

Put $\cos x = t$ $\Rightarrow$ $-\sin x\,dx = dt$, we get

or $I_2=6\int_0^{1} e^{t}\cos\frac{t}{2}dt$

$$I_2=6\left[(e^t\cos\frac{t}{2})_0^1+\frac{1}{2}\int_0^1 e^t\sin\frac{t}{2}dt\right]$$

$$=6\left[e\cos(1/2)-1+\frac{1}{2}\left\{(e^t\sin t/2)_0^1-\frac{1}{2}\int_0^1 e^t\cos t/2\,dt\right\}\right]$$

$$I_2=6\left[e\cos\left(\frac{1}{2}\right)-1+\frac{1}{2}\left\{e\sin(1/2)-\frac{1}{2}.\frac{1}{6}I_2\right\}\right]$$

$$I_2+\frac{1}{4}I_2=6\left[e\cos(1/2)+\frac{1}{2}e\sin(1/2)-1\right]$$

$$\Rightarrow I_2=\frac{24}{5}\left[e\cos(1/2)+\frac{1}{2}e\sin\left(\frac{1}{2}\right)-1\right]$$

10. $\int_0^{\pi/2}\sin x\,dx=\frac{\left(\frac{\pi}{2}-0\right)}{4}\left(\sin 0+\sin\frac{\pi}{2}+2\sin\frac{\pi}{4}\right)$

$$=\frac{\pi}{8}(1+\sqrt{2})$$

11. $f''(x)<0,\ \forall\ x\in(a,b)$, for $c\in(a,b)$

$$F(c)=\frac{c-a}{2}(f(a)+f(c))+\frac{b-c}{2}(f(b)+f(c))$$

$$=\frac{b-a}{2}f(c)+\frac{c-a}{2}f(a)+\frac{b-c}{2}f(b)$$

$$\Rightarrow F'(c)=\frac{b-a}{2}f'(c)+\frac{1}{2}f(a)-\frac{1}{2}f(b)$$

$$=\frac{1}{2}[(b-a)f'(c)+f(a)-f(b)]$$

$$F''(c)=\frac{1}{2}(b-a)f''(c)<0$$

$[\because f''(x)<0,\forall x\in(a,b)$ and $b>a]$

$\therefore$ F(c) is max. at the point (c, f(c)) where

$$F'(c)=0\Rightarrow f'(c)=\left(\frac{f(b)-f(a)}{b-a}\right)$$

12. $\lim_{x\to a}\frac{\int_a^x f(x)dx-\left(\frac{x-a}{2}\right)(f(x)+f(a))}{(x-a)^3}=0$

$$\lim_{h\to 0}\frac{\int_a^{a+h} f(x)dx-\frac{h}{2}(f(a+h)+f(a))}{h^3}=0$$

$$\Rightarrow \lim_{h\to 0}\frac{f(a+h)-\frac{1}{2}[f(a)+f(a+h)]-\frac{h}{2}(f'(a+h))}{3h^2}=0$$

[Using L'Hospital rule]

$$\Rightarrow \lim_{h\to 0}\frac{\frac{1}{2}f(a+h)-\frac{1}{2}f(a)-\frac{h}{2}f'(a+h)}{3h^2}=0$$

$$\Rightarrow \lim_{h\to 0}\frac{\frac{1}{2}f'(a+h)-\frac{1}{2}f'(a+h)-\frac{h}{2}f''(a+h)}{6h}=0$$

[Using L' Hospital rule]

$\Rightarrow \lim_{h\to 0}\frac{-f''(a+h)}{12}=0$ $\Rightarrow$ $f''(x)=0,\ \forall\ a\in R$

$\Rightarrow$ f(x) must be of max. degree 1

13. Let $I = \int_0^1 (1-x^{50})^{100}\,dx$ and $I' = \int_0^1 (1-x^{50})^{101}\,dx$

Then, $I' = \int_0^1 1.(1-x^{50})^{101}\,dx = (x(1-x^{50})^{101})_0^1$

$+101\int_0^1 50x^{50}(1-x^{50})^{100}\,dx$

$= 5050\int_0^1 x^{50}(1-x^{50})^{100}\,dx$

$-I' = 5050\int_0^1 -x^{50}(1-x^{50})^{100}\,dx$

$\Rightarrow 5050I - I' = 5050\int_0^1 (1-x^{50})^{100}\,dx$

$+5050\int_0^1 -x^{50}(1-x^{50})^{100}\,dx$

$\Rightarrow 5050\int_0^1 (1-x^{50})^{101}\,dx = 5050\ I'$

$\Rightarrow 5050\ I = 5051\ I' \Rightarrow\ 5050\dfrac{I}{I'} = 5051$

17. $S_n = \dfrac{1}{n}\sum_{k=1}^{n}\dfrac{1}{1 + k/n + k^2/n^2}$

$S_n < \int_0^1 \dfrac{dx}{x^2+x+1}$

($\because$ the function is decreasing)

$S_n < \int_0^1 \dfrac{dx}{\left(x+1/2\right)^2 + \left(\sqrt{3}/2\right)^2}$

$S_n < \dfrac{2}{\sqrt{3}}\left[\tan^{-1}\dfrac{2x+1}{\sqrt{3}}\right]_0^1$

$S_n < \dfrac{2}{\sqrt{3}}\left[\dfrac{\pi}{3} - \dfrac{\pi}{6}\right]$

$S_n < \dfrac{\pi}{3\sqrt{3}}$

Now $T_n - S_n = 1 - \dfrac{1}{3n} \Rightarrow T_n - S_n > \dfrac{2}{3}$

$\Rightarrow\ T_n > S_n + \dfrac{2}{3}$

as $S_n < \dfrac{\pi}{3\sqrt{3}}$ so $T_n > \dfrac{\pi}{3\sqrt{3}}$

18. $\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)dt\ , 0 \le x \le 1$

differentiating both the sides & squreing

$\Rightarrow\ 1-(f'(x))^2 = f^2(x) \Rightarrow \dfrac{f'(x)}{\sqrt{1-f^2(x)}} = 1$

$\Rightarrow\ \sin^{-1} f(x) = x + c$

$f(0) = 0$

$\Rightarrow\ f(x) = \sin x\ \ \Rightarrow\ \sin x \le x$ for $x \in [0, 1]$

$\Rightarrow\ f\left(\dfrac{1}{2}\right) < \dfrac{1}{2}$ and $f\left(\dfrac{1}{3}\right) < \dfrac{1}{3}$.

19. $I_n = \int_{-\pi}^{\pi}\dfrac{\sin nx}{\left(1+\pi^x\right)\sin x}\,dx$

$I_n = \int_{-\pi}^{\pi}\dfrac{\pi^x \sin nx}{\left(1+\pi^x\right)\sin x}\,dx$

$2I_n = \int_{-\pi}^{\pi}\dfrac{\sin nx}{\sin x}dx$ ..**(i)**

$2\,I_{n+2} = \int_{-\pi}^{\pi}\dfrac{\sin\left(n+2\right)x}{\sin x}\,dx$ ...**(ii)**

(ii) – (i)

$\Rightarrow\ 2(I_{n+2}-I_n) = \int_{-\pi}^{\pi}\cos\left(n+1\right)x = 0\ \ \Rightarrow\ \ I_{n+2} = I_n$

$\sum_{m=1}^{10} I_{2m} = 10\sum_{m=1}^{10} I_2 = \dfrac{10}{2}\int_{-\pi}^{\pi}\dfrac{\sin 2x}{\sin x}\,dx = 0$

Put n = 1 in equation **(i)**

$2I_1 = \int_{-\pi}^{\pi}\dfrac{\sin x dx}{\sin x} = 2\pi$

$I_1 = \pi$

$\sum_{m=1}^{10} I_{2m+1} = 10\pi$

20. $f(x) = \int_0^x f\left(t\right)dt$ ...**(i)**

$f'(x) = f(x)\ \ \Rightarrow\ \ f(x) = k.e^x$

From **(i)** $f(0) = 0$

$\Rightarrow\ f(0) = k.e^0\ \ \Rightarrow\ \ k = 0\ \ \Rightarrow\ \ f(x) = 0$

21. Applying L-Hospital rule,

$$\lim_{x\to 0}\frac{\int_0^x \frac{t\ell n(1+t)}{t^4+4}dt}{x^3} = \lim_{x\to 0}\frac{\frac{x\ell n(1+x)}{x^4+4}}{3x^2}$$

$$= \lim_{x\to 0}\frac{\ell n(1+x)}{3x(x^4+4)} = \frac{1}{12}$$

22. $I = \int_0^1 \frac{x^4\left(1-2x+x^2\right)^2}{1+x^2}dx$

$$I = \int_0^1 \frac{x^4\left\{\left(1+x^2\right)^2 - 4x\left(1+x^2\right)+4x^2\right\}}{1+x^2}dx$$

$$= \int_0^1\left(1+x^2\right)x^4dx - \int_0^1 4x^5dx + 4\int_0^1\frac{\left(x^6+1\right)-1}{1+x^2}dx$$

$$= \frac{1}{5}+\frac{1}{7}-4.\frac{1}{6}+4\int_0^1\frac{\left(x^2+1\right)^3-3x^2\left(1+x^2\right)}{1+x^2}dx - 4\int_0^1\frac{dx}{1+x^2}$$

$$= \frac{12}{35}-\frac{2}{3}+4\int_0^1\left(x^4+2x^2+1\right)dx - 12\int_0^1 x^2dx - \pi$$

$$= \frac{12}{35}-\frac{2}{3}+4\left(\frac{1}{5}+\frac{2}{3}+1\right)-4-\pi$$

$$= \frac{12}{35}-\frac{2}{3}+\frac{52}{15}-\pi = \frac{22}{7}-\pi$$

23. $f(x) = \begin{cases}\{x\} & \text{when } -9 \le x < -8;\ -7 \le x < -6, \ldots\ldots \\ 1-\{x\} & \text{when } -10 \le x \le -9;\ -8 \le x < -7, \ldots\ldots\end{cases}$

Since f(x) & $\cos\pi x$ both are periodic functions having period 2.

$$I = \frac{10\times\pi^2}{10}\left(\int_0^1(1-\{x\})\cos\pi x dx + \int_1^2\{x\}\cos\pi x dx\right)$$

$$= \pi^2\left(\int_0^1(1-x)\cos\pi x dx + \int_1^2(x-1)\cos\pi x dx\right)$$

$$= \pi^2\left(\int_0^1\cos\pi x dx - \int_1^2\cos\pi x dx + \int_1^2 x\cos\pi x dx - \int_0^1 x\cos\pi x dx\right)$$

$\Rightarrow\ I = 4$

24. $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4+1}\,dt$

$e^{-x}f'(x) - e^{-x}f(x) = \sqrt{x^4+1}$

$\Rightarrow\ f'(x) - f(x) = e^x\sqrt{x^4+1}$

$\Rightarrow\ \frac{dy}{dx} = y + e^x\sqrt{x^4+1}$ (say) ....(i)

considering y = f(x). so that x = f'(y)

$f^{-1}{}'(2) = \left(\frac{dx}{dy}\right)_{y=2}$ ....(ii)

for x = 0 $\Rightarrow$ f(x) = 2 i.e. y = 2

$\Rightarrow\ f^{-1}(2) = 0$

$\frac{dy}{dx} = 2 + 1\sqrt{1} = 3$

from (2), $f^{-1}{}'(2) = \frac{1}{3}$

25. $I = \int_{\sqrt{\ell n2}}^{\sqrt{\ell n3}}\frac{x\sin x^2}{\sin x^2 + \sin(\ell n6 - x^2)}dx$; put $x^2 = t$

$\Rightarrow$ 2xdx = dt

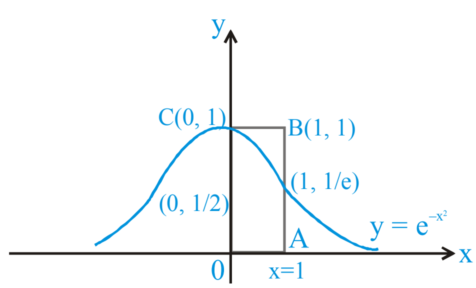
$\Rightarrow\ I = \frac{1}{2}\int_{\ell n2}^{\ell n3}\frac{\sin t}{\sin t + \sin(\ell n6 - t)}dt$ ....(i)

$\Rightarrow\ I = \frac{1}{2}\int_{\ell n2}^{\ell n3}\frac{\sin(\ell n6 - t)}{\sin(\ell n6 - t)+\sin t}dt$ ....(ii)

Adding equation (i) & (ii)

$\Rightarrow\ 2I = \frac{1}{2}\int_{\ell n2}^{\ell n3}dt\ \Rightarrow\ I = \frac{1}{4}\ell n\left(\frac{3}{2}\right)$

26. Area (OABC) = 1



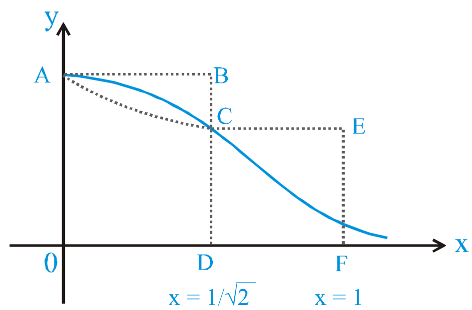
Shaded area is S.

Clearly S < 1

and $\int_0^1 e^{-x^2}dx > \int_0^1 e^{-x}dx$

$\Rightarrow \quad S > 1 - \dfrac{1}{e}$ ($\therefore$ **(B)** is correct)

Again $S \geq$ Area (trapezium ACDO)



$$\Rightarrow \quad S \geq \frac{1}{2}\left(1+\frac{1}{\sqrt{e}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad S \geq \frac{1}{2\sqrt{2}}\left(1+\frac{1}{\sqrt{e}}\right)$$

$\therefore$ C is wrong

Also $S \leq$ Sum of areas of rectangles ABDO & CEFD

$$\Rightarrow \quad S \leq \frac{1}{\sqrt{2}} \times 1 + \left(1-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{e}}\right)$$

$$\Rightarrow \quad S \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1-\frac{1}{\sqrt{2}}\right)$$

($\therefore$ **(D)** is correct)

27. $\displaystyle\int_{-\pi/2}^{\pi/2} x^2 \cos x\,dx + \int_{-\pi/2}^{\pi/2} \ell n\left(\frac{\pi+x}{\pi-x}\right)\cos x\,dx$

$$= \int_{-\pi/2}^{\pi/2} x^2 \cos x\,dx = 2\int_0^{\pi/2} \underset{\text{I}}{x^2}\ \underset{\text{II}}{\cos x}\,dx$$

$$= 2\left((x^2 \sin x)_0^{\pi/2} - 2\int_0^{\pi/2} x \sin x\,dx\right)$$

$$= 2\left(\frac{\pi^2}{4} - 2\left(-(x\cos x)_0^{\pi/2} + \int_0^{\pi/2} \cos x\,dx\right)\right)$$

$$= 2\left(\frac{\pi^2}{4} - 2\int_0^{\pi/2} \cos x\,dx\right)$$

$$= 2\left(\frac{\pi^2}{4} - 2\right) = \frac{\pi^2}{2} - 4$$

28.

$$L = \lim_{n\to\infty} \frac{1^a + 2^a + \ldots\ldots + n^a}{(n+1)^{a-1}\left[\underbrace{na + na + na + \ldots + na}_{n\text{ times}} + 1 + 2 + 3 + \ldots + n\right]}$$

$$= \lim_{n\to\infty} \frac{\sum_{r=1}^{n} r^a}{(n+1)^{a-1}\left[n^2 a + \frac{n(n+1)}{2}\right]}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^a}{n^a}\right)n^{a+1}}{(n+1)^{a-1}\left[n^2 a + \frac{n(n+1)}{2}\right]}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^a}{n^a}\right)}{\left(\frac{n+1}{n}\right)^{a-1}\left[\frac{n^2 a + \frac{n(n+1)}{2}}{n^2}\right]}$$

$$= \frac{\int_0^1 x^a\,dx}{\left(a+\frac{1}{2}\right)} = \frac{1}{60} \Rightarrow \frac{2}{(a+1)(2a+1)} = \frac{1}{60}$$

$$\Rightarrow \quad 2a^2 + 3a - 119 = 0 \qquad \Rightarrow \quad a = 7 \ \&\ -\frac{17}{2}$$

$a = -\dfrac{17}{2}$ will be rejected as $\displaystyle\int_0^1 x^{-\frac{17}{2}}\,dx$ is not defined.

42. Let $f(x) = \displaystyle\int_0^x \frac{t^2\,dt}{1+t^4} - 2x + 1$

$$f'(x) = \frac{x^2}{1+x^4} - 2$$

$$\Rightarrow \quad \frac{-2x^4 + x^2 - 2}{x^4+1} < 0\ \forall x \in R$$

$f(0) > 0, \quad f(1) < 0$

$\therefore$ One solution in (0, 1)

43. $I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1+e^x} dx$ ....(i)

$I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1+\frac{1}{e^x}} dx$

$= \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1+\frac{1}{e^x}} dx$ ....(ii)

(i) and (ii)

$2I = \int_{-\pi/2}^{\pi/2} x^2 \cos x \, dx$

$I = \int_{0}^{\pi/2} x^2 \cos x \, dx$ (even fn)

$= x^2 . \sin x \,|_0^{\pi/2} - \int_0^{\pi/2} 2x \sin x \, dx$

$= \frac{\pi^2}{4} - 2\left[(-x\cos x)_0^{\pi/2} - \int_0^{\pi/2}(-\cos x)dx\right]$

$= \frac{\pi^2}{4} - 2\left[0 + \sin x \,|_0^{\pi/2}\right]$

$= \frac{\pi^2}{4} - 2[1] = \frac{\pi^2}{4} - 2$

MOCK TEST

1. $I = \int_{-1}^{1}\left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right)dx = \int_1^3 \frac{\pi}{2} dx = (3-1)$

$\frac{\pi}{2} = \pi$

2. **(B)**

$p = \lim_{n\to\infty}\left[\frac{\prod_{r=1}^{n}(n^3+r^3)}{n^{3n}}\right]^{1/n}$

$\ell n\, p = \lim_{n\to\infty} . \frac{1}{n}\sum_{r=1}^{n} \ell n\left(1+\left(\frac{r}{n}\right)^3\right) = \int_0^1 \ell n\,(1+x^3)\,dx$

$= \ell n\, 2 - 3 + 3\lambda$

3. Let $5(4x-5) = t^2$ $\Rightarrow$ $20x - 25 = t^2$

$\Rightarrow x = \frac{t^2+25}{20}$

Also $20\,dx = 2t\,dt$ or $dx = \frac{t}{10}dt$

$\therefore I = \int_{\sqrt{15}}^{\sqrt{35}}\left(\sqrt{\frac{t^2+25}{10} - t} + \sqrt{\frac{t^2+25}{10} + t}\right)\frac{t}{10}\,dt$

$= \int_{\sqrt{15}}^{\sqrt{35}}\left(\frac{|t-5|}{\sqrt{10}} + \frac{|t+5|}{\sqrt{10}}\right)\frac{t}{10}\,dt$

$= \int_{\sqrt{15}}^{5}\left(\frac{-t+5}{\sqrt{10}} + \frac{t+5}{\sqrt{10}}\right)\frac{t}{10}\,dt$

$+ \int_{5}^{\sqrt{35}}\left(\frac{t-5}{\sqrt{10}} + \frac{t+5}{\sqrt{10}}\right)\frac{t}{10}\,dt = \frac{1}{\sqrt{10}}\left[\frac{t^2}{2}\right]_{\sqrt{15}}^{5}$

$+ \frac{1}{5\sqrt{10}}\int_5^{\sqrt{35}} t^2\,dt$

$= \frac{1}{\sqrt{10}}\left[\frac{25}{2} - \frac{15}{2}\right] + \frac{1}{15\sqrt{10}}\left[(\sqrt{35})^3 - 125\right]$

$= \frac{\sqrt{10}}{2} + \frac{7\sqrt{7} - 5\sqrt{5}}{3\sqrt{2}}$

4. (C)

Since $0 < x < 1 \Rightarrow x > x^2 > \frac{x^2}{2}$

$\Rightarrow -x < -x^2 < -\frac{x^2}{2} \Rightarrow e^{-x} < e^{-x^2} < e^{-\frac{x^2}{2}}$

$\Rightarrow e^{-x}\cos^2 x < e^{-x^2}\cos^2 x < e^{-\frac{x^2}{2}}\cos^2 x < e^{-\frac{x^2}{2}}$

$\therefore \int_0^1 e^{-x}\cos^2 x\,dx < \int_0^1 e^{-x^2}\cos^2 x\,dx$

$< \int_0^1 e^{-\frac{x^2}{2}}\cos^2 x\,dx < \int_0^1 e^{-\frac{x^2}{2}}\,dx$

$\therefore I_1 < I_2 < I_3 < I_4$

5. Given,

$f'(1) = \left(\frac{dy}{dx}\right)_{x=1} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$f'(2) = \left(\frac{dy}{dx}\right)_{x=2} = \tan\frac{\pi}{3} = \sqrt{3}$

$f'(3) = \left(\frac{dy}{dx}\right)_{x=3} = \tan\frac{\pi}{4} = 1$

Let, $I = \int_1^3 f'(x).f''(x)dx + \int_2^3 f''(x)dx = I_1 + I_2$

$\therefore I_1 = \int_1^3 f'(x).f''(x)dx$

$I_1 = f'(x).f'(x)\Big|_1^3 - \int_1^3 f''(x)f'(x)dx$

$2I_1 = \{f'(3)\}^2 - \{f'(1)\}^2$

$2I_1 = 1 - \frac{1}{3}$

$I_1 = \frac{1}{3}$

and, $I_2 = \int_2^3 f''(x)dx = f'(x)\Big|_2^3 = f'(3) - f'(2)$

$= 1 - \sqrt{3}$

$\therefore I = I_1 + I_2 = \frac{1}{3} + 1 - \sqrt{3} = \frac{4}{3} - \sqrt{3}$

6. (D)

$\lim_{n\to\infty} \frac{1}{\sqrt{4n^2}} + \frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \ldots\ldots\ldots + \frac{1}{\sqrt{4n^2 - (n-1)^2}}$

$= \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{\sqrt{4n^2 - r^2}} = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{n}.\frac{1}{\sqrt{4 - \left(\frac{r}{n}\right)^2}}$

$= \int_0^1 \frac{dx}{\sqrt{4 - x^2}} = \left(\sin^{-1}\frac{x}{2}\right)_0^1 = \frac{\pi}{6}$

7. $I = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{\sqrt{n}}{\sqrt{r}.n\left(3\sqrt{\frac{r}{n}} + 4\right)^2}$

$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{\sqrt{\frac{r}{n}}\left(3\sqrt{\frac{r}{n}} + 4\right)^2} \cdot \frac{1}{n} = \int_0^1 \frac{1}{\sqrt{x}(3\sqrt{x} + 4)^2}dx$

Put $z = 3\sqrt{x} + 4 \Rightarrow I = \frac{2}{3}\int_4^7 \frac{dz}{z^2} = \frac{1}{14}$

8. (B)

Put $x = a\cos^2\theta + b\sin^2\theta$

$\Rightarrow dx = 2(b - a)\sin\theta\cos\theta\,d\theta$

$\therefore I = \int_a^b (x - a)^3 (b - x)^4\,dx = 2(b - a)^8$

$\int_0^{\pi/2} \sin^7\theta\cos^9\theta\,d\theta \qquad = 2(b - a)^8 \int_0^{\pi/2} \sin^7\theta(1 - \sin^2\theta)^4$

$\cos\theta d\theta$

Let $\sin\theta = t \Rightarrow \cos\theta\; d\theta = dt$

$\Rightarrow I = 2(b - a)^8 \int_0^1 t^7(1 - t^2)^4 dt$

$= 2(b - a)^8 \int_0^1 t^7(1 - 4t^2 + 6t^4 - 4t^6 + t^8)dt$

$= \frac{(b - a)^8}{280}$

9. $I_n + I_{n+2} = \int_0^{\pi/4} (\tan^n x + \tan^{n+2} x)dx,\ n \ge 2$

$$= \int_0^{\pi/4} \tan^n x\ \sec^2 x\ dx = \frac{1}{n+1}$$

$\therefore\ I_2 + I_4 = \frac{1}{3},\quad I_3 + I_5 = \frac{1}{4},\quad I_4 + I_6 = \frac{1}{5}$ etc.

$\therefore\ \frac{1}{I_2+I_4}, \frac{1}{I_3+I_5}, \frac{1}{I_4+I_6}, ...$

are in A.P.

10. (D)

True

S_1 : $\cot^{-1} x = t$

when $x = -1$ $\quad t = \frac{3\pi}{4}$

when $x = 1$ $\quad t = \frac{\pi}{4}$ $\quad$ True

S_2 : True

S_3 : False $\quad$ (the discontinuity may not be removable)

S_4 : False

11. (B, C)

(A) $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$

(B) $\lim_{n\to\infty} \frac{1}{n} \sum_{r=n+1}^{2n} f\left(\frac{r}{n}\right) = \int_1^2 f(x)dx$

(C) $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r+n}{n}\right) = \int_0^1 f(1+x)dx$

$$= \int_1^2 f(t)dt = \int_1^2 f(x)dx$$

(D) $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} f\left(\frac{r}{n}\right) = \int_0^2 f(x)dx$.

12. (A, B)

$$I_n = \int_0^1 \frac{dx}{(1+x^2)^n} = \int_0^1 \left(1 + x^2\right)^{-n} dx$$

$$= \frac{x}{(1+x^2)^n}\Bigg|_0^1 - \int_0^1 (-n)(1+x^2)^{-n-1}.2x.\,xdx$$

$$= \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}} = \frac{1}{2^n} + 2n\int_0^1 \frac{1+x^2-1}{(1+x^2)^{n+1}}dx$$

$$= \frac{1}{2^n} + 2n\,I_n - 2n\,I_{n+1}$$

$\therefore\ 2n\,I_{n+1} = 2^{-n} + (2n-1)\,I_n$

$\therefore\ 2I_2 = \frac{1}{2} + I_1 = \frac{1}{2} + \tan^{-1}x\,\big|_0^1$

$$I_2 = \frac{1}{4} + \frac{\pi}{8}.$$

13. $f(x) = 2^{\{x\}}$

Clearly f(x) is periodic with period 1.

Now $\quad \int_0^1 2^{\{x\}}dx = \int_0^1 2^x dx = \left[\frac{2^x}{\ell n2}\right]_0^1 = \frac{1}{\ell n2} = \log_2 e$

Also $\quad \int_0^{100} 2^{\{x\}}dx = 100\int_0^1 2^{\{x\}}dx = 100\log_2 e$

$$\left[\text{Using } \int_0^{na} f(x)dx = n\int_0^a f(x)dx \text{ if a is the period of } f(x)\right]$$

14. (A,B,D)

$f(2-x) = f(2+x),\quad f(4-x) = f(4+x)$

$f(4+x) = f(4-x) = f(2+2-x) = f(2-(2-x) = f(x)$

$\therefore$ 4 is a period of f(x)

$$\int_0^{50} f(x)dx = \int_0^{48} f(x)dx + \int_0^2 f(x)dx = 12\int_0^4 f(x)dx + \int_0^2 f(x)dx$$

$$= 12\left(\int_0^2 f(x)\,dx + \int_0^2 f(4-x)\,dx\right) + 5$$

$$= 12\left(\int_0^2 f(x)\,dx + \int_0^2 f(4+x)\,dx\right) + 5$$

$$= 24\int_0^2 f(x)\,dx + 5 = 125$$

Also $\quad \int_0^{50} f(x)\,dx = \int_{-4}^{46} f(x)\,dx = \int_2^{52} f(x)\,dx = 125$

15. (A, D)

$$F'(x) = -\frac{2}{x^3}\int_4^x \{4t^2 - 2F'(t)\}\ dt + \frac{1}{x^2}\ (4x^2 - 2F'(x)$$

Put $x = 4$

$$F'(4) = 0 + \frac{1}{16}\ [4(16) - 2F'(4)] = 4 - \frac{F'(4)}{8}$$

$\Rightarrow\ F'(4) = \frac{32}{9}$ $\quad\therefore$ Option **(A)** is correct

For option **(D)**

$$F(x) = \frac{1}{x^2}\int_4^k [4t^2 - 2F'(t)]\,dt = \frac{1}{x^2}\left[\frac{4t^3}{3} - 2F(t)\right]_4^x$$

$$\therefore\ F(8) = \frac{1}{8^2}\left[\frac{4.8^3}{3} - 2F(8) - \frac{4.4^3}{3} + 2F(4)\right]$$

$$= \frac{32}{3} - \frac{F(8)}{32} - \frac{4}{3} + 2(0)$$

$$\therefore\ F(8) = \frac{32}{33}\left\{\frac{28}{3}\right\}$$

$$\therefore\ F'(4) = \frac{11F(8)}{28}$$

16. (C)

Statement-I

$$I = \int_0^{\pi/4} \frac{1}{\cos x}\sqrt{\frac{1-\sin x}{1+\sin x}}\,dx = \int_0^{\pi/4} \frac{1-\sin x}{\cos x \cdot \cos x}\,dx$$

$$= \int_0^{\pi/4} (\sec^2 x - \sec x \tan x)\,dx = [\tan x - \sec x]_0^{\pi/4}$$

$$= 2 - \sqrt{2}$$

Statement-II

$$I = \int_0^{\pi/4} \frac{\sec x\,dx}{1 + 2\sin^2 x} = \int_0^{\pi/4} \frac{\cos x}{(1 - \sin^2 x)(1 + 2\sin^2 x)}\,dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} \frac{dt}{(1-t^2)\,(1+2t^2)} = \frac{1}{3}\int_0^{\frac{1}{\sqrt{2}}} \left(\frac{1}{1-t^2} + \frac{2}{1+2t^2}\right) dt$$

$$= \left(\frac{1}{3}\frac{1}{2}\ \ell n\ \frac{1+t}{1-t} + \frac{2}{3\sqrt{2}}\tan^{-1}\sqrt{2}\ t\right)_0^{1/\sqrt{2}}$$

$$= \frac{1}{6}\ \ell n\ \frac{\sqrt{2}+1}{\sqrt{2}-1} + \frac{\sqrt{2}}{3}\ \left(\frac{\pi}{4}\right) = \frac{1}{6}\ \ell n\,(3 + 2\sqrt{2}) + \frac{\pi}{6\sqrt{2}}$$

17. (C)

Statement-I $\quad \frac{d}{dx}\ F(x) = \frac{e^{\sin x}}{x}, x > 0$

$$\Rightarrow\ F(x) = \int \frac{e^{\sin x}dx}{x}$$

$$I = \int_1^4 \frac{3e^{\sin x^3}}{x}dx = \int_1^4 \frac{3x^2 e^{\sin x^3}}{x^3}dx = \int_1^{64} \frac{e^{\sin t}}{t}dt = [F(x)]_1^{64}$$

$= F(64) - F(1)$

$\therefore\ F(64) - F(1) = F(k) - F(1)$

$\therefore\ k = 64$

Statement-II

$$I = \int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)\,dx = \int_{\operatorname{cosec}\theta}^{\sin\theta} f\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)dt$$

$$= \int_{\operatorname{cosec}\theta}^{\sin\theta} f(t)\,dt = -I$$

$\therefore\ I = 0$

18. (B)

Statement-I

$$I = \int_0^n \{x\}\,dx = n\int_0^1 x\,dx = n\left[\frac{x^2}{2}\right]_0^1 = \frac{n}{2}$$

$[\because \int_0^{nT} f(x)\,dx = n\,.\int_0^T f(x)\,dx$; where T is the period of f(x)]

Statement-II

$$\int_0^n [x]\,dx = \int_0^n (x - \{x\}\,dx = \int_0^n x\,dx - \int_0^n \{x\}\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^n - n\int_0^1 x \cdot dx = \frac{n^2}{2} - n \cdot \left[\frac{x^2}{2}\right]_0^1 = \frac{n}{2}\ (n-1)$$

19. (C)

$$\int_a^b x\ f(x)\ dx = \int_a^b (a + b - x)\ f(a + b - x)\ dx$$

$$= (a + b)\int_a^b f(a + b - x)\ dx - \int_a^b x\,f(a + b - x)\ dx$$

$\therefore$ statement-2 is true only when $f(a + b - x) = f(x)$ which holds in statement-1

$\therefore$ statement-2 is false and statement-1 is true.

20. (D)

$$\lim_{x\to 2}\frac{4\int_0^{f(x)} t^3\,dt}{x-2} = \lim_{x\to 2}\frac{4\,.\,f'(x)\,f(x)^3}{1} = 32$$

$\therefore$ statement-1 is false

statement-2 is true, which is a standard result.

21. (A) → q, (B) → s, (C) → t, (D) → t

(A) Let

$$I = \int_4^{10}\frac{[x^2]dx}{[x^2-28x+196]+[x^2]} = \int_4^{10}\frac{[x^2]dx}{[(x-14)^2]+[x^2]} \quad ....(i)$$

Also, $I = \int_4^{10}\frac{[(14-x)^2]dx}{[x^2]+[(14-x)^2]}$ ....(ii)

using $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Adding equation (i) and (ii)

$$2I = \int_4^{10} dx = [x]_4^{10} = 6$$

$$\therefore\quad I = \frac{6}{2} = 3$$

(B) $\int_{-1}^{2}\frac{|x|}{x}dx = \int_{-1}^{0}-1\,dx + \int_0^2 1\,dx = -[x]_{-1}^{0} + [x]_0^2$

$= -[0-(-1)]+[2-0] = 1$

(C) $\lim_{n\to\infty}\frac{1^{99}+2^{99}+......n^{99}}{n^{100}} = \lim_{n\to\infty}\frac{1}{n}$.

$$\left[\left(\frac{1}{n}\right)^{99}+\left(\frac{2}{n}\right)^{99}+....+\left(\frac{n}{n}\right)^{99}\right] = \lim_{n\to\infty}\frac{1}{n}\,.\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{99}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\left(\frac{r}{n}\right)^{99} = \int_0^1 x^{99}\,dx = \left[\frac{x^{100}}{100}\right]_0^1 = \frac{1}{100}$$

(D) $\frac{1}{\alpha} = 5050\int_{-1}^{1}|x|^{100}\,dx = 5050\times 2\int_0^1 x^{100}\,dx$

$$= 2\times 5050\left[\frac{x^{101}}{101}\right]_0^1$$

$$= 2\times 5050\left[\frac{1}{101}-0\right]$$

$$\frac{1}{\alpha} = 100 \quad\Rightarrow\quad \alpha = \frac{1}{100}$$

22. (A) → p, (B) → q, (C) → t, (D) → p,

(A) $L = \lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{\sqrt{n^2-r^2}}{n^2} = \lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{n}\sqrt{1-\left(\frac{r}{n}\right)^2}$

$$= \int_0^1\sqrt{1-x^2}\,dx = \frac{\pi}{4}$$

(B) $\int_0^{\frac{\pi}{4}}\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx =$

$$2\int_0^{\pi/4}\frac{1}{2}\,\frac{\sqrt{\tan x}+\sqrt{\cot x}}{\tan x+\cot x}(\tan x+\cot x)\,dx$$

$$= 2\int_0^{\pi/4}\frac{d\left(\sqrt{\tan x}-\sqrt{\cot x}\right)}{\left(\sqrt{\tan x}-\sqrt{\cot x}\right)^2+2}dx = 2\,.\,\frac{1}{\sqrt{2}}\tan^{-1}$$

$$\left(\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right)\Bigg|_0^{\pi/4} = \frac{\pi}{\sqrt{2}}$$

(C) $\int_{-1}^{1}\sin^3 x\cos^2 x\,dx = 0$ $\{\because$ the intergrand is odd$\}$

(D) $I = \int_0^{\frac{\pi}{2}}\frac{\sqrt{\sin^3 x}dx}{\sqrt{\sin^3 x}+\sqrt{\cos^3 x}} = \int_0^{\frac{\pi}{2}}\frac{\sqrt{\cos^3 x}dx}{\sqrt{\sin^3 x}+\sqrt{\cos^3 x}}$

$$\therefore\quad 2I = \int_0^{\pi/2}1\,dx = \frac{\pi}{2}$$

$$\therefore\quad I = \frac{\pi}{4}$$

23.

1. (A)

$$A = \int_1^{\infty}[\text{cosec}^{-1}x]\,dx = \int_1^{\text{cosec}1}1\,dx + \int_{\text{cosec}1}^{\infty}0\,dx = \text{cosec}\,1-1$$

2. (B)

$$= \int_1^{100} [\sec^{-1} x]dx \quad = \int_1^{\sec 1} 0 \; dx + \int_{\sec 1}^{100} 1 dx = 100 - \sec 1$$

3. (B)

$$\int_{\operatorname{cosec}1-1}^{100-\sec 1} [\tan^{-1} x] \, dx = 100 - \tan 1 - \sec 1$$

24.

1. (C)

$$I = \int_{-\pi/4}^{\pi/4} \ell n\left(\frac{\sin x + \cos x}{\cos x - \sin x}\right) dx$$

$$I = 0 \qquad (\because f(x) = -f(-x))$$

2. (B)

$$I = \int_{-\pi/4}^{\pi/4} \ell n(\sin x + \cos x)\; dx = \int_{-\pi/4}^{\pi/4} \ell n(\cos x - \sin x)\, dx$$

$$\therefore \quad 2I = \int_{-\pi/4}^{\pi/4} \ell n(\cos^2 x - \sin^2 x)\, dx$$

$$= \int_{-\pi/4}^{\pi/4} \ell n(\cos 2x)\, dx \qquad = 2\int_0^{\pi/4} \ell n(\cos 2x)\, dx$$

$$\therefore \quad I = \int_0^{\pi/4} \ell n(\cos 2x)\, dx = \frac{1}{2}\int_0^{\pi/2} \ell n(\cos t)\, dt$$

$$= \frac{1}{2}\left(-\frac{\pi}{2}\ell n 2\right) = -\frac{\pi}{4}\ \ell n 2$$

3. (D)

$$I = \int_0^{\frac{\pi}{4}} \ell n(\sin 2x)\, dx \qquad \text{let } 2x = t,\ 2dx = dt$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ell n \sin t \; dt = \frac{1}{2}\left(-\frac{\pi}{2}\ell n 2\right) = -\frac{\pi \ell n 2}{4}$$

25.

1. (A)

$$S = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{(n+r)\sqrt{r(2n+r)}}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n\left(1+\frac{r}{n}\right)\sqrt{\frac{r}{n}\left(2+\frac{r}{n}\right)}}$$

$$S = \int_0^1 \frac{dx}{(1+x)\sqrt{2x+x^2}} = \int_0^1 \frac{dx}{(1+x)\sqrt{(1+x)^2 - 1}}$$

$$S = \left[\sec^{-1}(1+x)\right]_0^1 = \sec^{-1} 2 - \sec^{-1} 1 = \frac{\pi}{3}$$

2. (A)

$$f(x) = \tan^{-1} x$$

$$T_n = \frac{1}{n}\, f\left(\frac{n}{n}\right) = \frac{1}{n}\, f(1) = \frac{\pi}{4n}$$

3. (B)

$$S = \lim_{n\to\infty} \sum_{r=0}^{2n-1} \frac{1}{n}\sec^2\left(\frac{r}{n}\right) = \int_0^2 \sec^2 x\; dx = \left[\tan x\right]_0^2 = \tan 2$$

26. (2) $\displaystyle \lim_{n\to\infty} \frac{1}{n}$

$$\left[\cos^{2p}\frac{\pi}{2n} + \cos^{2p}\frac{2\pi}{2n} + \cos^{2p}\frac{3\pi}{2n} + \ldots\ldots + \cos^{2p}\frac{n\pi}{2^n}\right]$$

$$= \int_0^1 \cos^{2p}\frac{\pi x}{2} dx \qquad \text{put } \frac{\pi x}{2} = t$$

$$= \int_0^{\pi/2} \cos^{2p} t \,.\, \frac{2}{\pi}\, dt = \frac{2}{\pi}\int_0^{\pi/2} \cos^{2p} t\, dt$$

$$= \frac{2}{\pi}\left[\frac{2p-1}{2p}\cdot\frac{2p-3}{2p-2}\cdot \ldots\ldots\ldots\ldots \cdot \frac{1}{2}, \frac{\pi}{2}\right]$$

27. Let $\displaystyle L = \int_{-\infty}^{\infty} \frac{dx}{1+x^2+x^4+\ldots.x^{2m}} = 2\int_0^{\infty} \frac{x^2-1}{x^{2m+2}-1}\, dx$

$$\therefore \quad \lim_{m\to\infty} L = 2\int_0^{\infty} \lim_{m\to\infty} \frac{x^2-1}{x^{2m+2}-1}\, dx$$

$$= 2\int_0^1 \lim_{m\to\infty} \frac{x^2-1}{x^{2m+2}-1}\, dx + 2\int_1^{\infty} \lim_{m\to\infty} \frac{x^2-1}{x^{2m+2}-1}\, dx$$

$$= 2\int_0^1 (1-x^2)\, dx + 2\int_1^{\infty} 0\, dx = \frac{4}{3}$$

28. (2)

$$\int_0^{2\pi} \frac{dx}{a + b\cos x + c\sin x} = \int_0^{\pi} \frac{dx}{a + b\cos x + c\sin x} + \int_0^{\pi} \frac{dx}{a + b\cos x - c\sin x}$$

$$= 2\int_0^{\infty} \frac{dt}{(a-b)\,t^2 + 2ct + a + b} + 2\int_0^{\infty} \frac{dt}{(a-b)\,t^2 - 2ct + a + b}$$

$$= \frac{2}{a-b}\int_0^{\infty} \frac{dt}{t^2 + \frac{2c}{a-b}t + \frac{a+b}{a-b}} + \frac{2}{a-b}\int_0^{\infty} \frac{dt}{t^2 - \frac{2c}{a-b}t + \frac{a+b}{a-b}}$$

$$= \frac{2}{a-b}\int_0^{\infty} \frac{dt}{\left(t + \frac{c}{a-b}\right)^2 + \frac{a^2-b^2-c^2}{(a-b)^2}}$$

$$+ \frac{2}{a-b}\int_0^{\infty} \frac{dt}{\left(t - \frac{c}{a-b}\right)^2 + \frac{a^2-b^2-c^2}{(a-b)^2}}$$

$$= \frac{2}{a-b} \frac{a-b}{\sqrt{a^2-b^2-c^2}} \tan^{-1} \frac{(a-b)t + c}{\sqrt{a^2-b^2-c^2}} \Bigg|_0^{\infty}$$

$$+ \frac{2}{a-b} \cdot \frac{a-b}{\sqrt{a^2-b^2-c^2}} \tan^{-1} \frac{(a-b)t - c}{\sqrt{a^2-b^2-c^2}} \Bigg|_0^{\infty}$$

$$= \frac{2}{\sqrt{a^2-b^2-c^2}} \left(\frac{\pi}{2} + \tan^{-1} \frac{c}{\sqrt{a^2-b^2-c^2}}\right)$$

$$+ \frac{2}{\sqrt{a^2-b^2-c^2}} \left(\frac{\pi}{2} + \tan^{-1} \frac{-c}{\sqrt{a^2-b^2-c^2}}\right)$$

$$= \frac{2\pi}{\sqrt{a^2-b^2-c^2}}$$

29. Let $L = \lim\limits_{n\to\infty} \frac{1}{n^{2m}} [(n^2+1^2)^m (n^2+2^2)^m \ldots\ldots (2n^2)^m]^{1/n}$

$$\Rightarrow L = \lim_{n\to\infty} \left(\frac{(n^2+1^2)(n^2+2^2)\ldots.(2n^2)}{n^{2n}}\right)^{\frac{m}{n}}$$

$$\Rightarrow \log_e L = \lim_{n\to\infty} \log_e$$

$$\left[\frac{1}{n^{2n}}\{(n^2+1^2)(n^2+2^2)\ldots.(2n^2)\}\right]^{\frac{m}{n}} = \lim_{n\to\infty} \frac{m}{n}$$

$$\left(\left(\sum_{k=1}^{n} \log_e(n^2+k^2)\right) - 2n\log_e n\right)$$

$$= \lim_{n\to\infty} \frac{m}{n} \sum_{k=1}^{n} (\log_e(n^2+k^2) - 2\log_e n)$$

$$= \lim_{n\to\infty} m \sum_{k=1}^{n} \frac{\log_e(n^2+k^2) - 2\log_e n}{n}$$

$$\Rightarrow \log_e L = m\left(\ell n 2 + \frac{\pi}{2} - 2\right)$$

$$\Rightarrow L = e^{m\left(\ell n 2 + \frac{\pi}{2} - 2\right)} = \left(\frac{2\sqrt{e^{\pi}}}{e^2}\right)^m$$

30. (4)

Given the $f(x+y) = f(x) + f(y)$ ....**(i)**

Putting $x = 0$ and $y = 0$ in **(i)**, we get

$f(0+0) = f(0) + f(0)$

$\Rightarrow f(0) = 0$ ....**(ii)**

Now $\quad f'(x) = \lim\limits_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x) + f(h) - f(x)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} f'(h) = f'(0)$$

$\Rightarrow f(x) = \int f'(0)\,dx = x f'(0) + c$ ....**(ii)**

Putting $x = 0$ in **(ii)**, we get

$f(0) = 0 + c$

$\Rightarrow c = 0 \qquad [\because f(0) = 0$ from **(ii)**$]$

$\Rightarrow f(x) = xf'(0)$ ....**(ii)**

Thus $I_n = n\int_0^n f(x)\,dx = n\int_0^n xf'(0)\,dx$

$$\Rightarrow I_n = \frac{n^3 \cdot f'(0)}{2}$$

therefore $I_1 + I_2 + I_3 + I_4 + I_5 = \frac{f'(0)}{2}(1^3+2^3+3^3+4^3+5^3)$

$$\Rightarrow 450 = \frac{f'(0)}{2} \cdot \left\{\frac{5.(5+1)}{2}\right\}^2$$

$\Rightarrow f'(0) = 4$

$\therefore f(x) = 4x$ (from equation **(iii)**)