HINTS & SOLUTIONS

EXERCISE - 1

Single Choice

2. $\int \frac{dx}{\sin x.\sin(x+\alpha)} = \frac{1}{\sin\alpha}\int \frac{\sin(\alpha+x-x)}{\sin x\ \sin(x+\alpha)}dx$

$= \text{cosec}\ \alpha \int \frac{\sin(x+\alpha)\cos x - \cos(x+\alpha)\sin x}{\sin\ x \sin(x+\alpha)}$

$= \text{cosec}\ \alpha \left[\int \cot x\, dx - \int \cot(x+\alpha)\right] + C$

$= \text{cosec}\ \alpha\ [\log|\sin x| - \log|\sin(x+\alpha)|] + C$

$= \text{cosec}\ \alpha \log\left|\frac{\sin x}{\sin(x+\alpha)}\right| + C$

3. $x.\int \frac{dx}{x} = x(\ln|x| + C) = x\, ln|x| + Cx$

5. $\int \frac{2x+1}{(x^2+4x+1)^{3/2}}dx = \int \frac{2x+1}{x^3\left(1+\frac{4}{x}+\frac{1}{x^2}\right)^{3/2}}dx$

$= \int \frac{2x^{-2}+x^{-3}}{\left(1+\frac{4}{x}+\frac{1}{x^2}\right)^{3/2}}dx$

now put $\frac{1}{x^2}+\frac{4}{x}+1 = t^2$

7. $I = \int \frac{dx}{x(1+x^7)} - \int \frac{x^6}{1+x^7}dx$

8. $\int \left(\frac{\cos^8 x - \sin^8 x}{1-2\sin^2 x\cos^2 x}\right)dx$

$= \int \frac{(\cos^4 x+\sin^4 x)(\cos^2 x - \sin^2 x)}{(1-2\sin^2 x\cos^2 x)}dx$

$= \int (\cos^2 x - \sin^2 x)\,dx = \int \cos 2x dx = \frac{\sin 2x}{2} + C$

9. $\int \left\{\ln(1+\sin x) + x\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}dx$

$= \int \ell n(1+\sin x)\,dx + \int x\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$

$= x.\ell n\,(1+\sin x) - \int \frac{1}{1+\sin x}\cos x.\ x\,dx + \int x\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$

$x.\ell n\,(1+\sin x) - \int \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}.\ x\,dx + \int x\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$

$= x\,\ell n\,(1+\sin x) + C$

10. $\left[\frac{1}{4}\tan^{-1}\frac{x+3}{4} + \frac{1}{2\cdot 6}\ln\left|\frac{x-9}{x+3}\right|\right]$

$= 3\tan^{-1}\left(\frac{x+3}{4}\right) + \ln\left|\frac{x-9}{x+3}\right| \Rightarrow \lambda = 3,\ \mu = 1 \Rightarrow 4$

12. $\int \frac{x^{1/3}}{(x^4-1)^{4/3}}dx = \frac{1}{4}\int \frac{4x^{-5}}{(1-x^{-4})^{4/3}}dx$

$= \frac{-3}{4}(1-x^{-4})^{-1/3} + C$ (Put $1 - x^{-4} = t$)

14. $\int \frac{x\,dx}{\sqrt{(1+x^2)\left(1+\sqrt{1+x^2}\right)}}$; put $1+\sqrt{1+x^2} = t^2$

15. $f(x) = \int \frac{2\sin x - \sin 2x}{x^3}dx$

$f'(x) = \frac{2\sin x - \sin 2x}{x^3} = \frac{2\sin x}{x}.\frac{1-\cos x}{x^2}$

$= 2\left(\frac{\sin x}{x}\right).\frac{2\sin^2\frac{x}{2}}{\frac{x^2}{4}\times 4} = \frac{2\times 2}{4}\left(\frac{\sin x}{x}\right).\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2$

$\lim_{x\to 0} f'(x) = \lim_{x\to 0}.\left(\frac{\sin x}{x}\right) \lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = 1$

16. reduction formula of $(\sin x)^n$

18. $I = \int \sqrt{\sec x - 1}\, dx = \int \frac{\sqrt{1-\cos x}}{\sqrt{\cos x + 1 - 1}} dx$

$= \int \frac{\sqrt{2}\sin\frac{x}{2}dx}{\sqrt{\left(\sqrt{2}\cos\frac{x}{2}\right)^2 - 1}}$ put $\sqrt{2}\cos\frac{x}{2} = t$

$\Rightarrow \sin\frac{x}{2}dx = -\sqrt{2}\,dt$

$\Rightarrow I = -2\int \frac{dt}{\sqrt{t^2-1}} = -2\ell n\left(t + \sqrt{t^2-1}\right)$

$-2\ell n\left(\sqrt{2}\cos\frac{x}{2} + \sqrt{2\cos^2\frac{x}{2} - 1}\right) + C_1$

$= -2 \times \frac{1}{2}\ell n\, 2 - 2\,\ell n\left(\cos\frac{x}{2} + \sqrt{\cos^2\frac{x}{2} - \frac{1}{2}}\right) + C_1$

$= -2\,\ell n\left(\cos\frac{x}{2} + \sqrt{\cos^2\frac{x}{2} - \frac{1}{2}}\right) + C$

19. $I = \int \frac{(e^x + \cos x + 1) - (e^x + \sin x + x)}{e^x + \sin x + x} dx$

$= ln(e^x + \sin x + x) - x + C$

$\therefore\ f(x) = e^x + \sin x + x$ and $g(x) = -x$

$f(x) + g(x) = e^x + \sin x$

22. use I.B.P. taking $ln\left(x + \sqrt{1+x^2}\right)$ as the first function

and $\frac{x}{\sqrt{1+x^2}}$ as the second function

23. $\int \frac{(x^4+1)}{x(x^4+1+2x^2)}dx = \int\left(\frac{-2x}{x^4+1+2x^2} + \frac{1}{x}\right)dx$

$= \ell n|x| - \int\left(\frac{2x}{(x^2+1)^2}\right)dx = \ell n|x| + \frac{1}{x^2+1} + c$

24. $\int\left(\frac{x}{(1+x^5)}\right)^{3/2} dx = \int \frac{x^{-6}}{(1+x^{-5})^{3/2}}dx$

{Let $t = 1 + x^{-5}$; $dt = -5x^{-6}dx$}

$= -\frac{1}{5}\int\frac{dt}{(t)^{3/2}} = \frac{2}{5}(1+x^{-5})^{-1/2} + C$

25. $\int \frac{1 - \frac{1}{x^2}}{\left(x^2 + \frac{1}{x^2} + 3\right)\tan^{-1}\left(x + \frac{1}{x}\right)}dx$;

$\int \frac{\left(1 - \frac{1}{x^2}\right)dx}{\left(\left(x + \frac{1}{x}\right)^2 + 1\right)\tan^{-1}\left(x + \frac{1}{x}\right)}dx$

27. $I_n = \int \cot^n x\, dx = \int \cot^{n-2} x.(\mathrm{cosec}^2 x - 1)\, dx$

$I_n = -\frac{u^{n-1}}{n-1} - I_{n-2}$

or $I_n + I_{n-2} = -\frac{u^{n-1}}{n-1}$ (Put $n = 2, 3, 4, \ldots.. 10$)

$I_2 + I_0 = -\frac{u}{1}$

$I_3 + I_1 = -\frac{u^2}{2}$

$I_4 + I_2 = -\frac{u^3}{3}$

$\vdots \quad \vdots \quad \vdots$

$I_{10} + I_9 = -\frac{u^9}{9}$

---

adding $I_0 + I_1 + 2(I_2 + I_3 + \ldots. + I_8) + I_9 + I_{10}$

$= -\left(u + \frac{u^2}{2} + \ldots\ldots\ldots + \frac{u^9}{9}\right) \Rightarrow$ **(B)**

29. $I = \int \frac{dx}{x(x^{2007}+1)} = \int \frac{x^{2007} + 1 - x^{2007}}{x(x^{2007}+1)}dx$

$= \int\left(\frac{1}{x} - \frac{x^{2006}}{1+x^{2007}}\right)dx$

$= \ln x - \frac{1}{2007}\ln(1+x^{2007})$

$= \frac{\ln x^{2007} - \ln(1+x^{2007})}{2007}$

$= \frac{1}{2007}\ln\left(\frac{x^{2007}}{1+x^{2007}}\right) + C$

$p+q+r = 6021$

31. $e^{\sqrt{\sin x}} = t \quad \frac{e^{\sqrt{\sin x}}}{2\sqrt{\sin x}}\cos x$;

$e^{\sqrt{\sin x}}\sqrt{\cot x}\sqrt{\cos x}\,dx = 2\,dt$

Hence $\int 2\,dt = 2t + C \quad \Rightarrow \quad 2e^{\sqrt{\sin x}} + C$

33. $\int \frac{x^4-4}{x^3\sqrt{\frac{4}{x^2}+1+x^2}}dx = \int \frac{x-4x^{-3}}{\sqrt{4x^{-2}+1+x^2}}dx$

$= \frac{1}{2}\int t^{-1/2}dt$ [put $t = 4x^{-2}+1+x^2$]

$= \sqrt{\frac{4}{x^2}+1+x^2} + C = \frac{\sqrt{4+x^2+x^4}}{x} + C$

34. $I = \int \frac{x^2(1-\ln x)}{x^4\left(\left(\frac{\ln x}{x}\right)^4 - 1\right)}dx = \int \frac{1-\ln x}{x^2\left(\left(\frac{\ln x}{x}\right)^4 - 1\right)}dx$

put $\frac{\ln x}{x} = t \quad \Rightarrow \quad \frac{1-\ln x}{x^2} = dt$

$I = \int \frac{dt}{(t^4-1)} = \int \frac{dt}{(t^2+1)(t^2-1)}$

$= \frac{1}{2}\int \frac{(t^2+1)-(t^2-1)}{(t^2+1)(t^2-1)}dt$

$I = \frac{1}{2}\left(\int \frac{dt}{t^2-1} - \int \frac{dt}{t^2+1}\right) = \frac{1}{2}\left(\frac{1}{2}\ln\frac{t-1}{t+1} - \tan^{-1}t\right)$

$= \frac{1}{4}\ln\left(\frac{\ln x - x}{\ln x + x}\right) - \frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right) + C$

36. $\int \frac{2x+3}{(x^2+3x)(x^2+3x+2)+1}dx$

put $x^2+3x = t \quad \Rightarrow \quad (2x+3)\,dx = dt$

$\int \frac{dt}{t(t+2)+1}; \int \frac{dt}{(t+1)^2} = C - \frac{1}{t+1} = C - \frac{1}{x^2+3x+1}$

$\Rightarrow \quad a = 1, b = 3, c = 1$

$\Rightarrow \quad a+b+c = 5$

38. $I = \int \left(\sin(100x+x)\cdot(\sin x)\right)^{99}dx$

$= \int\left((\sin(100x)\cos x + \cos 100x\cdot\sin x)(\sin x)^{99}\right)dx$

$= \int \underbrace{\sin(100x)}_{I}\underbrace{\cos x\cdot(\sin x)^{99}}_{II}dx + \int \cos(100x)\cdot(\sin x)^{100}dx$

$= \frac{\sin(100x)(\sin x)^{100}}{100} - \frac{100}{100}\int \cos(100x)(\sin x)^{100}dx$

$+ \int \cos(100x)(\sin x)^{100}dx$

$\Rightarrow \quad \frac{\sin(100x)(\sin x)^{100}}{100} + C$

EXERCISE - 2

Part # I : Multiple Choice

1. $\sqrt{1+2\tan x\sec x+2\tan^2 x} = |\sec x + \tan x|$

$\therefore \int |\sec x + \tan x|\, dx = I$

$I = \ell n|\sec x + \tan x| + \ell n|\sec x| + c$ .....(i)

$I = \ell n|\sec^2 x + \sec x \tan x| + c$

$I = -\ell n|\sec x - \tan x| + \ell n|\sec x| + c$ .....(ii)

$I = \ell n|1 + \tan x(\sec x + \tan x)| + c$ .....(iv)

2. $I = \int e^{3x}\cos 4x\,dx = e^{3x}\left(A\sin 4x + B\cos 4x\right) + C$ ....(i)

$I = \frac{1}{4}e^{3x}\sin 4x - \int \frac{3}{4}e^{3x}\sin 4x\,dx$

$= \frac{1}{4}e^{3x}\sin 4x + \frac{3}{16}e^{3x}\cos 4x - \int \frac{9}{16}e^{3x}\cos 4x\,dx$

$\frac{25}{16} I = \frac{1}{16}\left(4e^{3x}\sin 4x + 3e^{3x}\cos 4x\right)$

comparing with equation (i)

$\Rightarrow A = \frac{4}{25}, B = \frac{3}{25} \qquad \Rightarrow \frac{A}{B} = \frac{4}{3}$

$\Rightarrow 3A = 4B \qquad \Rightarrow 4A + 3B = 1$

3. $f'(x) = 3x^2.\sin\frac{1}{x} - x.\cos\frac{1}{x}$

$\Rightarrow f(x) = \int\left(3x^2.\sin\frac{1}{x} - x\cos\frac{1}{x}\right)dx$

$= x^3\sin\frac{1}{x}.-\int \cos\frac{1}{x}\left(-\frac{1}{x^2}\right)x^3\,dx - \int x\cos\frac{1}{x}\,dx$

$= x^3\sin\frac{1}{x} + C$

since $f\left(\frac{1}{\pi}\right) = 0 + C \quad \Rightarrow \quad C = 0$

$\Rightarrow f(x) = \begin{cases} x^3\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

f(x) is clearly continuous and differentiable at x = 0 zero with f'(0) = 0.

$f''(0) = \lim_{h\to 0}\frac{3h^2\sin\frac{1}{h} - h\cos\frac{1}{h}}{h}$

$= 3h\sin\frac{1}{h} - \cos\frac{1}{h}$

This limit does't exist, hence f'(x) is non-differentiable at x = 0.

Also $\lim_{x\to 0} f'(x) = 0$.

Thus f'(x) is continuous at x = 0.

5. put $ln(\tan x) = t$

6. $I = \int 2^{mx}.3^{nx}\,dx = \int\left(2^m.3^n\right)^x dx = \frac{2^{mx}.3^{nx}}{\ell n(2^m.3^n)} + C$

13. $I_n = \int \cot^n x\,dx = \int \cot^{n-2} x(\text{cosec}^2 x - 1)\,dx = \frac{-\cot^{n-1} x}{n-1} - I_{n-2} + C_1$

$\Rightarrow I_n + I_{n-2} = \frac{-\cot^{n-1} x}{n-1} + C_{10}$

Now, $I_0 + I_1 + 2(I_2 + ... + I_8) + I_9 + I_{10}$

$= (I_2 + I_0) + (I_3 + I_1) + (I_4 + I_2) + (I_5 + I_3) + (I_6 + I_4) + (I_7 + I_5) + (I_8 + I_6) + (I_9 + I_7) + (I_{10} + I_8)$

$= -\left(\frac{\cot x}{1} + \frac{\cot^2 x}{2} + ... + \frac{\cot^9 x}{9}\right) + C$

$\therefore A = -1.$

14. $J + K = \int \frac{1+\sin x+\cos x}{1+\sin x+\cos x}dx \qquad J + K = x + C$ ....(1)

$\Rightarrow$ (C)

again $J - K = \int \frac{(\sin^2 x - \cos^2 x) + \sin x - \cos x}{1+\sin x+\cos x}dx$

$= \int \frac{(\sin x - \cos x) + (\sin x + \cos x + 1)}{1+\sin x+\cos x}dx$

$J - K = -\cos x - \sin x + C$ ....(2)

hence $J = K - (\sin x + \cos x) + C \Rightarrow$ (B)

Also (1)+(2)

$2J = x - (\cos x + \sin x) + C$

$$J = \frac{1}{2}[x - \sin x - \cos x] + C$$

and (1) – (2)

$$2K = x + (\sin x + \cos x) + C$$

$$K = \frac{1}{2}(x + \sin x + \cos x) + C$$

from (1) $J = x - K + C \Rightarrow$ **(C)**

Part # II : Assertion & Reason

1. Let $D(x) = \lambda_1 f_1(x) + \lambda_2 f_2(x) + \lambda_3 f_3(x)$

where $\lambda_1 = (b_2c_3 - b_3c_2)$, $\lambda_2 = a_2c_3 - a_3c_2$,

$\lambda_3 = a_2b_3 - a_3b_2$

then $\int D(x)\,dx = \int \lambda_1 f_1(x)\,dx + \int \lambda_2 f_2(x)\,dx$

$$+ \int \lambda_3 f_3(x)\,dx + C \quad \text{... (i)}$$

$$= \begin{vmatrix} \int f_1(x)dx & \int f_2(x)dx & \int f_3(x)dx \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + C$$

Thus statement-I is true and follows from statement-II which we have applies at Eq. (1)

4. Statement-1 is true, statement-2 is false

e.g. $F(x) = \int \frac{1}{\sqrt{x}}dx$,

although $\frac{1}{\sqrt{x}}$ is discontinuous at $x = 0$

but $\int \frac{1}{\sqrt{x}}dx = 2\sqrt{x}$ is continuous at $x = 0$.

5. $I = \int ((\log_x e) - (\log_x e)^2)\,dx = \int \left(\frac{1}{\ell nx} - \frac{1}{(\ell nx)^2}\right)dx$

put $\ell n\, x = t \Rightarrow x = e^t \Rightarrow dx = e^t\,dt$

$$\Rightarrow I = \int \left(\frac{1}{t} - \frac{1}{t^2}\right)e^t\,dt = \frac{e^t}{t} + C = \frac{x}{\ell nx} + C = x\log_x e + C$$

7. The statement-II is false since in

$$\int \frac{dx}{x - 3y} = \log(x - 3y) + C,$$

we are assuming that y is a constant.

We will now prove the statement-I.

From the given relation $(x - y)^2 = \frac{x}{y}$,

and $2\log(x - y) = \log x - \log y$ ... **(i)**

Also, $\frac{dy}{dx} = \left(-\frac{y}{x}\right) \cdot \frac{x + y}{x - 3y}$.

To prove the integral relation, it is sufficient to show

that $\frac{d}{dx}$ RHS. $= \frac{1}{x - 3y}$

Now, $\text{RHS} = \frac{1}{2}\log\left[\frac{x}{y} - 1\right] \quad \left(\because (x - y)^2 = \frac{x}{y}\right)$

$$= \frac{1}{2}[\log(x - y) - \log y]$$

$$= \frac{1}{2}\left[\frac{\log x - \log y}{2} - \log y\right] \quad \text{[From Eq. (i)]}$$

$$= \frac{1}{4}[\log x - 3\log y]$$

$$\Rightarrow \frac{d}{dx}\text{RHS.} = \frac{1}{4}\left[\frac{1}{x} - \frac{3}{y}\frac{dy}{dx}\right]$$

$$= \frac{1}{4}\left[\frac{1}{x} - \frac{3}{y}\left(-\frac{y}{x}\right)\frac{x + y}{x - 3y}\right] = \frac{1}{x - 3y}$$

Thus, statement-I is true.

EXERCISE - 3

Part # I : Matrix Match Type

2. **(A)** $v = 0 \Rightarrow a = b,$

Also $a + b = u \Rightarrow a = \frac{u}{2}$

Now, $I = \frac{1}{a}\int \frac{dx}{1+\cos x} = \frac{2}{u}\int \frac{dx}{2\cos^2 \frac{x}{2}}$

$= \frac{2}{u}\int \frac{1}{2}\sec^2 \frac{x}{2} dx = \frac{2}{u} \tan \frac{x}{2} + C$

(B) $v > 0 \Rightarrow a > b$

Now, $I = \int \frac{dx}{a + b\cos x} = \int \frac{dx}{(a-b) + 2b\cos^2 \frac{x}{2}}$

$= \int \frac{\sec^2 \frac{x}{2} dx}{(a+b) + (a-b)\tan^2 \frac{x}{2}}$

put $\tan \frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2 \frac{x}{2} dx = dt$

$\Rightarrow I = \int \frac{2\,dt}{(a+b) + (a-b) + t^2}$

$= \frac{2}{a-b}\int \frac{dt}{t^2 + \left(\frac{a+b}{a-b}\right)}$ ....(i)

$= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\, t\right) + C$

$= \frac{2}{\sqrt{uv}} \tan^{-1}\left(\sqrt{\frac{v}{u}} \tan \frac{x}{2}\right) + C$

(C) $v < 0 \Rightarrow a - b < 0 \Rightarrow b - a > 0$

Now $I = \frac{2}{b-a}\int \frac{dt}{\frac{a+b}{b-a} - t^2}$ (using equation (1) of part (B))

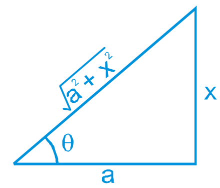
$= \frac{2}{b-a}\,\frac{1}{2}\sqrt{\frac{b-a}{b+a}} \ln \left| \frac{\sqrt{\frac{b+a}{b-a}} + t}{\sqrt{\frac{b+a}{b-a}} - t} \right| + C$

$= \frac{1}{\sqrt{-uv}} \ln \left| \frac{\sqrt{u} + \sqrt{-v}\tan \frac{x}{2}}{\sqrt{u} - \sqrt{-v}\tan \frac{x}{2}} \right| + C$

5. **(A)** $\int \frac{dx}{(a^2 + x^2)^{3/2}}$, put $x = a\tan\theta$

$= \int \frac{a\sec^2 \theta\, d\theta}{a^3 \sec^3 \theta} = \frac{1}{a^2}\sin\theta$

$= \frac{x}{a^2\sqrt{a^2 + x^2}} + C$



(B) $\int \frac{x^2}{\sqrt{a^2 - x^2}} dx = -\int\left(\frac{-x^2 + a^2}{\sqrt{a^2 - x^2}} - \frac{a^2}{\sqrt{a^2 - x^2}}\right)dx$

$= -\int \sqrt{a^2 - x^2}\,dx + a^2\int \frac{dx}{\sqrt{a^2 - x^2}}$

$= -\frac{x}{2}\sqrt{a^2 - x^2} - \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + a^2 \sin^{-1}\frac{x}{a}$

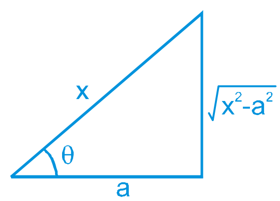
$= -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

(C) $\int \frac{dx}{(x^2 - a^2)^{3/2}}$ Put $x = a\sec\theta$

$= \int \frac{a\tan\theta\sec\theta}{a^3(\tan^3 \theta)} d\theta = \frac{1}{a^2}\int \frac{\cos\theta}{\sin^2\theta} d\theta$

$= \frac{1}{a^2}\left(\frac{-1}{\sin\theta}\right) + C$

$= c - \frac{x}{a^2\sqrt{x^2 - a^2}}$



(D) $\int \frac{1}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) = \frac{1}{a}\cos^{-1}\left(\frac{a}{|x|}\right)$

$= c + \frac{\pi}{2} - \frac{1}{a}\sin^{-1}\left(\frac{a}{|x|}\right)$

$= C - \frac{1}{a}\sin^{-1}\frac{a}{|x|}$

Part # II : Comprehension

Comprehension–2

1. $I = \int (\sqrt{\tan x} + \sqrt{\cot x})\, dx$

$= 2\int \frac{(\sqrt{\tan x} + \sqrt{\cot x})\,(\tan x + \cot x)}{2(\tan x + \cot x)}\, dx$

$I = 2\int \frac{d(\sqrt{\tan x} - \sqrt{\cot x})}{(\sqrt{\tan x} - \sqrt{\cot x})^2 + 2}$

$= \sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}}\right) + C$

2. If $x \in \left(0, \frac{\pi}{2}\right)$, then $I = \int (\sqrt{\tan x} + \sqrt{\cot x})\, dx$

$= \int \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right) dx$

$= \int \frac{\sin x + \cos x}{\sqrt{\sin x \,.\, \cos x}}\, dx = \int \frac{\sqrt{2}\,(\sin x - \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}}\, dx$

$= \sqrt{2}\,\sin^{-1}(\sin x - \cos x) + C$

3. If $x \in \left(\pi, \frac{3\pi}{2}\right)$, then $I = \int (\sqrt{\tan x} + \sqrt{\cot x})\, dx$

$= \int \left(\frac{\sqrt{-\sin x}}{\sqrt{-\cos x}} + \frac{\sqrt{-\cos x}}{\sqrt{-\sin x}}\right) dx$

$= -\int \frac{\sin x + \cos x}{\sqrt{\sin x \,.\, \cos x}}\, dx = -\int \frac{\sqrt{2}\,(\sin x - \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}}\, dx$

$= \sqrt{2}\,\sin^{-1}(\cos x - \sin x) + C$

Comprehension # 4

1. $I_n = \frac{x}{(x^2 + a^2)^n} + 2n\int \frac{x^2}{(x^2 + a^2)^{n+1}}\, dx$

$= \frac{x}{(x^2 + a^2)^n} + 2n\left[\int\left(\frac{x^2 + a^2}{(x^2 + a^2)^{n+1}} - \frac{a^2}{(x^2 + a^2)^{n+1}}\right) dx\right]$

$= \frac{x}{(x^2 + a^2)^n} + 2n\int \frac{1}{(x^2 + a^2)^n}\, dx$

$-2na^2\int \frac{1}{(x^2 + a^2)^{n+1}}\, dx$

$= \frac{x}{(x^2 + a^2)^n} + 2n(I_n - a^2 I_{n+1})$

Whence $I_{n+1} + \frac{1-2n}{2n}\,\frac{1}{a^2}\, I_n = \frac{1}{2na^2}\cdot\frac{x}{(x^2 + a^2)^n}$

2. $I_{n,\,-m} = \frac{\sin^{n-1} x}{(m-1)\cos^{m-1} x} - \frac{n-1}{m-1}\int \frac{\sin^{n-2} x}{\cos^{m-2} x}\, dx$

$= \frac{\sin^{n-1} x}{(m-1)\cos^{m-1} x} - \frac{n-1}{m-1}\, I_{n-2,\,2-m}$

3. $u_{n+1} = \int \frac{x^{n+1}}{\sqrt{ax^2 + 2bx + c}}\, dx$

$= \frac{1}{2a}\int \frac{x^n(2ax + 2b) - 2bx^n}{\sqrt{ax^2 + 2bx + c}}\, dx$

$= \frac{1}{2a}\int \frac{x^n(2ax + 2b)}{\sqrt{ax^2 + 2bx + c}}\, dx - \frac{b}{a}\, u_n$

$= I_n - \frac{b}{a}\, u_n$, where ... (i)

$I_n = \frac{1}{2a}\int \frac{x^n(2ax + 2b)}{\sqrt{ax^2 + 2bx + c}}\, dx$

$= \frac{1}{2a}\, x^n\, 2\sqrt{ax^2 + bx + c}$

$-\int nx^{n-1}\, 2\sqrt{ax^2 + 2bx + c}\; dx$

$= \frac{x^n}{a}\sqrt{ax^2 + 2bx + c} -$

$\frac{n}{a}\int \frac{x^{n-1}(ax^2 + 2bx + c)}{\sqrt{ax^2 + bx + c}}\, dx$ ...(ii)

from (i) and (ii) be get
$(n + 1)a\,u_{n+1} + (2n + 1)bu_n + ncu_{n-1}$
$= x^n\sqrt{ax^2 + 2bx + c}$

EXERCISE - 4

Subjective Type

1. $\operatorname{cosec}(b-a).\ell n\left|\frac{\sin(x-b)}{\sin\ (x-a)}\right|+c$

2. $\frac{1}{3}\left(x+\sqrt{x^2+2}\right)^{3/2}-\frac{2}{\left(x+\sqrt{x^2+2}\right)^{1/2}}+c$

3. $\sqrt{x}\sqrt{1-x}-2\sqrt{1-x}+\text{arc}\cos\sqrt{x}+c$

4. $\text{arc sec}x-\frac{\ell n x}{\sqrt{x^2-1}}+c$

5. $-\frac{x+1}{x^5+x+1}+c$

6.

$$\cos a.\text{arc}\cos\left(\frac{\cos x}{\cos\ a}\right)-\sin a.\ell n\left(\sin x+\sqrt{\sin^2 x-\sin^2 a}\right)+c$$

7. Let $I=\int\frac{d\left(x^2+1\right)}{\sqrt{x^2+2}}=\int\frac{2x}{\sqrt{x^2+2}}dx$

Put $x^2+2=t$

$\Rightarrow$ $2x\,dx=dt$

$\Rightarrow$ $I=\int\frac{dt}{\sqrt{t}}=2t^{1/2}+C=2\sqrt{x^2+2}+C$

8. $\int\frac{\cot x dx}{(1-\sin x)(\sec x+1)}=\int\frac{\cos^2 x\ dx}{\sin x(1-\sin x)(1+\cos x)}$

$=\int\frac{(1+\sin x)}{\sin x(1+\cos x)}dx$

$=\int\frac{dx}{\sin x(1+\cos x)}+\int\frac{1}{1+\cos x}dx$

$=\int\frac{\sin x\ dx}{(1-\cos^2 x)(1+\cos x)}+\tan\frac{x}{2}+C$

$=\int\frac{-dt}{(1+t)(1-t^2)}+\tan\frac{x}{2}+C$

$=\frac{1}{2}\ell n\left|\tan\frac{x}{2}\right|+\frac{1}{4}\sec^2\frac{x}{2}+\tan\frac{x}{2}+C$

9. $\frac{3}{2}\tan^{-1}x-\frac{1}{2}\ell n(1+x)+\frac{1}{4}\ell n(1+x^2)+\frac{x}{1+x^2}+c$

10. $\tan^{-1}\left(\frac{\sqrt{2}\sin 2x}{\sin x+\cos x}\right)+c$

11. $\frac{(x^2+1)\sqrt{x^2+1}}{9x^3}.\left[2-3\ \ell n\left(1+\frac{1}{x^2}\right)\right]$

12.

$$-\frac{1}{\sqrt{\sin\ \alpha}}\ell n\left[\cot x+\cot\alpha+\sqrt{\cot^2 x+2\cot\alpha\ \cot x-1}\right]+c$$

13. Put $x=\tan\theta$ $\Rightarrow$ $dx=\sec^2\theta d\theta$

$$I=\int\frac{\tan^2\theta\sec^2\theta}{\left(\frac{\sin\theta}{\cos\theta}\sin(\tan\theta)+\cos(\tan\theta)\right)^2}d\theta$$

$$=\int\frac{\tan^2\theta}{\cos^2(\tan\theta-\theta)}d\theta$$

Put $\tan\theta-\theta=u$ $\Rightarrow$ $\tan^2\theta d\theta=du$

$$I=\int\frac{du}{\cos^2 u}=\int\sec^2 u du=\tan(\tan\theta-\theta)$$

$$=\tan(x-\tan^{-1}x)=\frac{\tan x-x}{1+x\tan x}$$

$$=\frac{\sin x-x\cos x}{\cos x+x\sin x}+C$$

14. $\int\frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}dx=\int\frac{e^x(1+1-x^2)}{(1-x)\sqrt{1-x^2}}dx$

$$=\int e^x\left(\underbrace{\frac{1}{(1-x)\sqrt{1-x^2}}}_{f'(x)}+\underbrace{\frac{\sqrt{1-x^2}}{1-x}}_{f(x)}\right)dx$$

$$=e^x\sqrt{\frac{1+x}{1-x}}+C$$

15. $I = \int x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\,dx$ put $x^2 = a^2 \cos 2\theta$

$\Rightarrow$ $2x\,dx = -2a^2 \sin 2\theta\, d\theta$

$\Rightarrow$ $I = \int \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\,(-a^2 \sin 2\theta)\,d\theta$

$= -a^2 \int \frac{\sin\theta}{\cos\theta} \times 2\sin\theta\cos\theta\,d\theta = -a^2 \int (1-\cos 2\theta)\,d\theta$

$= -a^2\theta + \frac{a^2 \sin 2\theta}{2} + C_1$

$= \frac{a^2}{2}\sqrt{1-\frac{x^4}{a^4}} - \frac{a^2}{2}\cos^{-1}\left(\frac{x^2}{a^2}\right) + C_1$

$= \frac{1}{2}\sqrt{a^4 - x^4} + \frac{a^2}{2}\sin^{-1}\left(\frac{x^2}{a^2}\right) + C$

16. $f'(x) = \frac{1}{1+x^2} + \frac{1}{2}\left[\frac{1}{1+x} + \frac{1}{1-x}\right]$

$= \frac{1}{1+x^2} + \frac{1}{1-x^2} = \frac{2}{1-x^4}$

Now $\int \frac{1}{2}\left(\frac{2}{1-x^4}\right) d(x^4) = \int \frac{4x^3}{1-x^4}\,dx$

$= -\ell n(1-x^4) + C$

17. $I = \int \frac{\ell n(\cos x + \sqrt{\cos 2x})}{\sin^2 x}\,dx$

$= \int \operatorname{cosec}^2 x\,\ell n\left[(\sin x)(\cot x + \sqrt{\cot^2 x - 1})\right] dx$

$= \int \ell n(\sin x)\operatorname{cosec}^2 x\,dx$

$+ \int \ell n(\cot x + \sqrt{\cot^2 x - 1}).\operatorname{cosec}^2 x\,dx$

$= -\ell n(\sin x)\cot x + \int \cot^2 x\,dx - \int \ell n(t + \sqrt{t^2 - 1})\,dt$

[put cotx = t]

$= -\ell n(\sin x)\cot x - \cot x - x - \left[\ell n\left(t + \sqrt{t^2-1}\right)\right] t$

$- \int t.\frac{1 + \frac{2t}{2\sqrt{t^2-1}}}{t + \sqrt{t^2-1}}\,dt$

$= -\ell n(\sin x)\cot x - \cot x - x - t\ell n(t + \sqrt{t^2-1}) + \int \frac{t}{\sqrt{t^2-1}}\,dt$

$= -\ell n(\sin x)\cot x - \cot x - x - t\,\ell n(t + \sqrt{t^2-1}) + \sqrt{t^2-1}$

$= -\ell n(\sin x)\cot x - \cot x - x - \cot x\,\ell n\left(\cot x + \frac{\sqrt{\cos 2x}}{\sin x}\right) + \frac{\sqrt{\cos 2x}}{\sin x}$

$= \frac{\sqrt{\cos 2x}}{\sin x} - x - \cot x - \cot x\,\ell n(e(\cos x + \sqrt{\cos 2x})) + C$

18. $-\left(1 + \frac{1}{x^4}\right)^{1/4} + c$

19. $I = \int \frac{\sqrt{\cos 2x}}{\sin x}\,dx$

$= \int \frac{\sqrt{\cos^2 x - \sin^2 x}}{\sin x}\,dx = \int \sqrt{\cot^2 x - 1}\,dx$

Put cotx = secθ

$\Rightarrow$ $-\operatorname{cosec}^2 x\,dx = \sec\theta\tan\theta\,d\theta$

$\therefore$ $I = \int \sqrt{\sec^2\theta - 1}.\frac{\sec\theta.\tan\theta}{-(1+\sec^2\theta)}\,d\theta$

$= -\int \frac{\sec\theta.\tan^2\theta}{1+\sec^2\theta}\,d\theta = -\int \frac{\sin^2\theta}{\cos\theta + \cos^3\theta}\,d\theta$

$= -\int \frac{1-\cos^2\theta}{\cos\theta(1+\cos^2\theta)}\,d\theta$

$= -\int \frac{(1+\cos^2\theta) - 2\cos^2\theta}{\cos\theta(1+\cos^2\theta)}\,d\theta$

$= -\int \sec\theta\,d\theta + 2\int \frac{\cos\theta}{1+\cos^2\theta}\,d\theta$

$= -\log|\sec\theta + \tan\theta| + 2\int \frac{\cos\theta}{2-\sin^2\theta}\,d\theta$

$= -\log|\sec\theta + \tan\theta| + 2\int \frac{dt}{2-t^2}$ (put sinθ = t)

$= -\log|\sec\theta + \tan\theta| + 2.\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+\sin\theta}{\sqrt{2}-\sin\theta}\right| + C$

$$= -\log|\cot x + \sqrt{\cot^2 x - 1}| + \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right| + C$$

$$= \frac{1}{\sqrt{2}}\log\left[\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right] - \log(\cot x + \sqrt{\cot^2 x - 1}) + c$$

20. $\int f(x)\,dx = \int\left(\ell n(\ell nx) + \frac{1}{(\ell nx)^2}\right)dx$

Put $\ell nx = t \Rightarrow x = e^t \Rightarrow dx = e^t\,dt$

$$= \int e^t\left(\ell nt + \frac{1}{t^2}\right)dt = \int e^t\left(\ell nt - \frac{1}{t} + \frac{1}{t} + \frac{1}{t^2}\right)dt$$

$$= e^t\left(\ell nt - \frac{1}{t}\right) + C \Rightarrow y = x\left(\ell n(\ell nx) - \frac{1}{\ell nx}\right) + C$$

$\because$ it passes through (e, e) $\Rightarrow$ C = 2

Hence, $y = x\left[\ell n(\ell nx) - \frac{1}{\ell nx}\right] + 2$

21. $\sin^{-1}\left(\frac{1}{2}\sec^2\frac{x}{2}\right) + c$

22. $3x - \ell n\left(\sqrt{x^2+x+1}\right) + \sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$

23. $\int\frac{\sec^2 x\,dx}{\tan^2 x + 2\tan x} = \int\frac{dt}{t^2+2t}$

[Put $\tan x = t$]

$$= \frac{1}{2}\int\left(\frac{1}{t} - \frac{1}{t+2}\right)dt = \frac{1}{2}\ell n\left|\frac{\tan x}{\tan x + 2}\right| + C$$

24. $\sin^{-1}\left(\frac{ax^2+b}{cx}\right) + k$

25. $\int(\sin x)^{-11/3}\cos x^{-1/3}dx = \int\frac{1}{(\sin^{11}x\cos x)^{1/3}}dx$

$$= \int\frac{\operatorname{cosec}^4 x}{(\cot x)^{1/3}}dx = \int\frac{(1+\cot^2 x)\operatorname{cosec}^2 x}{(\cot x)^{1/3}}dx$$

$= -\int(t^{-1/3} + t^{5/3})dt$ [Put $\cot x = t$]

$$= -\left[\frac{t^{2/3}}{2/3} + \frac{t^{8/3}}{8/3}\right] + C = \frac{-3}{8}[4\cot^{2/3}x + \cot^{8/3}x] + C$$

26. $(a+x)\arctan\sqrt{\frac{x}{a}} - \sqrt{ax} + c$

27. $I = \int\frac{2\sin 2\phi - \cos\phi}{6 - \cos^2\phi - 4\sin\phi}d\phi$

$$= \int\frac{(4\sin\phi - 1)\cos\phi}{6 - (1-\sin^2\phi) - 4\sin\phi}d\phi$$

put $\sin\phi = t \Rightarrow \cos\phi\,d\phi = dt$

$$\Rightarrow I = \int\frac{(4t-1)\,dt}{5+t^2-4t} = 2\int\frac{(2t-4)+7/2}{t^2-4t+5}dt$$

$$= 2[\ell n|t^2-4t+5|] + 7\int\frac{1}{(t-2)^2+(1)^2}dt$$

$$= 2[\ell n|t^2-4t+5|] + 7\tan^{-1}\left(\frac{t-2}{1}\right) + C$$

$$= 2\ell n\left|\sin^2\phi - 4\sin\phi + 5\right| + 7\tan^{-1}(\sin\phi - 2) + C$$

28. $\frac{3}{2}x^{2/3} - \frac{12}{7}x^{7/12} + 2x^{1/2} - \frac{12}{5}x^{5/12} + 3x^{1/3} + 6x^{1/6} - 12x^{1/12}$
$+ 12\log|x^{1/12} + 1| - 4x^{1/4} + c$

29. $\int\frac{f(x)dx}{x^2(x+1)^3} = \int\left(\frac{1}{x^2} - \frac{x}{(x+1)^3}\right)dx$

$$\int\frac{f(x)dx}{x^2(x+1)^3} = \int\frac{3x^2+3x+1}{x^2(x+1)^3}dx$$

$\Rightarrow f(x) = 3x^2 + 3x + 1$

$f'(0) = 3$

30. $\int\cos 2\theta\,\ell n\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)d\theta$

$$= \ell n\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\frac{\sin 2\theta}{2} - \int\frac{\sin 2\theta}{2}\cdot\frac{2}{\cos 2\theta}d\theta$$

$$= \frac{1}{2}\ell n\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\sin 2\theta - \frac{1}{2}\ell n(\sec 2\theta) + C$$

31. $I=\int\left[\left(\frac{x}{e}\right)^x+\left(\frac{e}{x}\right)^x\right]\ell nx\ dx=\int\left(1+\frac{1}{\left(\frac{x}{e}\right)^{2x}}\right)\ell nx\left(\frac{x}{e}\right)^x dx$

$$\left\{\text{Put}\left(\frac{x}{e}\right)^x=t\Rightarrow \ell nx\left(\frac{x}{e}\right)^x dx=dt\right\}$$

$$I=\int\left(1+\frac{1}{t^2}\right)dt=t-\frac{1}{t}+C=\left(\frac{x}{e}\right)^x-\left(\frac{e}{x}\right)^x+C$$

32. $\int\frac{x}{(7x-10-x^2)^{3/2}}dx=\int\frac{x}{[(x-5)(2-x)]^{3/2}}dx$

(put $x=5\cos^2\alpha+2\sin^2\alpha$)

$$=\int\frac{(5\cos^2\alpha+2\sin^2\alpha)(-3\sin 2\alpha)}{[(-3\sin^2\alpha)(-3\cos^2\alpha)]^{3/2}}d\alpha$$

$$=\int\frac{(5\cos^2\alpha+2\sin^2\alpha)(-3\sin 2\alpha)}{27(\sin\alpha\cos\alpha)^3}d\alpha$$

$$=\frac{-6}{27}\int\frac{(5\cos^2\alpha+2\sin^2\alpha)}{(\sin\alpha\cos\alpha)^2}d\alpha$$

$$=\frac{-2}{9}\int(5\,\text{cosec}^2\alpha+2\sec^2\alpha)d\alpha$$

$$=\frac{-2}{9}\left[-5\cot\alpha+2\tan\alpha\right]$$

$$=\frac{-2}{9}\int(-5\cot\alpha+2\tan\alpha)\ d\alpha$$

$$=\frac{10}{9}\sqrt{\frac{2-x}{x-5}}-\frac{4}{9}\sqrt{\frac{x-5}{2-x}}+C\quad\frac{2(7x-20)}{9\sqrt{7x-10-x^2}}+c$$

33. $\int\frac{1+x\cos x}{x\left(1-x^2e^{2\sin x}\right)}dx$

Put $xe^{\sin x}=t\Rightarrow(xe^{\sin x}.\cos x+e^{\sin x})\,dx=dt$

$\Rightarrow e^{\sin x}(x\cos x+1)\,dx=dt$

$$\Rightarrow\quad I=\int\frac{dt}{t(1-t^2)}=\int\frac{1}{t(1-t)(1+t)}dt$$

Let $\frac{1}{t(1-t)(1+t)}=\frac{A}{t}+\frac{B}{(1-t)}+\frac{C}{1+t}$

$1=A(1-t)(1+t)+B(t)(1+t)+C(t)(1-t)$

Put $t=0\quad\Rightarrow\quad A=1$

Put $t=1\quad\Rightarrow\quad B=1/2$

Put $t=-1\quad\Rightarrow\quad C=-1/2$

$$\Rightarrow\quad I=\int\left\{\frac{1}{t}+\frac{1}{2(1-t)}-\frac{1}{2(1+t)}\right\}dt$$

$$=\ell n|t|-\frac{1}{2}\ell n|1-t|-\frac{1}{2}\ell n|1+t|+K$$

$$=\ell n|x\,e^{\sin x}|-\frac{1}{2}\log|1-x^2e^{2\sin x}|+K$$

34. $\int\frac{e^{\cos x}(x\sin^3x+\cos x)}{\sin^2x}dx$

put $\cos x=t$, we get $-\sin x\,dx=dt$

$$dx=\frac{-dt}{\sqrt{1-t^2}}$$

$$\int e^t\left(\cos^{-1}t-\frac{1}{\sqrt{1-t^2}}+\frac{1}{\sqrt{1-t^2}}+\frac{t}{(1-t^2)^{3/2}}\right)dt$$

$$=e^t\left(\cos^{-1}t+\frac{1}{\sqrt{1-t^2}}\right)+C=e^{\cos x}(x+\text{cosec }x)+C$$

35. $I=\int(\sqrt{\tan x}+\sqrt{\cot x})dx=\int\frac{\tan x+1}{\sqrt{\tan x}}dx$

put $\tan x=t^2\ \Rightarrow\ (\sec^2x)dx=2t\,dt\ \Rightarrow\ dx=\frac{2t}{1+t^4}dt$

$$I=\int\frac{t^2+1}{\sqrt{t^2}}.\frac{2t}{t^4+1}dt=2\int\frac{t^2+1}{t^4+1}dt$$

$$=2\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}-2+2}dt=2\int\frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+\left(\sqrt{2}\right)^2}dt$$

$$=2\int\frac{du}{u^2+\left(\sqrt{2}\right)^2},\quad\text{where}\quad u=t-\frac{1}{t}$$

$$\Rightarrow\quad I=\frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C$$

$$=\sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right)+C$$

EXERCISE - 5

Part # I : AIEEE/JEE-MAIN

8. $\int \frac{5\tan x}{\tan x - 2}dx = \int \frac{5\sin x}{\sin x - 2\cos x}dx$

$5\sin x = A(\sin x - 2\cos x) + B\frac{d}{dx}(\sin x - 2\cos x)$

$= A(\sin x - 2\cos x) + B(\cos x + 2\sin x)$

$\left.\begin{matrix} A + 2B = 5 \\ -2A + B = 0 \end{matrix}\right\} A = 1, B = 2$

Now $\int \frac{5\sin x}{\sin x - 2\cos x}dx$

$= \int\left(\frac{(\sin x - 2\cos x)}{\sin x - 2\cos x} + \frac{2(\cos x + 2\sin x)}{\sin x - 2\cos x}\right)dx$

$= x + 2\log|\sin x - 2\cos x| + K$

$a = 2$

9. put $x^3 = t$ $\Rightarrow$ $3x^2dx = dt$ $I = \frac{1}{3}\int f(t)tdt$

$\frac{1}{3}\left[t\int f(t)dt - \int(1.\int f(t)dt)dt\right]$

$= \frac{1}{3}x^3\Psi(x^3) - \int \Psi(x^3)x^2dx$

12. $\int \frac{2x^{12} + 5x^9}{\left(x^5 + x^3 + 1\right)^3}dx$

$\int \frac{2x^{12} + 5x^9}{x^{15}\left(1 + x^{-2} + x^{-5}\right)^3}dx$

$\int \frac{2x^{-3} + 5x^{-6}}{\left(x^{-5} + x^{-2} + 1\right)^3}dx$

$x^{-5} + x^{-2} + 1 = t$

$(+5x^{-6} + 2x^{-3})\,dx = -dt$

$\int \frac{dt}{t^3} = -\left(\frac{t^{-2}}{-2}\right) + C = \frac{1}{2t^2} + C = \frac{x^{10}}{2\left(x^5 + x^3 + 1\right)^2} + C$

Part # II : IIT-JEE ADVANCED

1. $\int \sin^{-1}\left(\frac{2x + 2}{\sqrt{4x^2 + 8x + 13}}\right)dx$

$= \int \sin^{-1}\left[\frac{x + 1}{\sqrt{x^2 + 2x + \frac{13}{4}}}\right]dx$

$= \int \sin^{-1}\left[\frac{x + 1}{\sqrt{(x + 1)^2 + (3/2)^2}}\right]dx$

Put $x + 1 = 3/2\tan\theta$, $dx = \frac{3}{2}\sec^2\theta d\theta$

$= \int \sin^{-1}\left[\frac{\left(\frac{3}{2}\tan\theta\right)}{\sqrt{\frac{9}{4}\tan^2\theta + \frac{9}{4}}}\right]\frac{3}{2}\sec^2\theta d\theta$

$= \frac{3}{2}\int \sin^{-1}\left[\frac{\sin\theta}{\cos\theta}\frac{\cos\theta}{1}\right]\sec^2\theta d\theta$

$= \frac{3}{2}\int \theta\sec^2\theta d\theta = \frac{3}{2}\left[\theta\tan\theta - \int\tan\theta d\theta\right]$

$\frac{3}{2}[\theta\tan\theta - \log|\sec\theta|] + C$

$I = \frac{3}{2}\left[\frac{2}{3}(x + 1)\tan^{-1}\left[\frac{2}{3}(x + 1)\right] - \log\sqrt{1 + \frac{4}{9}(x + 1)^2}\right] + C$

$= (x + 1)\tan^{-1}\left(\frac{2x + 2}{3}\right) - \frac{3}{4}\log(9 + 4x^2 + 8x + 4) + \frac{3}{4}\log 9 + C$

$= (x + 1)\tan^{-1}\left(\frac{2x + 2}{3}\right) - \frac{3}{4}\log(4x^2 + 8x + 13) + C$

2. $I = \int (x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6)^{1/m}dx$

$= \int (x^{3m} + x^{2m} + x^m)\left[\frac{2x^{3m} + 3x^{2m} + 6x^m}{x^m}\right]^{1/m}dx$

$= \int\left(\frac{x^{3m} + x^{2m} + x^m}{x}\right)(2x^{3m} + 3x^{2m} + 6x^m)^{1/m}dx$

$$= \int (x^{3m-1} + x^{2m-1} + x^{m-1})(2x^{3m} + 3x^{2m} + 6x^m)^{1/m} dx$$

Put $2x^{3m} + 3x^{2m} + 6x^m = y$

$$\therefore I = \frac{1}{6m}\int y^{1/m} dy = \frac{1}{6m}\frac{y^{(1/m)+1}}{\left(\frac{m+1}{m}\right)} + C$$

$$= \frac{1}{6}\frac{y^{\frac{m+1}{m}}}{(m+1)} + C$$

$$= \frac{1}{6}\frac{(2x^{3m} + 3x^{2m} + 6x^m)^{\frac{m+1}{m}}}{m+1} + C$$

4. Here $ff(x) = \frac{f(x)}{[1+f(x)^n]^{1/n}} = \frac{x}{(1+2x^n)^{1/n}}$

and $fff(x) = \frac{x}{(1+3x^n)^{1/n}}$

$\therefore \quad g(x) = (fofo......of)(x) = \frac{x}{(1+nx^n)^{1/n}}$

Let $I = \int x^{n-2} g(x) dx = \int \frac{x^{n-1}}{(1+nx^n)^{1/n}} dx$

$$= \frac{1}{n^2}\int \frac{n^2 x^{n-1}}{(1+nx^n)^{1/n}} dx = \frac{1}{n^2}\int \frac{\frac{d}{dx}(1+nx^n)}{(1+nx^n)^{1/n}} dx$$

$$= \frac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}} + K$$

6. $J - I = \int \left(\frac{e^{-x}}{e^{-4x} + e^{-2x} + 1} - \frac{e^x}{e^{4x} + e^{2x} + 1} \right) dx$

$$= \int \frac{e^{3x} - e^x}{e^{4x} + e^{2x} + 1} dx = \int \frac{e^x(e^{2x} - 1)}{e^{4x} + e^{2x} + 1} dx$$

Let $e^x = t \qquad e^x dx = dt$

$$= \int \frac{t^2 - 1}{t^4 + t^2 + 1} dt = \int \frac{1 - 1/t^2}{(t + 1/t)^2 - 1} dt$$

$$= \frac{1}{2}\ell n\left|\frac{t + \frac{1}{t} - 1}{t + \frac{1}{t} + 1}\right| + C = \frac{1}{2}\ell n\left(\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1}\right) + C$$

7. Let $I = \int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$

$$= \int \frac{\sec x(\sec x + \tan x)\sec x}{(\sec x + \tan x)^{11/2}} dx$$

Put $\sec x + \tan x = t$

$\Rightarrow \quad (\sec x \tan x + \sec^2 x)\, dx = dt$

Also $\because \sec^2 x - \tan^2 x = 1$

$\Rightarrow \quad \sec x - \tan x = \frac{1}{t}$

$\therefore \quad \sec x = \frac{1}{2}\left(t + \frac{1}{t}\right)$

$$\therefore \; I = \frac{1}{2}\int \frac{\left(1 + \frac{1}{t}\right)dt}{t^{11/2}} = \frac{1}{2}.\int \left(t^{-9/2} + t^{-13/2}\right) dt$$

$$= \frac{1}{2}\left(-\frac{2t^{-7/2}}{7} - \frac{2t^{-11/2}}{11}\right) + K$$

$$= -\frac{1}{t^{11/2}}\left(\frac{1}{11} + \frac{t^2}{7}\right) + K$$

$$= -\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$$

MOCK TEST

1. (B)

$$\frac{d}{dx}(A\,\ell n|\cos x+\sin x-2|+Bx+C)$$

$$=A\frac{\cos x-\sin x}{\cos x+\sin x-2}+B$$

$$=\frac{A\cos x-A\sin x+B\cos x+B\sin x-2B}{\cos x+\sin x-2}$$

$\therefore\quad 2=A+B\,,-1=-A+B\,,\lambda=-2B$

$\therefore\quad A=3/2, B=1/2, \lambda=-1$

2. $\displaystyle\int x.\frac{\ell n(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx$

$$I\quad=\int \ell n\,(x+\sqrt{1+x^2})\,.\,\frac{x}{\sqrt{1+x^2}}dx$$

$$=\sqrt{1+x^2}\;\ell n\,(x+\sqrt{1+x^2})\;-\int\frac{1}{\sqrt{1+x^2}}.\sqrt{1+x^2}\;dx$$

(using integration by parts)

$$=\sqrt{1+x^2}\;\ell n\,(x+\sqrt{1+x^2})-x+C$$

Alternate:

Let $x+\sqrt{1+x^2}=e^t$ then $\sqrt{1+x^2}-x$

$$=e^{-t}\quad\frac{dx}{\sqrt{1+x^2}}=dt$$

$$I=\int x\;t\;\;dt\;=\int\frac{(e^t-e^{-t})t\,dt}{2}\;=\frac{1}{2}\;t(e^t+e^{-t})$$

$$-\frac{1}{2}(e^t-e^{-t})+C=\sqrt{1+x^2}\;\;\ell n\,(x+\sqrt{1+x^2})-x+C.$$

3. (B)

$$\frac{2x}{(x-1)(x-4)}=\frac{D}{x-1}+\frac{E}{x-4}$$

$$2x=D(x-4)+E(x-1)$$

$\therefore\quad D=-2/3, E=8/3$

$$\therefore\quad\int\frac{e^{x-1}}{(x-1)(x-4)}2x\,dx=\int e^{x-1}\left(\frac{-2/3}{x-1}+\frac{8/3}{x-4}\right)dx$$

$$=-\frac{2}{3}F(x-1)+\frac{8}{3}e^3F(x-4)+C$$

$\therefore\quad A=-2/3, B=8/3\,e^3$

4. (A)

$$I_n=\int(\log x)^n dx$$

$$\therefore\quad I_{n-1}=\int(\log x)^{n-1}dx$$

Now $\displaystyle I_n=\int(\log x)^n 1.dx$

$$=(\log x)^n.\;x-n\int(\log x)^{n-1}.\frac{1}{x}.x\;dx$$

$$I_n=n(\log x)^n-n\,I_{n-1}$$

$$I_n+n\,I_{n-1}=n\,(\log x)^n$$

5. $\displaystyle\int\frac{\sin^8x-\cos^8x}{\sin^4x+\cos^4x}dx=\int(\sin^4x-\cos^4x)\,dx$

$$=\int(\sin^2x-\cos^2x)\,dx\;=-\frac{1}{2}\sin 2x+C$$

6. (A)

$$A=\begin{bmatrix}x & x\\ x & x\end{bmatrix}$$

$$\Rightarrow\quad A^2=\begin{bmatrix}2x^2 & 2x^2\\ 2x^2 & 2x^2\end{bmatrix},$$

$$A^3=\begin{bmatrix}2^2x^2 & 2^2x^2\\ 2^2x^2 & 2^2x^2\end{bmatrix}\;\text{........ so on}$$

$$e^A=I+A+\frac{A^2}{2!}+\frac{A^3}{3!}\;.......$$

$$e^A=\begin{bmatrix}1+x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+...... & x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+......\\ x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+......... & 1+x+\frac{2x^2}{2!}+\frac{2^2x^3}{3!}+..\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}e^{2x}+1 & e^{2x}-1\\ e^{2x}-1 & e^{2x}+1\end{bmatrix}$$

$\Rightarrow\quad f(x)=e^{2x}+1\;\&\;g(x)=e^{2x}-1$

Now solve all the options

7. $\displaystyle\int\{\sec^2x+\tan^2x+2\tan x\sec x\}^{1/2}dx$

$$=\int(\sec x+\tan x)\,dx$$

$=\ell n\,|(\sec x+\tan x)|+\ell n\,|\sec x|+C$

$=\ell n\,|\sec x\,(\sec x+\tan x)|+C$

8. (C)

$$\int e^{(x\sin x+\cos x)}\left(\frac{x^4\cos^3 x - x\sin x+\cos x}{x^2\cos^2 x}\right).dx$$

$$= \int (e^{(x\sin x+\cos x)}\, x^2\cos x)\,dx$$

$$- \int \underset{\text{I}}{e^{x\sin x+\cos x}}\underset{\text{II}}{\left(\frac{x\sin x-\cos x}{(x\cos x)^2}\right)}.dx$$

Applying Integration by parts

$$= \left\{x.e^{x\sin x+\cos x} - \int e^{x\sin x+\cos x}.dx\right\}$$

$$-\left\{e^{x\sin x+\cos x}.\frac{1}{x\cos x} - \int e^{x\sin x+\cos x}.dx\right\}$$

$$= e^{x\sin x+\cos x}\left\{x-\frac{1}{x\cos x}\right\}+C$$

9. $I = 2\int \sin x\,.\,\text{cosec}\,4x\,dx = 2\int \frac{\sin x\,dx}{4\sin x\cos x\cos 2x}$

$$= \frac{1}{2}\int \frac{\cos x dx}{(1-\sin^2 x)(1-2\sin^2 x)}$$

Put $\sin x = t$, then $\cos x\,dx = dt$

$$\Rightarrow I = \frac{1}{2}\int \frac{dt}{(1-t^2)(1-2t^2)}$$

$$= \frac{1}{2}\left[-\int \frac{1}{1-t^2}dt + \int \frac{2}{1-2t^2}dt\right] \text{ [By partial fraction]}$$

$$= \frac{1}{2}\left[-\frac{1}{2}\ell n\left|\frac{1+t}{1-t}\right| + \frac{2}{2\sqrt{2}}\ell n\left|\left(\frac{1+\sqrt{2}t}{1-\sqrt{2}t}\right)\right|\right]$$

$$= \frac{1}{2\sqrt{2}}\ell n\left|\left(\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right)\right| - \frac{1}{4}\ell n\left|\left(\frac{1+\sin x}{1-\sin x}\right)\right| + C$$

10. (B)

S_1 : $a = 0 \Rightarrow \int \sec x\,dx = \log\left(\tan\frac{x}{2}\right)+C$

$\Rightarrow$ f is a log function for $a = 0$

S_2 : $|a| < 1$

$$\Rightarrow \frac{2}{1-a}\int \frac{\frac{1}{2}\sec^2\frac{x}{2}dx}{\left(\frac{a+1}{1-a}\right)-\tan^2\frac{x}{2}} = \frac{2}{1-a}\int \frac{dt}{\left(\frac{a+1}{1-a}\right)-t^2}$$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}}\ell n\left(\frac{\sqrt{a+1}+\sqrt{1-a}\times\tan\frac{x}{2}}{\sqrt{a+1}-\sqrt{1-a}\times\tan\frac{x}{2}}\right)+c \text{ , f is a log}$$

function for $|a| < 1$

S_3 : $a = 1$

$$\Rightarrow \int \frac{dx}{1+\cos x} = \int \frac{1}{2}\sec^2\frac{x}{2}\,dx = \tan\frac{x}{2}+C$$

$\Rightarrow$ f is a polynomial for $a = 1$

S_4 : False

11. $I = \int \frac{3\cot 3x - \cot x}{\tan x - 3\tan 3x}dx = pf(x)+g(x)+C$ ....**(i)**

Using

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x} \text{ and } \cot 3x = \frac{\cot^3 x - 3\cot x}{3\cot^2 x - 1}$$

$$I = \int \frac{1-3\tan^2 x+2-2}{3-\tan^2 x}dx$$

$$I = \int 1.dx - 2\int \frac{1+\tan^2 x}{3-\tan^2 x}dx = x - 2\int \frac{\sec^2 x}{3-\tan^2 x}dx + C$$

$$= x - \frac{2}{2\sqrt{3}}\ell n\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + C$$

$$= 1.x + \frac{1}{\sqrt{3}}\ell n\left|\frac{\sqrt{3}-\tan x}{\sqrt{3}+\tan x}\right| + C$$

comparing with R.H.S. of equation **(i)**, we get result.

12. $I = \int \frac{(x-1)dx}{x^2\sqrt{2x^2-2x+1}} = \int \frac{(x-1)\,dx}{x^3\sqrt{2-\frac{2}{x}+\frac{1}{x^2}}}$

Put $2-\frac{2}{x}+\frac{1}{x^2} = t^2$, then $\left(\frac{x-1}{x^3}\right)dx = t\,dt$

$$\Rightarrow I = \int \frac{t\,dt}{t} = t + C = \sqrt{2-\frac{2}{x}+\frac{1}{x^2}}+C$$

$$= \frac{\sqrt{2x^2-2x+1}}{x}+C$$

So $f(x) = 2x^2 - 2x + 1$ and $g(x) = x$

13. $I_n = \int \cot^n x \, dx = \int \cot^{n-2} x(\operatorname{cosec}^2 x - 1)\, dx$

$$= \frac{-\cot^{n-1} x}{n-1} - I_{n-2} + C_1$$

$$\Rightarrow \quad I_n + I_{n-2} = \frac{-\cot^{n-1} x}{n-1} + C_1$$

Now, $I_0 + I_1 + 2(I_2 + ... + I_8) + I_9 + I_{10}$

$= (I_2 + I_0) + (I_3 + I_1) + (I_4 + I_2) + (I_5 + I_3) + (I_6 + I_4) + (I_7 + I_5) + (I_8 + I_6) + (I_9 + I_7) + (I_{10} + I_8)$

$$= -\left(\frac{\cot x}{1} + \frac{\cot^2 x}{2} + ... + \frac{\cot^9 x}{9}\right) + C$$

$\therefore$ A = – 1.

14. $I = \int e^{3x} \cos 4x \, dx = e^{3x}\left(A\sin 4x + B\cos 4x\right) + C$ ......(i)

$$I = \frac{1}{4}e^{3x} \sin 4x - \int \frac{3}{4}e^{3x} \sin 4x \; dx = \frac{1}{4}e^{3x} \sin 4x + \frac{3}{16}e^{3x} \cos 4x - \int \frac{9}{16}e^{3x} \cos 4x \, dx$$

$$\frac{25}{16} I = \frac{1}{16}\left(4e^{3x} \sin 4x + 3e^{3x} \cos 4x\right)$$

comparing with equation (i)

$\Rightarrow \quad A = \frac{4}{25}, B = \frac{3}{25}$

$\Rightarrow \quad \frac{A}{B} = \frac{4}{3}$

$\Rightarrow \quad 3A = 4B$

$\Rightarrow \quad 4A + 3B = 1$

15. $I = \int 2^{mx} . 3^{nx} dx = \int (2^m . 3^n)^x dx = \frac{2^{mx} . 3^{nx}}{\ell n (2^m . 3^n)} + C$

16. (A)

$\int ((\log_x e) - (\log_x e)^2)\, dx$

$$= \int \left(\frac{1}{\ell nx} - \frac{1}{(\ell nx)^2}\right) dx = \int \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t \, dt$$

(where $t = \ell n x$)

$$= \frac{e^t}{t} + C = \frac{x}{\ell nx} + C = x \log_x e + C$$

17. (A)

Since $y = f^{-1}(x)$, therefore, $x = f(y)$.

$\therefore$ $dx = f'(y)\, dy$. Thus $\int f^{-1}(x)\, dx = \int y f'(y)\, dy + c$

$\therefore$ statement-2 is true

statement-1 is true for $f^{-1}(x) = \sin^{-1} x$ in statement-2

19. (D)

Since $\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$,

$\therefore$ $d(\cot^{-1}x) = -\, d(\tan^{-1}x)$.

Thus $\int 2^{\tan^{-1} x} \; d(\cot^{-1}x) = -\int 2^{\tan^{-1} x} \; d(\tan^{-1}x)$

$$= -\frac{2^{\tan^{-1} x}}{\ell n\, 2} + c$$

$\therefore$ **Statement-I** is false

Statement-II is true

20. (A)

Statement-II

$\int e^{g(x)} \; (f(x)\, g'(x) + f'(x)\, dx$

$= \int f(x).e^{g(x)}.g'(x) \; dx + \int e^{g(x)}.f'(x)\, dx$

$= f(x) \,.\, e^{g(x)} - \int f'(x)\, e^{g(x)} \; dx + \int e^{g(x)}.f'(x) \; dx$

$= e^{g(x)}\, f(x) + c$

Statement-I

Since $g(x) = \sin^{-1} x$, $f(x) = \sqrt{1-x^2}$

$$\therefore \quad \int e^{\sin^{-1} x}\left(1 - \frac{x}{\sqrt{1-x^2}}\right) dx$$

$$= \int e^{\sin^{-1} x}\left(\frac{1}{\sqrt{1-x^2}}.\sqrt{1-x^2} + \frac{-x}{\sqrt{1-x^2}}\right)$$

$= e^{\sin^{-1} x}.\sqrt{1-x^2} + c$ by statement-2

21. (A) → (q), (B) → (r), (C) → (p), (D) → (r)

(A) If $\frac{\pi}{4} < x < \frac{3\pi}{8}$, then $\sin x > \cos x$

$$\therefore \quad \int \frac{\sin x - \cos x}{|\sin x - \cos x|} dx = \int 1 \,.\, dx = x + c$$

(B) $\int \frac{x^2\,dx}{(x^3+1)(x^3+2)} = \frac{1}{3}\int 3x^2\left(\frac{1}{x^3+1}-\frac{1}{x^3+2}\right)dx$

$= \frac{1}{3}\,\ell n\left|\frac{x^3+1}{x^3+2}\right| + c$

$\therefore\ f(x) = \ell n|x|$

(C) $\int \sin^{-1}x\cos^{-1}x\,dx = \int\left[\frac{\pi}{2}\sin^{-1}x-(\sin^{-1}x)^2\right]dx$

$\Rightarrow\ \frac{\pi}{2}(x\sin^{-1}x+\sqrt{1-x^2}) - (x(\sin^{-1}x)^2 + \sin^{-1}x$

$\sqrt{1-x^2} - x) + c$ By parts

$\Rightarrow\ \sin^{-1}x\left[\frac{\pi}{2}x - x\sin^{-1}x - 2\sqrt{1-x^2}\right] + \frac{\pi}{2}$

$\sqrt{1-x^2} + 2x + c$

$\therefore\ f^{-1}(x) = \sin^{-1}x,\ f(x) = \sin x$

(D) $\int \frac{dx}{x\,\ell n|x|} = \ell n|\ell n|x|| + c$

$\therefore\ f(x) = \ell n|x|$

22. (A) → (t), (B) → (t), (C) → (r), (D) → (s)

(A) $F(x) = \int \frac{x+\sin x}{1+\cos x}dx = \int\left(x\,\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)dx$

$= x\tan x/2 + c$

Since $0 = F(0)$

$\therefore$ $c = 0$ and $F(\pi/2) = \pi/2$

(B) $F(x) = \int e^{\sin^{-1}x}\left(1 - \frac{x}{\sqrt{1-x^2}}\right)dx$

$= \int e^{\sin^{-1}x}\left(\frac{1}{\sqrt{1-x^2}}\sqrt{1-x^2} - \frac{x}{\sqrt{1-x^2}}\right)dx$

$F(x) = e^{\sin^{-1}x}\sqrt{1-x^2} + c$

$F(0) = 1 + c \quad \Rightarrow \quad c = 0 \qquad (\because F(0) = 1)$

$F(1/2) = e^{\pi/6}.\frac{\sqrt{3}}{2} = \frac{k\sqrt{3}}{\pi}e^{\pi/6}$

$\therefore\ k = \frac{\pi}{2}$

(C) $F(x) = \int \frac{dx}{(x^2+1)(x^2+9)} = \frac{1}{8}\int\left(\frac{1}{x^2+1} - \frac{1}{x^2+9}\right)dx$

$= \frac{1}{8}\left[\tan^{-1}x - \frac{1}{3}\tan^{-1}\frac{x}{3}\right] + c$

$F(0) = c \quad \Rightarrow \quad c = 0$

$\therefore\ f(\sqrt{3}) = \frac{1}{8}\left(\frac{\pi}{3} - \frac{1}{3}.\frac{\pi}{6}\right) = \frac{5\pi}{144} = \frac{5k}{36}\ \therefore\ k = \frac{\pi}{4}$

(D) $F(x) = \int \frac{\sqrt{\tan x}}{\sin x\cos x}dx = \int (\tan x)^{\frac{-1}{2}}\sec^2 x\,dx$

$= 2\sqrt{\tan x} + c$

$F(0) = c$

$\therefore\ c = 0$

$\therefore\ F(\pi/4) = 2 = \frac{2k}{\pi} \qquad \therefore\ k = \pi$

23. 1. (A) 2. (D) 3. (B)

Differentiating both sides

$\frac{x^3-6x^2+11x-6}{\sqrt{x^2+4x+3}} = (Ax^2+Bx+C)\frac{(x+2)}{\sqrt{x^2+4x+3}}$

$+ (2Ax+B)\sqrt{x^2+4x+3} + \frac{\lambda}{\sqrt{x^2+4x+3}}$

$x^3 - 6x^2 + 11x - 6 = (Ax^2+Bx+C)(x+2) + (2Ax+B)$
$(x^2+4x+3) + \lambda$

compairng coefficients of like powers of x

$x^3 \ :\ 1 = A + 2A$

$\Rightarrow\ A = 1/3$

$x^2 \ :\ -6 = 2A + B + 8A + B$

$2B = -6 - 10.\frac{1}{3}$

$\Rightarrow\ B = -\frac{14}{3}$

$x \ :\ 11 = 2B + C + 6A + 4B$

$C = 11 - 6\left(-\frac{14}{3}\right) - 6.\frac{1}{3} = 11 + 28 - 2 = 37$

Constant terms : $-6 = 2C + 3B + \lambda$

$\lambda = -6 - 2.37 - 3.\left(-\frac{14}{3}\right) = -6 - 74 + 14 = -66.$

24.

1. $I = \int (\sqrt{\tan x} + \sqrt{\cot x})\, dx$

$$= 2\int \frac{(\sqrt{\tan x} + \sqrt{\cot x})\,(\tan x + \cot x)}{2(\tan x + \cot x)}\, dx$$

$$I = 2\int \frac{d(\sqrt{\tan x} - \sqrt{\cot x})}{(\sqrt{\tan x} - \sqrt{\cot x})^2 + 2}$$

$$= \sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}}\right) + C$$

2. If $x \in \left(0, \frac{\pi}{2}\right)$, then I

$$= \int (\sqrt{\tan x} + \sqrt{\cot x})\, dx$$

$$= \int \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right) dx$$

$$= \int \frac{\sin x + \cos x}{\sqrt{\sin x \,.\, \cos x}}\, dx = \int \frac{\sqrt{2}\, d\,(\sin x - \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}}$$

$$= \sqrt{2}\ \sin^{-1}(\sin x - \cos x) + C$$

3. If $x \in \left(\pi, \frac{3\pi}{2}\right)$, then

$$I = \int (\sqrt{\tan x} + \sqrt{\cot x})\, dx$$

$$= \int \left(\frac{\sqrt{-\sin x}}{\sqrt{-\cos x}} + \frac{\sqrt{-\cos x}}{\sqrt{-\sin x}}\right) dx$$

$$= -\int \frac{\sin x + \cos x}{\sqrt{\sin x \,.\, \cos x}}\, dx$$

$$= -\int \frac{\sqrt{2}\, d\,(\sin x - \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}}$$

$$= \sqrt{2}\ \sin^{-1}(\cos x - \sin x) + C$$

25.

1. (A)

Let $P = \sin^3 x \cos^3 x$

$$\frac{dP}{dx} = 3\sin^2 x \cos^4 x - 3\sin^4 x \cos^2 x = 3\sin^2 x\,(1 - \sin^2 x)$$

$\cos^2 x - 3\sin^4 x \cos^2 x$

$= 3\sin^2 x \cos^2 x - 6\sin^4 x \cos^2 x$

$\therefore\ P = 3 I_{2,2} - 6 I_{4,2}$ $\quad\therefore\ I_{4,2} = \frac{1}{6}(-P + 3 I_{2,2})$

2. (A)

Let $P = \sin^5 x \cos^3 x$

$\therefore\ \frac{dP}{dx} = 5\sin^4 x \cos^4 x - 3\sin^6 x \cos^2 x$

$= 5\sin^4 x\,(1 - \sin^2 x)\cos^2 x - 3\sin^6 x \cos^2 x$

$= 5\sin^4 x \cos^2 x - 8\sin^6 x \cos^2 x$

$\therefore\ P = 5 I_{4,2} - 8 I_{6,2}$

$\therefore\ I_{4,2} = \frac{1}{5}(P + 8 I_{6,2})$

3. (B)

Let $P = \sin^5 x \cos^3 x$

$\therefore\ \frac{dP}{dx} = 5\sin^4 x \cos^4 x - 3\sin^6 x \cos^2 x$

$= 5\sin^4 x \cos^4 x - 3\sin^4 x\,(1 - \cos^2 x)\cos^2 x$

$= 8\sin^4 x \cos^4 x - 3\sin^4 x \cos^2 x$

$\therefore\ P = 8 I_{4,4} - 3 I_{4,2}$

$\therefore\ I_{4,2} = \frac{1}{3}(-P + 8 I_{4,4})$

26. $\int \cos 2x\, \ell n\,(1 + \tan x)\, dx = \frac{1}{2}\sin 2x\, \ell n\,(1 + \tan x) -$

$$\frac{1}{2}\int \frac{\sec^2 x}{(1 + \tan x)}\ \sin 2x\, dx$$

$$= \frac{1}{2}\sin 2x\, \ell n\,(1 + \tan x) - \int \frac{\tan x + 1 - 1}{1 + \tan x}\, dx$$

$$= \frac{1}{2}\sin 2x\, \ell n\,(1 + \tan x) - x + \frac{1}{2}$$

$$\int \frac{2\cos x + \sin x - \sin x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\sin 2x\, \ell n\,(1 + \tan x) - x + \frac{x}{2} + \frac{1}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x}\, dx$$

$$= \frac{1}{2}\left[\sin 2x\, \ell n\,(1 + \tan x) - x + \ell n\,|\sin x + \cos x|\right] + C$$

27. (1)

$$\int \frac{x\cos\alpha+1}{(x^2+2x\cos\alpha+1)^{3/2}}\,dx = \int \frac{x\cos\alpha+1}{\left((x+\cos\alpha)^2+\sin^2\alpha\right)^{3/2}}\,dx$$

Put $x + \cos\alpha = \sin\alpha \tan\theta$

$$= \int \frac{\cos\alpha\,(\sin\alpha\tan\theta-\cos\alpha)+1}{(\sin^2\alpha\tan^2\theta+\sin^2\alpha)^{3/2}}\ \sin\alpha\ .\ \sec^2\theta\ d\theta$$

$$= \int \frac{\cos\alpha\ \sin\alpha\tan\theta+\sin^2\alpha}{\sin^3\alpha\sec^3\theta}\ \sin\alpha\ \sec^2\theta\ d\theta$$

$$= \int \frac{\cos\alpha\tan\theta+\sin\alpha}{\sin\alpha}\ \cos\theta\ d\theta$$

$$= \int (\cot\alpha.\sin\theta+\cos\theta)\ d\theta\ = -\cot\alpha\cos\theta + \sin\theta + c$$

$$= -\cot\alpha\ \frac{\sin\alpha}{\sqrt{x^2+2x\cos\alpha+1}} + \frac{x+\cos\alpha}{\sqrt{x^2+2x\cos\alpha+1}} + c$$

$$= \frac{x}{\sqrt{x^2+2x\cos\alpha+1}} + c$$

28. $I = \int \sqrt{\frac{\operatorname{cosec}x-\cot x}{\operatorname{cosec}x+\cot x}}\cdot\frac{\sec x\ dx}{\sqrt{1+2\sec x}}$

$$= \int \sqrt{\frac{1-\cos x}{1+\cos x}}\cdot\frac{\sec x\ dx}{\sqrt{1+2\sec x}} = \int \tan\frac{x}{2}\cdot\frac{dx}{\sqrt{\cos x}\sqrt{\cos x+2}}$$

$$= \int \tan\frac{x}{2}\cdot\frac{\sec^2\frac{x}{2}\ dx}{\sqrt{1-\tan^2\frac{x}{2}}\sqrt{3+\tan^2\frac{x}{2}}}$$

$$= \int \frac{\tan\frac{x}{2}\sec^2\frac{x}{2}dx}{\sqrt{2-\sec^2\frac{x}{2}}\sqrt{2+\sec^2\frac{x}{2}}} \qquad = \int \frac{\tan\frac{x}{2}\sec^2\frac{x}{2}dx}{\sqrt{2^2-\left(\sec^2\frac{x}{2}\right)^2}}$$

Let $\sec^2\frac{x}{2} = t$, then $\sec^2\frac{x}{2}\tan\frac{x}{2}dx = dt$

$$\Rightarrow\ I = \int \frac{dt}{\sqrt{2^2-t^2}} = \sin^{-1}\left(\frac{t}{2}\right) + C$$

$$= \sin^{-1}\left(\frac{1}{2}\sec^2\frac{x}{2}\right) + C$$

29. (2)

$$I = \int \left(\frac{x-1}{x+1}\right)\frac{dx}{x\sqrt{x+1+\frac{1}{x}}}$$

So, $I = \int \frac{(x-1)\,dx}{(x+1)x\sqrt{x+1+\frac{1}{x}}}$

$$= \int \frac{\left(1-\frac{1}{x}\right)\left(1+\frac{1}{x}\right)dx}{(x+1)\left(1+\frac{1}{x}\right)\sqrt{x+1+\frac{1}{x}}}$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}$$

Put $x + 1 + \frac{1}{x} = t^2$

$$\left(1-\frac{1}{x^2}\right)dx = 2t\ dt$$

$$= \int \frac{2t\ dt}{(t^2+1)t} = 2\tan^{-1}t + c$$

$$= 2\tan^{-1}\left(\sqrt{x+\frac{1}{x}+1}\right) + c$$

30. $I = \int e^x\left(\frac{1+nx^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}\right)dx$

$$= \int e^x\left(\sqrt{\frac{1+x^n}{1-x^n}} + \frac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}\right)dx$$

Let $f(x) = \sqrt{\frac{1+x^n}{1-x^n}}$

$$\Rightarrow\ f'(x) = \frac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}$$

$$\therefore\ I = e^x.\sqrt{\frac{1+x^n}{1-x^n}} + C$$

DCAM classes
Dynamic Classes for Academic Mastery