Binomial Theorem and Principle of Mathematical Induction

An algebric expression consisting of two terms with positive and negative sign between them is called **binomial expression**.

Binomial Theorem for Positive Integer

If n is any positive integer, then

$$(x + a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots + {}^nC_n a^n.$$

i.e.

$$(x + a)^n = \sum_{r=0}^{n} {}^nC_r \, x^{n-r} a^r \qquad \ldots\text{(i)}$$

where, x and a are real numbers and ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called **binomial coefficients**.

Also, here Eq. (i) is called **Binomial theorem**.

$${}^nC_r = \frac{n!}{r!(n-r)!} \text{ for } 0 \le r \le n.$$

Properties of Binomial Theorem for Positive Integer

(i) Total number of terms in the expansion of $(x + a)^n$ is $(n + 1)$ i.e. finite number of terms.

(ii) The sum of the indices of x and a in each term is n.

(iii) The above expansion is also true when x and a are complex numbers.

(iv) The coefficient of terms equidistant from the beginning and the end are equal. These coefficients are known as the binomial coefficients i.e. ${}^nC_r = {}^nC_{n-r}$, $r = 0, 1, 2, \ldots, n$.

(v) The values of the binomial coefficients steadily increase to maximum and then steadily decrease.

(vi) In the binomial expansion of $(x + a)^n$, the r th term from the end is $(n - r + 2)$th term from the beginning.

(vii) If n is a positive integer, then number of terms in $(x+y+z)^n$ is $\frac{(n+1)(n+2)}{2}$.

Some Special Cases

(i) $(x-a)^n = {}^nC_0\, x^n - {}^nC_1 x^{n-1}a + {}^nC_2 x^{n-2}a^2 - {}^nC_3 x^{n-3}a^3 + \ldots + (-1)^n\, {}^nC_n\, a^n$

i.e. $(x-a)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r \cdot x^{n-r} \cdot a^r$

(ii) $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_r\, x^r + \ldots + {}^nC_n\, x^n$

i.e. $(1+x)^n = \sum_{r=0}^{n} {}^nC_r \cdot x^r$

(iii) $(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \ldots + (-1)^r\, {}^nC_r\, x^r + \ldots + (-1)^n\, {}^nC_n x^n$

i.e. $(1-x)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r \cdot x^r$

(iv) The coefficient of x^r in the expansion of $(1+x)^n$ is ${}^nC_r$ and in the expansion of $(1-x)^n$ is $(-1)^r\, {}^nC_r$.

(v) (a) $(x+a)^n + (x-a)^n = 2\,({}^nC_0 x^n a^0 + {}^nC_2 x^{n-2} a^2 + \ldots)$

(b) $(x+a)^n - (x-a)^n = 2\,({}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + \ldots)$

(vi) (a) If n is odd, then $(x+a)^n + (x-a)^n$ and $(x+a)^n - (x-a)^n$ both have the same number of terms equal to $\left(\frac{n+1}{2}\right)$.

(b) If n is even, then $(x+a)^n + (x-a)^n$ has $\left(\frac{n}{2}+1\right)$ terms.

and $(x+a)^n - (x-a)^n$ has $\left(\frac{n}{2}\right)$ terms.

General Term in a Binomial Expansion

(i) General term in the expansion of $(x+a)^n$ is

$$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$$

(ii) General term in the expansion of $(x-a)^n$ is

$$T_{r+1} = (-1)^r\, {}^nC_r\, x^{n-r}\, a^r$$

(iii) General term in the expansion of $(1+x)^n$ is

$$T_{r+1} = {}^nC_r\, x^r$$

(iv) General term in the expansion of $(1-x)^n$ is

$$T_{r+1} = (-1)^r\ {}^nC_r\, x^r$$

Some Important Results

(i) Coefficient of x^m in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$ is the coefficient of T_{r+1}, where $r = \frac{np-m}{p+q}$.

(ii) The term independent of x in the expansion of $\left(ax^p + \frac{b}{x^q}\right)^n$ is the coefficient of T_{r+1}, where $r = \frac{np}{p+q}$.

(iii) If the coefficient of rth, $(r+1)$ th and $(r+2)$ th term of $(1+x)^n$ are in AP, then $n^2 - (4r+1)n + 4r^2 = 2$

(iv) In the expansion of $(x+a)^n$,

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \times \frac{a}{x}$$

(v) (a) The coefficient of x^{n-1} in the expansion of

$$(x-1)(x-2)\ldots.(x-n) = -\frac{n(n+1)}{2}$$

(b) The coefficient of x^{n-1} in the expansion of

$$(x+1)(x+2)\ldots.(x+n) = \frac{n(n+1)}{2}$$

(vi) If the coefficient of pth and qth terms in the expansion of $(1+x)^n$ are equal, then $p+q = n+2$.

(vii) If the coefficients of x^r and x^{r+1} in the expansion of $\left(a + \frac{x}{b}\right)^n$ are equal, then $n = (r+1)(ab+1) - 1$.

(viii) The number of terms in the expansion of

$$(x_1 + x_2 + \ldots + x_r)^n \text{ is } {}^{n+r-1}C_{r-1}.$$

Middle Term in a Binomial Expansion

(i) If n is even in the expansion of $(x+a)^n$ or $(x-a)^n$, then the middle term is $\left(\frac{n}{2}+1\right)$th term.

(ii) If n is odd in the expansion of $(x+a)^n$ or $(x-a)^n$, then the middle terms are $\frac{(n+1)}{2}$ th term and $\frac{(n+3)}{2}$ th term.

Note When there are two middle terms in the expansion, then their binomial coefficients are equal.

Greatest Coefficient

Binomial coefficient of middle term is the greatest binomial coefficient.

(i) If n is even, then in $(x+a)^n$, the greatest coefficient is ${}^nC_{n/2}$.

(ii) If n is odd, then in $(x+a)^n$, the greatest coefficient is ${}^nC_{\frac{n-1}{2}}$ $\left(\text{or } {}^nC_{\frac{n+1}{2}}\right)$.

Greatest Term

In the expansion of $(x+a)^n$,

(i) If $\frac{n+1}{\left|\frac{x}{a}\right|+1}$ is an integer $= p$ (say), then greatest terms are T_p and T_{p+1}.

(ii) If $\frac{n+1}{\left|\frac{x}{a}\right|+1}$ is not an integer with m as integral part of $\frac{n+1}{\frac{x}{a}+1}$, then T_{m+1} is the greatest term.

Divisibility Problems

From the expansion, $(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$

We can conclude that

(i) $(1+x)^n - 1 = {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$ is divisible by x i.e. it is a multiple of x.

or $$(1+x)^n - 1 = M(x)$$

(ii) $(1+x)^n - 1 - nx = {}^nC_2x^2 + {}^nC_3x^3 + \ldots + {}^nC_nx^n = M(x^2)$

(iii) $(1+x)^n - 1 - nx - \frac{n(n-1)}{2}x^2 = {}^nC_3x^3 + {}^nC_4x^4 + \ldots + {}^nC_nx^n = M(x^3)$

Important Results on Binomial Coefficients

If $C_0, C_1, C_2, \ldots, C_n$ are the coefficients of $(1+x)^n$, then

(i) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(ii) $\dfrac{{}^nC_r}{{}^{n-1}C_{r-1}} = \dfrac{n}{r}$

(iii) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

(iv) $C_0 + C_1 + C_2 + \ldots + C_n = 2^n$

(v) $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$

(vi) $C_0 - C_2 + C_4 - C_6 + \ldots = (\sqrt{2})^n \cos\dfrac{n\pi}{4}$

(vii) $C_1 - C_3 + C_5 - C_7 + \ldots = (\sqrt{2})^n \sin\dfrac{n\pi}{4}$

(viii) $C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^n C_n = 0$

(ix) $C_1 - 2\cdot C_2 + 3\cdot C_3 - \ldots = 0$

(x) $C_0 + 2\cdot C_1 + 3\cdot C_2 + \ldots + (n+1)\cdot C_n = (n+2)\,2^{n-1}$

(xi) $C_0C_r + C_1C_{r+1} + \ldots + C_{n-r}C_n = {}^{2n}C_{n-r}$

$$= {}^{2n}C_{n+r} = \frac{(2n)!}{(n-r)!\,(n+r)!}$$

(xii) $C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = {}^{2n}C_n = \dfrac{(2n)!}{(n!)^2}$

(xiii) $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots + (-1)^n \cdot C_n^2 = \begin{cases} 0, & \text{if } n \text{ is odd.} \\ (-1)^{n/2} \cdot {}^nC_{n/2}, & \text{if } n \text{ is even.} \end{cases}$

(xiv) $C_1{}^2 - 2C_2{}^2 + 3C_3^2 - \ldots + (-1)^n\, n \cdot C_n^2$

$$= (-1)^{\frac{n}{2}-1} \cdot \frac{n}{2} \cdot \frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}, \text{ when } n \text{ is even.}$$

(xv) $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \ldots + \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{(n+1)}$

(xvi) $C_0 - \dfrac{C_1}{2} + \dfrac{C_2}{3} - \dfrac{C_3}{4} + \ldots + (-1)^n \dfrac{C_n}{n+1} = \dfrac{1}{n+1}$

(xvii) $C_0 + \frac{C_1}{2} + \frac{C_2}{2^2} + \frac{C_3}{2^3} + \ldots + \frac{C_n}{2^n} = \left(\frac{3}{2}\right)^n$

(xviii) $\sum_{r=0}^{n} (-1)^r \, {}^nC_r \left\{\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \ldots \text{ upto } m \text{ terms}\right\}$

$$= \frac{2^{mn} - 1}{2^{mn}(2^n - 1)}$$

Multinomial Theorem

For any $n \in N$,

(i) $(x_1 + x_2)^n = \sum_{r_1 + r_2 = n} \frac{n!}{r_1! r_2!} x_1^{r_1} x_2^{r_2}$

(ii) $(x_1 + x_2 + \ldots + x_n)^n = \sum_{r_1 + r_2 + \ldots + r_k = n} \frac{n!}{r_1! r_2! \ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$

(iii) The general term in the above expansion is

$$\frac{n!}{r_1! r_2! \ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$$

(iv) The greatest coefficient in the expansion of $(x_1 + x_2 + \ldots + x_m)^n$ is $\frac{n!}{(q!)^{m-r}[(q+1)!]^r}$, where q and r are the quotient and remainder respectively, when n is divided by m.

Some Important Results

(i) If n is a positive integer and $a_1, a_2, \ldots, a_m \in C$, then the coefficient of x^r in the expansion of $(a_1 + a_2 x + a_3 x^2 + \ldots + a_m x^{m-1})^n$ is

$$\sum \frac{n!}{n_1! n_2! \ldots n_m!} a_1^{n_1} x \, a_2^{n_2} \ldots a_m^{n_m}.$$

(ii) Total number of terms in the expansion of $(a + b + c + d)^n$ is $\frac{(n+1)(n+2)(n+3)}{6}$.

R-f Factor Relations

If $(\sqrt{A} + B)^n = I + f$ where I and n are positive integers, n being odd and $0 \le f < 1$, then $(I + f)f = k^n$, where $A - B^2 = k > 0$ and $\sqrt{A} - B < 1$.

Binomial Theorem for Any Index

If n is any rational number, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \dots, |x| < 1$$

(i) In the above expansion, if n is any positive integer, then the series in RHS is finite and if n is negative/ rational number, then there are infinite number of terms exist.

(ii) General term in the expansion of $(1+x)^n$ is

$$T_{r+1} = \frac{n(n-1)(n-2)\dots[n-(r-1)]}{r!}x^r.$$

(iii) Expansion of $(x+a)^n$ for any rational index

***Case* I** When $x > a$ i.e. $\frac{a}{x} < 1$

In this case, $(x+a)^n = \left\{x\left(1+\frac{a}{x}\right)\right\}^n = x^n\left(1+\frac{a}{x}\right)^n$

$$= x^n\left\{1 + n\cdot\frac{a}{x} + \frac{n(n-1)}{2!}\left(\frac{a}{x}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{a}{x}\right)^3 + \dots\right\}$$

***Case* II** When $x < a$ i.e. $\frac{x}{a} < 1$

In this case, $(x+a)^n = \left\{a\left(1+\frac{x}{a}\right)\right\}^n = a^n\left(1+\frac{x}{a}\right)^n$

$$= a^n\left\{1 + n\cdot\frac{x}{a} + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{x}{a}\right)^3 + \dots\right\}$$

(iv) $$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1\cdot 2}x^2 + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3 + \dots + \frac{n(n+1)(n+2)\dots(n+r-1)}{r!}x^r + \dots$$

(v) $$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \dots + (-1)^r\frac{n(n+1)(n+2)\dots(n+r-1)}{r!}x^r + \dots$$

(vi) $$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \dots + (-1)^r\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots$$

(vii) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots + (-1)^r x^r + \ldots$

(viii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots$

(ix) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots + (-1)^r (r+1) x^r + \ldots$

(x) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots + (r+1) x^r + \ldots$

(xi) $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots \infty$

(xii) $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots \infty$

(xiii) $(1+x)^n = 1 + nx$, if $x^2, x^3, \ldots$ are all very small as compared to x.

Principle of Mathematical Induction

In an algebra, there are certain results that are formulated in terms of n, where n is a positive integer. Such results can be proved by specific technique, which is known as the principle of Mathematical Induction.

Statement

A sentence or description which can be judged either true or false, is called the statement.

e.g. (i) 3 divides 9.

(ii) Lucknow is the capital of Uttar Pradesh.

1. First Principle of Mathematical Induction

Let $P(n)$ be a statement involving natural number n. To prove statement $P(n)$ is true for all natural number, we follow following process

(i) Prove that $P(1)$ is true.

(ii) Assume $P(k)$ is true.

(iii) Using (i) and (ii) prove that statement is true for $n = k + 1$, i.e. $P(k+1)$ is true.

This is first principle of Mathematical Induction.

2. Second Principle of Mathematical Induction

In second principle of Mathematical Induction following steps are used:

(i) Prove that $P(1)$ is true.

(ii) Assume $P(n)$ is true for all natural numbers such that $2 \le n < k$.

(iii) Using (i) and (ii), prove that $P(k+1)$ is true.