DCAM classes

Dynamic Classes for Academic Mastery

Complex Numbers

Complex Number

A number of the form $z = x + iy$, where $x, y \in R$, is called a complex number. Here, the symbol i is used to denote $\sqrt{-1}$ and it is called iota.

The set of complex numbers is denoted by C.

Real and Imaginary Parts of a Complex Number Let $z = x + iy$ be a complex number, then x is called the **real part** and y is called the **imaginary part** of z and it may be denoted as $\text{Re}(z)$ and $\text{Im}(z)$, respectively.

Purely Real and Purely Imaginary Complex Number A complex number z is a purely real, if its imaginary part is 0.

i.e. $\text{Im}(z) = 0$. And purely imaginary, if its real part is 0 i.e. $\text{Re}(z) = 0$.

Zero Complex Number A complex number is said to be zero, if its both real and imaginary parts are zero.

Equality of Complex Numbers

Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal, iff $a_1 = a_2$ and $b_1 = b_2$ i.e. $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

Iota

Mathematician Euler, introduced the symbol i (read as iota) for $\sqrt{-1}$ with property $i^2 + 1 = 0$. i.e. $i^2 = -1$. He also called this symbol as the imaginary unit. Integral power of iota (i) are given below.

$$i = \sqrt{-1}, i^2 = -1, i^3 = -i, i^4 = 1$$

So, $$i^{4n+1} = i, i^{4n+2} = -1, i^{4n+3} = -i, i^{4n+4} = 1$$

In other words, $$i^n = \begin{cases} (-1)^{n/2}, \text{if } n \text{ is an even integer} \\ (-1)^{\frac{n-1}{2}} \cdot i, \text{if } n \text{ is an odd integer} \end{cases}$$

Algebra of Complex Numbers

1. Addition of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers, then their sum will be defined as

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Properties of Addition of Complex Numbers

(i) **Closure Property** Sum of two complex numbers is also a complex number.

(ii) **Commutative Property** $z_1 + z_2 = z_2 + z_1, \forall\ z_1, z_2, z_3 \in C$

(iii) **Associative Property** $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, $\forall\ z_1, z_2, z_3 \in C$

(iv) **Existence of Additive Identity** $z + 0 = z = 0 + z$

Here, 0 is additive identity element.

(v) **Existence of Additive Inverse** $z + (-z) = 0 = (-z + z)$

Here, $(-z)$ is additive inverse or negative of complex number z.

2. Subtraction of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers, then the difference $z_1 - z_2$ is defined as

$$\begin{aligned} z_1 - z_2 &= (x_1 + iy_1) - (x_2 + iy_2) \\ &= (x_1 - x_2) + i(y_1 - y_2) \end{aligned}$$

Note The difference of two complex numbers $z_1 - z_2$, follows the closure property, but this operation is neither commutative nor associative, like in real numbers. Also, there does not exist any identity element for this operation and so inverse element also does not exists.

3. Multiplication of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers, then their multiplication is defined as

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Properties of Multiplication of Complex Numbers

(i) **Closure Property** Product of two complex numbers is also a complex number.

(ii) **Commutative Property** $z_1 z_2 = z_2 z_1\ \forall\ z_1, z_2 \in C$.

(iii) **Associative Property** $(z_1\, z_2)\, z_3 = z_1 (z_2\, z_3)\ \forall\ z_1, z_2, z_3 \in C$.

(iv) **Existence of Multiplicative Identity** $z \cdot 1 = z = 1 \cdot z$

Here, 1 is multiplicative identity element of z.

(v) **Existence of Multiplicative Inverse** For every non-zero complex number z there exists a complex number z_1 such that $z \cdot z_1 = 1 = z_1 \cdot z$.

Then, complex number z_1 is called multiplicative inverse element of complex number z.

(vi) **Distributive Property** For each $z_1, z_2, z_3 \in C$

(a) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ [left distribution]

(b) $(z_2 + z_3)z_1 = z_2 z_1 + z_3 z_1$ [right distribution]

4. Division of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers, then their division $\frac{z_1}{z_2}$ is defined as

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)}{(x_2 + iy_2)} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$

provided, $z_2 \neq 0$.

Note The division of two complex numbers $\frac{z_1}{z_2}$, follows the closure property, but this operation is neither commutative nor associative, like in real numbers. Also, there does not exist any identity element for this operation and so inverse element also does not exists.

Identities Related to Complex Numbers

For any complex numbers z_1, z_2, we have

(i) $(z_1 + z_2)^2 = z_1^2 + 2\, z_1 z_2 + z_2^2$

(ii) $(z_1 - z_2)^2 = z_1^2 - 2z_1 z_2 + z_2^2$

(iii) $(z_1 + z_2)^3 = z_1^3 + 3z_1^2 z_2 + 3z_1 z_2^2 + z_2^3$

(iv) $(z_1 - z_2)^3 = z_1^3 - 3z_1^2 z_2 + 3z_1 z_2^2 - z_2^3$

(v) $z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)$

These identities are similar as the algebraic identities in real numbers.

Conjugate of a Complex Number

If $z = x + iy$ is a complex number, then conjugate of z is denoted by $\bar{z}$, i.e. $\bar{z} = x - iy$

Properties of Conjugate of Complex Numbers

For any complex number z, z_1, z_2, we have

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2\,\text{Re}(z), z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary.

(iii) $z - \bar{z} = 2i\,[\text{Im}(z)], z - \bar{z} = 0 \Leftrightarrow z$ is purely real.

(iv) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(v) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(vi) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

(vii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

(viii) $z_1\,\bar{z}_2 \pm \bar{z}_1\,z_2 = 2\,\text{Re}(\bar{z}_1\,z_2) = 2\,\text{Re}(z_1\,\bar{z}_2)$

(ix) $(\bar{z})^n = \overline{(z^n)}$

(x) If $z = f(z_1)$, then $\bar{z} = f(\bar{z}_1)$

(xi) If $z = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, then $\bar{z} = \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \end{vmatrix}$

where, $a_i, b_i, c_i; (i = 1, 2, 3)$ are complex numbers.

(xii) $z\,\bar{z} = \{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2$

Reciprocal/Multiplicative Inverse of a Complex Number

Let $z = x + iy$ be a non-zero complex number, then the multiplicative inverse

$$z^{-1} = \frac{1}{z} = \frac{1}{x + iy} = \frac{1}{x + iy} \times \frac{x - iy}{x - iy}$$

[on multiply and divide by conjugate of $z = x + iy$]

$$= \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + \frac{i(-y)}{x^2 + y^2}$$

Modulus (or Absolute value) of a Complex Number

If $z = x + iy$, then modulus or magnitude of z is denoted by $|z|$ and is given by

$$|z| = \sqrt{x^2 + y^2}$$

Geometrically it represents a distance of point $z(x, y)$ from origin.

Note In the set of non-real complex number, the order relation is not defined i.e. $z_1 > z_2$ or $z_1 < z_2$ has no meaning but $|z_1| > |z_2|$ or $|z_1| < |z_2|$ has got its meaning, since $|z_1|$ and $|z_2|$ are real numbers.

Properties of Modulus of Complex Numbers

(i) $|z| \geq 0$

(ii) (a) $|z| = 0$, iff $z = 0$ i.e. $\text{Re}(z) = 0 = \text{Im}(z)$ (b) $|z| > 0$, iff $z \neq 0$

(iii) $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$

(iv) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(v) $z\bar{z} = |z|^2$

(vi) $|z_1 z_2| = |z_1||z_2|$

In general, $|z_1 z_2 z_3 \ldots z_n| = |z_1||z_2||z_3|\ldots|z_n|$

(vii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, provided $z_2 \neq 0$

(viii) $|z_1 \pm z_2| \leq |z_1| + |z_2|$

In general, $|z_1 \pm z_2 \pm z_3 \pm \ldots \pm z_n| \leq |z_1| + |z_2| + |z_3| + \ldots + |z_n|$

(ix) $|z_1 \pm z_2| \geq ||z_1| - |z_2||$

(x) $|z^n| = |z|^n$

(xi) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$ i.e. greatest and least possible value of $|z_1 + z_2|$ is $|z_1| + |z_2|$ and $||z_1| - |z_2||$ respectively.

(xii) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2) = 2\,\text{Re}(z, \bar{z}_2)$

(xiii) $|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1$$

$$= |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1\bar{z}_2)$$

$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$

(xiv) $|z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) = |z_1|^2 + |z_2|^2 - (z_1\bar{z}_2 + \bar{z}_1 z_2)$

$$= |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1\bar{z}_2)$$

$$= |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$

(xv) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

(xvi) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary.

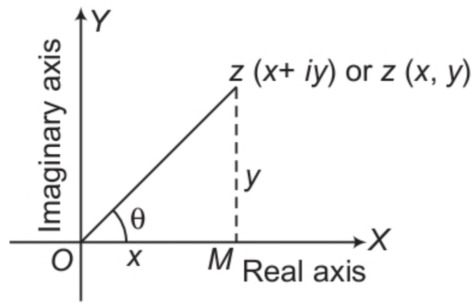
(xvii) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$ where $a, b \in R$.

(xviii) z is **unimodulus**, if $|z| = 1$

Argand Plane and Argument of a Complex Number

Argand Plane

Any complex number $z = x + iy$ can be represented geometrically by a point (x, y) in a plane, called **Argand plane** or **Gaussian plane**.



There exists a one-one correspondence between the points of the plane and the members of the set C of all complex numbers.

The length of the line segment OP is the modulus of z,

i.e. $|z| =$ length of $OP = \sqrt{x^2 + y^2}$.

Argument

The angle made by the line joining point z to the origin, with the positive direction of real axis is called argument of that complex number. It is denoted by the symbol arg (z) or amp (z).

$$\arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Argument of z is not unique, general value of the argument of z is $2n\pi + \theta$, where n is an integer. But arg (0) is not defined.

A purely real number is represented by a point on real axis.

A purely imaginary number is represented by a point on imaginary axis.

Principal Value of Argument

The value of the argument which lies in the interval $(-\pi, \pi]$ is called principal value of argument.

(i) If $x > 0$ and $y > 0$, then $\arg(z) = \theta$

(ii) If $x < 0$ and $y > 0$, then $\arg(z) = \pi - \theta$

(iii) If $x < 0$ and $y < 0$, then $\arg(z) = -(\pi - \theta)$

(iv) If $x > 0$ and $y < 0$, then $\arg(z) = -\theta$

where, $\theta = \tan^{-1}\left|\frac{y}{x}\right|$.

Properties of Argument

(i) $\arg(\bar{z}) = \begin{cases} \pi, & \text{if } z \text{ is purely negative real number} \\ -\arg(z), & \text{otherwise} \end{cases}$

(ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi, (k = 0, 1 \text{ or } -1)$

In general,

$$\arg(z_1 z_2 z_3 \ldots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \ldots + \arg(z_n) + 2k\pi, (k \text{ is an integer})$$

(iii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi \; (k = 0, 1 \text{ or } -1)$

(iv) $\arg(z_1 \bar{z}_2) = \arg(z_1) - \arg(z_2) + 2k\pi, (k = 0, 1 \text{ or } -1)$

(v) $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg(z) + 2k\pi, (k = 0, 1 \text{ or } -1)$

(vi) $\arg(z^n) = n \arg(z) + 2k\pi$, ($k$ is an integer)

(vii) If $\arg\left(\frac{z_2}{z_1}\right) = \theta$, then $\arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta, (k = 0, 1 \text{ or } -1)$

(viii) If $\arg(z) = 0 \Rightarrow z$ is real

(ix) $\arg(z) - \arg(-z) = \begin{cases} \pi, & \text{if } \arg(z) > 0 \\ -\pi, & \text{if } \arg(z) < 0 \end{cases}$

(x) If $|z_1 + z_2| = |z_1 - z_2|$, then

$$\arg\left(\frac{z_1}{z_2}\right) \Rightarrow \arg(z_1) - \arg(z_2) = \frac{\pi}{2}$$

(xi) If $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) = \arg(z_2)$

(xii) If $|z - 1| = |z + 1|$, then $\arg(z) = \pm\frac{\pi}{2}$

(xiii) If $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$, then $|z| = 1$

(xiv) (a) If $z = 1 + \cos\theta + i\sin\theta$, then

$$\arg(z) = \frac{\theta}{2} \text{ and } |z| = 2\cos\frac{\theta}{2}$$

(b) If $z = 1 + \cos\theta - i\sin\theta$, then

$$\arg(z) = -\frac{\theta}{2} \text{ and } |z| = 2\cos\frac{\theta}{2}$$

(c) If $z = 1 - \cos\theta + i\sin\theta$, then

$$\arg(z) = \frac{\pi}{2} - \frac{\theta}{2} \text{ and } |z| = 2\sin\frac{\theta}{2}$$

(d) If $z = 1 - \cos\theta - i\sin\theta$, then

$$\arg(z) = \frac{\theta}{2} - \frac{\pi}{2} \text{ and } |z| = 2\sin\frac{\theta}{2}$$

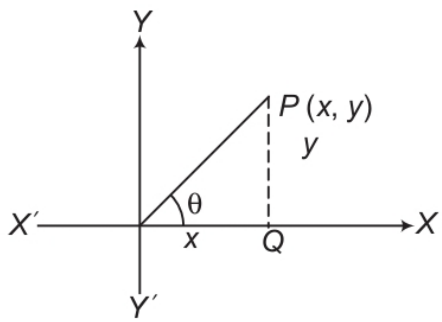
(xv) If $|z_1| \le 1, |z_2| \le 1$, then

(a) $|z_1 - z_2|^2 \le (|z_1| - |z_2|)^2 + [\arg(z_1) - \arg(z_2)]^2$

(b) $|z_1 + z_2|^2 \le (|z_1| + |z_2|)^2 - [\arg(z_1) - \arg(z_2)]^2$

Polar Form of a Complex Number

If $z = x + iy$ is a complex number, then z can be written as $z = r(\cos\theta + i\sin\theta)$, where $\theta = \arg(z)$ and $r = \sqrt{x^2 + y^2}$ this is called polar form.



If the general value of the argument is considered, then the polar form of z is $z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$, where n is an integer.

Eulerian Form of a Complex Number

If $z = x + iy$ is a complex number, then it can be written as

$$z = re^{i\theta}$$

where, $r = |z|$ and $\theta = \arg(z)$

This is called Eulerian form and $e^{i\theta} = \cos\theta + i\sin\theta$ and

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

De-Moivre's Theorem

A simplest formula for calculating powers of complex numbers in the standard polar form is known as De-Moivre's theorem.

If $n \in I$ (set of integers), then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ and if $n \in Q$ (set of rational numbers), then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

Properties of De-Moivre's Theorem

(i) If $\frac{p}{q}$ is a rational number, then

$$(\cos\theta + i\sin\theta)^{p/q} = \left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)$$

(ii) $\dfrac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$

(iii) More generally, for a complex number $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$\begin{aligned} z^n &= r^n(\cos\theta + i\sin\theta)^n \\ &= r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta} \end{aligned}$$

(iv) $(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right)\right]$

(v) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n)$

$$= \cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$$

(vi) $(\sin\theta \pm i\cos\theta)^n \neq \sin n\theta \pm i\cos n\theta$

(vii) $(\cos\theta + i\sin\phi)^n \neq \cos n\theta + i\sin n\phi$

Note

(i) $\cos 0 + i\sin 0 = 1$

(ii) $\cos\pi + i\sin\pi = -1$

(iii) $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

(iv) $\cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i$

Cube Roots of Unity

Cube roots of unity are $1, \omega, \omega^2$,

where $\omega = \dfrac{-1}{2} + i\dfrac{\sqrt{3}}{2} = e^{i2\pi/3}$ and $\omega^2 = \dfrac{-1}{2} - \dfrac{i\sqrt{3}}{2}$

$$\arg(\omega) = \frac{2\pi}{3} \text{ and } \arg(\omega^2) = \frac{4\pi}{3}$$

Properties of Cube Roots of Unity

(i) $1 + \omega^2 + \omega^{2r} = \begin{cases} 0, & \text{if } r \text{ is not a multiple of 3.} \\ 3, & \text{if } r \text{ is a multiple of 3.} \end{cases}$

(ii) $\omega^3 = 1$

(iii) $\omega^{3r} = 1, \omega^{3r+1} = \omega$ and $\omega^{3r+2} = \omega^2$, where $r \in I$.

(iv) Cube roots of unity lie on the unit circle $|z|=1$ and divide its circumference into 3 equal parts.

(v) It always forms an equilateral triangle.

(vi) Cube roots of -1 are $-1, -\omega, -\omega^2$.

Some Important Identities

(i) $x^2+x+1=(x-\omega)(x-\omega^2)$

(ii) $x^2-x+1=(x+\omega)(x+\omega^2)$

(iii) $x^2+xy+y^2=(x-y\omega)(x-y\omega^2)$

(iv) $x^2-xy+y^2=(x+\omega y)(x+y\omega^2)$

(v) $x^2+y^2=(x+iy)(x-iy)$

(vi) $x^3+y^3=(x+y)(x+y\omega)(x+y\omega^2)$

(vii) $x^3-y^3=(x-y)(x-y\omega)(x-y\omega^2)$

(viii) $x^2+y^2+z^2-xy-yz-zx=(x+y\omega+z\omega^2)(x+y\omega^2+z\omega)$

or $(x\omega+y\omega^2+z)(x\omega^2+y\omega+z)$

or $(x\omega+y+z\omega^2)(x\omega^2+y+z\omega)$

(ix) $x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

nth Roots of Unity

The nth roots of unity, it means any complex number z, which satisfies the equation $z^n=1$ or $z=(1)^{1/n}$

or $$z=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}, \text{ where } k=0,1,2,\dots,(n-1)$$

Properties of nth Roots of Unity

(i) nth roots of unity form a GP with common ratio $e^{i2\pi/n}$.

(ii) Sum of nth roots of unity is always 0.

(iii) Sum of pth powers of nth roots of unity is n, if p is a multiple of n.

(iv) Sum of pth powers of nth roots of unity is zero, if p is not a multiple of n.

(v) Product of nth roots of unity is $(-1)^{n-1}$.

(vi) The nth roots of unity lie on the unit circle $|z|=1$ and divide its circumference into n equal parts.

Square Root of a Complex Number

If $z = x + iy$, then

$$\sqrt{z} = \sqrt{x+iy} = \pm\left[\frac{\sqrt{|z|+x}}{2} + i\,\frac{\sqrt{|z|-x}}{2}\right], \text{ for } y > 0$$

$$= \pm\left[\sqrt{\frac{|z|+x}{2}} - i\sqrt{\frac{|z|-x}{2}}\right], \text{ for } y < 0$$

Logarithm of a Complex Number

Let $z = x + iy$ be a complex number and in polar form of z is $re^{i\theta}$, then

$$\log(x+iy) = \log(re^{i\theta}) = \log(r) + i\theta$$

$$= \log(\sqrt{x^2+y^2}) + i\tan^{-1}\frac{y}{x}$$

or $$\log(z) = \log(|z|) + i\,\text{amp}(z),$$

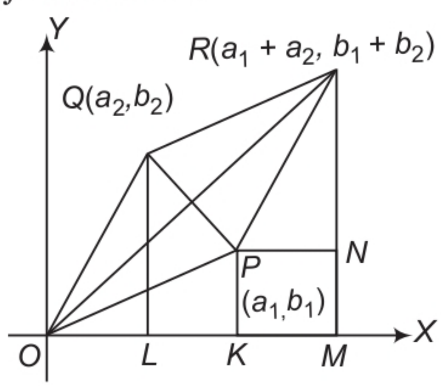
In general, $$z = re^{i(\theta + 2n\pi)}$$

$$\log(z) = \log|z| + i\arg(z) + 2n\pi i$$

Geometry of Complex Numbers

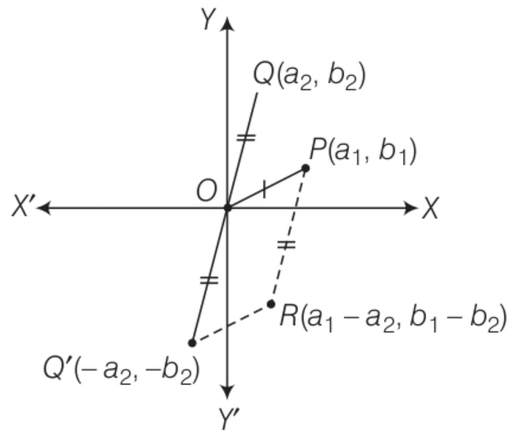
1. Geometrical Representation of Addition

If two points P and Q represent complex numbers z_1 and z_2, respectively, in the argand plane, then the sum $z_1 + z_2$ is represented by the extremity R of the diagonal OR of parallelogram $OPRQ$ having OP and OQ as two adjacent sides.



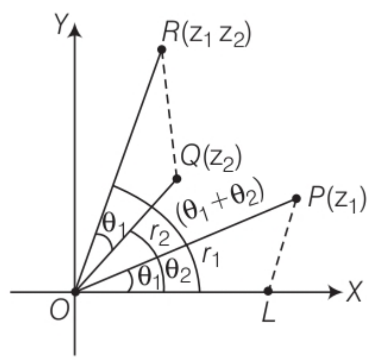
2. Geometrical Representation of Subtraction

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers represented by points $P(a_1, b_1)$ and $Q(a_2, b_2)$ in the argand plane. Q' represents the complex number $(-z_2)$. Complete the parallelogram $OPRQ'$ by taking OP and OQ' as two adjacent sides.



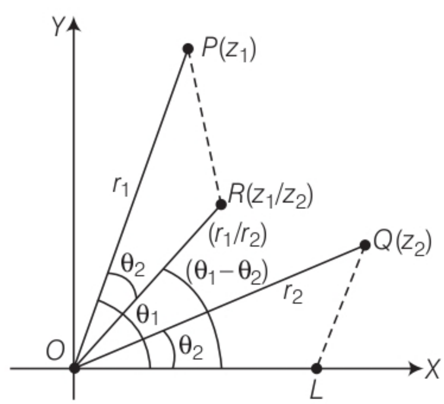
The sum of z_1 and $-z_2$ is represented by the extremity R of the diagonal OR of parallelogram $OPRQ'$. R represents the complex number $z_1 - z_2$.

3. **Geometrical Representation of Multiplication**



R has the polar coordinates $(r_1r_2, \theta_1 + \theta_2)$ and it represents the complex numbers z_1z_2.

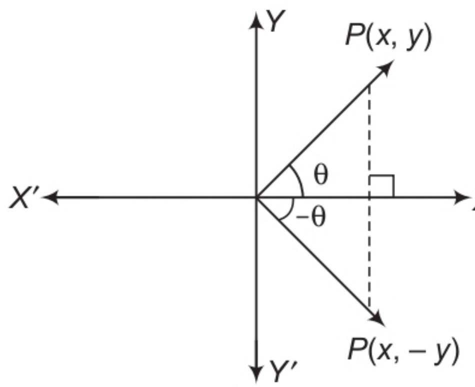
4. **Geometrical Representation of the Division**



R has the polar coordinates $\left(\frac{r_1}{r_2}, \theta_1 - \theta_2\right)$ and it represents the complex number z_1/z_2.

5. **Geometrical Representation of the Conjugate of Complex Numbers**

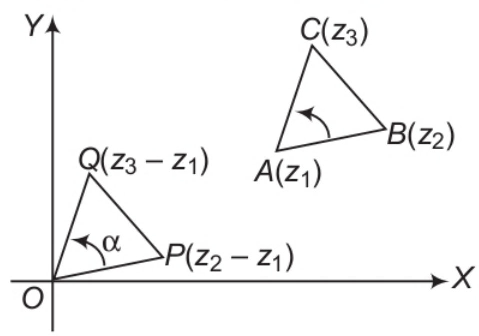
If a point P represents a complex number z, then its conjugate $\bar{z}$ is represented by the image of P in the real axis.



Geometrically, the point $(x, -y)$ is the mirror image of the point (x, y) on the real axis.

Concept of Rotation

Let z_1, z_2 and z_3 be the vertices of a ΔABC described in anti-clockwise sense. Draw OP and OQ parallel and equal to AB and AC, respectively.



Then, point P is $z_2 - z_1$ and Q is $z_3 - z_1$. If OP is rotated through angle α in anti-clockwise, sense it coincides with OQ.

$\therefore$
$$\text{amp}\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \alpha$$

Applications of Complex Numbers in Coordinate Geometry

Distance between Complex Points

(i) Distance between the points $A(z_1)$ and $B(z_2)$ is given by

$$AB = |z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

(ii) The point $P(z)$ which divides the join of segment AB internally in the ratio $m:n$ is given by

$$z = \frac{mz_2 + nz_1}{m+n}$$

If P divides the line externally in the ratio $m:n$, then

$$z = \frac{mz_2 - nz_1}{m-n}$$

Triangle in Complex Plane

(i) Let ABC be a triangle with vertices $A(z_1), B(z_2)$ and $C(z_3)$, then

(a) **Centroid of the ΔABC** is given by

$$z = \frac{1}{3}(z_1 + z_2 + z_3)$$

(b) **Incentre of the ΔABC** is given by

$$z = \frac{az_1 + bz_2 + cz_3}{a+b+c}$$

(ii) **Area of the triangle** with vertices $A(z_1), B(z_2)$ and $C(z_3)$ is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$$

For an equilateral triangle,

$$z_1^2 + z_2^2 + z_3^2 = z_2z_3 + z_3z_1 + z_1z_2$$

(iii) The triangle whose vertices are the points represented by complex numbers z_1, z_2 and z_3 is equilateral, if

$$\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$$

i.e. $$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

Straight Line in Complex Plane

(i) The general equation of a straight line is $\bar{a}z + a\bar{z} + b = 0$, where a is a complex number and b is a real number.

(ii) The complex and real slopes of the line $\bar{a}z + a\bar{z} + b = 0$ are $-\dfrac{a}{\bar{a}}$ and $-i\left(\dfrac{a+\bar{a}}{a-\bar{a}}\right)$.

(iii) The equation of straight line through z_1 and z_2 is $z = tz_1 + (1 - t) z_2$, where t is real.

(iv) If z_1 and z_2 are two fixed points, then $|z - z_1| = |z - z_2|$ represents perpendicular bisector of the line segment joining z_1 and z_2.

(v) Three points z_1, z_2 and z_3 are collinear, if

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

This is also, the equation of the line passing through z_1, z_2 and z_3 and slope is defined to be $\dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$.

(vi) **Length of Perpendicular** The length of perpendicular from a point z_1 to $\bar{a}z + a\bar{z} + b = 0$ is given by $\dfrac{|\bar{a}z_1 + a\bar{z}_1 + b|}{2|a|}$

(vii) The equation of a line parallel to the line $a\bar{z} + \bar{a}z + b = 0$ is $\bar{a}z + a\bar{z} + \lambda = 0$, where $\lambda \in R$.

(viii) The equation of a line perpendicular to the line $a\bar{z} + \bar{a}z + b = 0$ is $a\bar{z} - \bar{a}z + i\lambda = 0$, where $\lambda \in R$.

(ix) The equation of the perpendicular bisector of the line segment joining points $A(z_1)$ and $B(z_2)$ is

$$z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = |z_1|^2 - |z_2|^2$$

(x) If z is a variable point in the argand plane such that $\arg(z) = \theta$, then locus of z is a straight line through the origin inclined at an angle θ with X-axis.

(xi) If z is a variable point and z_1 is fixed point in the argand plane such that $(z - z_1) = \theta$, then locus of z is a straight line passing through the point z_1 and inclined at an angle θ with the X-axis.

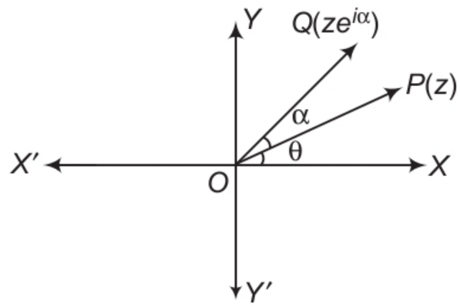
(xii) If z is a variable point and z_1, z_2 are two fixed points in the argand plane, such that

(a) $|z - z_1| + |z - z_2| = |z_1 - z_2|$, then locus of z is the line segment joining z_1 and z_2.

(b) $||z - z_1| - |z - z_2|| = |z_1 - z_2|$, then locus of z is a straight line joining z_1 and z_2 but z does not lie between z_1 and z_2.

(c) $\arg\left(\frac{z - z_1}{z - z_2}\right) = 0$ or π, then locus z is a straight line passing through z_1 and z_2.

(xiii)



(a) $ze^{i\alpha}$ is the complex number whose modulus is $|z|$ and argument $\theta + \alpha$.

(b) Multiplication by $e^{-i\alpha}$ to z rotates the vector **OP** in clockwise sense through an angle α.

(xiv) If z_1, z_2 and z_3 are the affixes of the points A, B and C in the argand plane, then

(a) $\angle BAC = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

(b) $\frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right|(\cos\alpha + i\sin\alpha)$, where $\alpha = \angle BAC$.

(xv) If z_1, z_2, z_3 and z_4 are the affixes of the points A, B, C and D, respectively in the argand plane.

(a) AB is inclined to CD at the angle $\arg\left(\frac{z_2 - z_1}{z_4 - z_3}\right)$.

(b) If CD is inclined at 90° to AB, then $\arg\left(\frac{z_2 - z_1}{z_4 - z_3}\right) = \pm\frac{\pi}{2}$.

Circle in Complex Plane

(i) An equation of the circle with centre at z_0 and radius r is
$|z - z_0| = r$ or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + z_0\bar{z}_0 - r^2 = 0$

(a) $|z - z_0| < r$, represents interior of the circle.

(b) $|z - z_0| > r$, represents exterior of the circle.

(c) $|z - z_0| \le r$ is the set of points lying inside and on the circle $|z - z_0| = r$. Similarly, $|z - z_0| \ge r$ is the set of points lying outside and on the circle $|z - z_0| = r$.

(d) **General equation of a circle is**

$$z\bar{z} + a\bar{z} + \bar{a}z + b = 0$$

where, a is a complex number and b is a real number. Centre of the circle $= -a$

Radius of the circle $= \sqrt{a\bar{a} - b}$ or $\sqrt{|a|^2 - b}$

(e) Four points z_1, z_2, z_3 and z_4 are concyclic, if $\dfrac{(z_4 - z_1)(z_2 - z_3)}{(z_4 - z_3)(z_2 - z_1)}$ is purely real.

(ii) $\dfrac{|z - z_1|}{|z - z_2|} = k \Rightarrow \begin{cases} \text{Circle, if } k \neq 1 \\ \text{Perpendicular bisector, if } k = 1 \end{cases}$

(iii) The equation of a circle described on the line segment joining z_1 and z_2 as diameter is $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$.

(iv) $\arg \dfrac{z - z_1}{z - z_2} = \beta$, then locus is the arc of a circle for which the segment joining z_1 and z_2 is a chord.

(v) If z_1 and z_2 are the fixed complex numbers, then the locus of a point z satisfying $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \pm\pi/2$ is a circle having z_1 and z_2 at the end points of a diameter.

(vi) If $\arg\left(\dfrac{z + 1}{z - 1}\right) = \dfrac{\pi}{2}$, then z lies on circle of radius unity and centre as origin.

(vii) If $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$, then locus of z is a circle with z_1 and z_2 as the extremities of diameter.

Conic in Complex Plane

Let z_1 and z_2 be two fixed points, and k be a positive real number.

(i) If $k > |z_1 - z_2|$, then $|z - z_1| + |z - z_2| = k$ represents an ellipse with foci at $A(z_1)$ and $B(z_2)$ and length of the major axis is k.

(ii) If $k < |z_1 - z_2|$, then $||z - z_1| - |z - z_2|| = k$ represents hyperbola with foci at $A(z_1)$ and $B(z_2)$.

Important Points to be Remembered

(i) $\sqrt{-a} \times \sqrt{-b} \neq \sqrt{ab}$

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is possible only, iff atleast one of the quantity either a or b is/are non-negative. e.g. $i^2 = \sqrt{-1} \times \sqrt{-1} \neq \sqrt{1}$

(ii) i is neither positive, zero nor negative.

(iii) Argument of 0 is not defined.

(iv) Argument of purely imaginary number is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

(v) Argument of purely real number is 0 or π.

(vi) If $\left|z+\frac{1}{z}\right|=a$, then greatest value of $|z|=\frac{a+\sqrt{a^2+4}}{2}$ and least value of $|z|=\frac{-a+\sqrt{a^2+4}}{2}$

(vii) The value of $i^i = e^{-\pi/2}$

(viii) The non-real complex numbers do not possess the property of order,

i.e. $x+iy <$ (or) $> c+id$ is not defined.

(ix) The area of the triangle on the argand plane formed by the complex numbers z, iz and $z+iz$ is $\frac{1}{2}|z|^2$.

(x) If ω_1 and ω_2 are the complex slope of two lines on the argand plane, then the lines are

(a) perpendicular, if $\omega_1+\omega_2=0$.

(b) parallel, if $\omega_1=\omega_2$.