Determinants

Determinant

Every square matrix A is associated with a number, called its determinant and it is denoted by Δ (read as delta) or det (A) or $|A|$.

Only square matrices have determinants. The matrices which are not square do not have determinants.

(i) **First Order Determinant**

If $A = [a]$, then det $(A) = |A| = a$.

(ii) **Second Order Determinant**

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$|A| = a_{11}a_{22} - a_{21}a_{12}$

(iii) **Third Order Determinant**

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\text{or } |A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

e.g. The expansion of the determinant $A = \begin{vmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \\ 2 & -5 & 4 \end{vmatrix}$ is

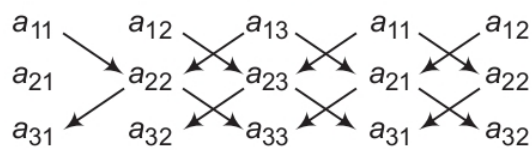
$$A = 1\begin{vmatrix} -2 & 1 \\ -5 & 4 \end{vmatrix} - 3\begin{vmatrix} 4 & 1 \\ 2 & 4 \end{vmatrix} - 2\begin{vmatrix} 4 & -2 \\ 2 & -5 \end{vmatrix}$$

$$= 1(-8+5) - 3(16-2) - 2(-20+4)$$

$$= -3 - 42 + 32 = -13$$

Evaluation of Determinant of Square Matrix of Order 3 by Sarrus Rule

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then determinant can be formed by enlarging the matrix by adjoining the first two columns on the right and draw lines as show below parallel and perpendicular to the diagonal.



The value of the determinant, this will be the sum of the product of element in line parallel to the diagonal minus the sum of the product of elements in line perpendicular to the line segment. Thus,

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

Note This method doesn't work for determinants of order greater than 3.

Properties of Determinants

(i) The value of the determinant remains unchanged, if rows are changed into columns and columns are changed into rows.
e.g. $$|A'| = |A|$$

(ii) If $A = [a_{ij}]_{n \times n}$, $n > 1$ and B be the matrix obtained from A by interchanging two of its rows or columns, then
$$\det(B) = -\det(A)$$

(iii) If two rows (or columns) of a square matrix A are proportional, then $|A| = 0$.

(iv) $|B| = k|A|$, where B is the matrix obtained from A, by multiplying one row (or column) of A by k.

(v) $|kA| = k^n|A|$, where A is a matrix of order $n \times n$.

(vi) If each element of a row (or column) of a determinant is the sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

e.g. $$\begin{vmatrix} a_1 + a_2 & b & c \\ p_1 + p_2 & q & r \\ u_1 + u_2 & v & w \end{vmatrix} = \begin{vmatrix} a_1 & b & c \\ p_1 & q & r \\ u_1 & v & w \end{vmatrix} + \begin{vmatrix} a_2 & b & c \\ p_2 & q & r \\ u_2 & v & w \end{vmatrix}$$

(vii) If the same multiple of the elements of any row (or column) of a determinant are added to the corresponding elements of any other row (or column), then the value of the new determinant remains unchanged,

e.g.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+ka_{31} & a_{12}+ka_{32} & a_{13}+ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

(viii) If each element of a row (or column) of a determinant is zero, then its value is zero.

(ix) If any two rows (or columns) of a determinant are identical, then its value is zero.

(x) If each element of row (or column) of a determinant is expressed as a sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

(xi) If r rows (or r columns) become identical, when a is substituted for x, then $(x-a)^{r-1}$ is a factor of given determinant.

Important Results on Determinants

(i) $|AB| = |A||B|$, where A and B are square matrices of the same order.

(ii) $|A^n| = |A|^n$

(iii) If A, B and C are square matrices of the same order such that ith columns (or rows) of A is the sum of i th columns (or rows) of B and C and all other columns (or rows) of A, B and C are identical, then $|A| = |B| + |C|$

(iv) $|I_n| = 1$, where I_n is identity matrix of order n.

(v) $|O_n| = 0$, where O_n is a zero matrix of order n.

(vi) If $\Delta(x)$ be a 3rd order determinant having polynomials as its elements.

(a) If $\Delta(a)$ has 2 rows (or columns) proportional, then $(x-a)$ is a factor of $\Delta(x)$.

(b) If $\Delta(a)$ has 3 rows (or columns) proportional, then $(x-a)^2$ is a factor of $\Delta(x)$.

(vii) A square matrix A is **non-singular**, if $|A| \neq 0$ and **singular**, if $|A| = 0$.

(viii) Determinant of a skew-symmetric matrix of odd order is zero and of even order is a non-zero perfect square.

(ix) In general, $|B+C| \neq |B| + |C|$

(x) Determinant of a diagonal matrix

= Product of its diagonal elements

(xi) Determinant of a triangular matrix

= Product of its diagonal elements

(xii) A square matrix of order n is non-singular, if its rank $r = n$ i.e. if $|A| \neq 0$, then rank $(A) = n$

(xiii) If $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ a & b & c \end{vmatrix}$, then

(a) $\sum\limits_{x=1}^{n} \Delta(x) = \begin{vmatrix} \sum\limits_{x=1}^{n} f_1(x) & \sum\limits_{x=1}^{n} f_2(x) & \sum\limits_{x=1}^{n} f_3(x) \\ \sum\limits_{x=1}^{n} g_1(x) & \sum\limits_{x=1}^{n} g_2(x) & \sum\limits_{x=1}^{n} g_3(x) \\ a & b & c \end{vmatrix}$

(b) $\prod\limits_{x=1}^{n} \Delta(x) = \begin{vmatrix} \prod\limits_{x=1}^{n} f_1(x) & \prod\limits_{x=1}^{n} f_2(x) & \prod\limits_{x=1}^{n} f_3(x) \\ \prod\limits_{x=1}^{n} g_1(x) & \prod\limits_{x=1}^{n} g_2(x) & \prod\limits_{x=1}^{n} g_3(x) \\ a & b & c \end{vmatrix}$

(xiv) If A is a non-singular matrix, then $|A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$.

(xv) Determinant of a orthogonal matrix = 1 or – 1.

(xvi) Determinant of a hermitian matrix is purely real.

(xvii) If A and B are non-zero matrices and $AB = O$, then it implies $|A| = O$ and $|B| = O$.

Minors and Cofactors

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then the **minor** M_{ij} of the element a_{ij} is the determinant obtained by deleting the ith row and jth column,

i.e. $$M_{11} = \text{minor of } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = \text{minor of } a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

and M_{13} = minor of $a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

The **cofactor** of the element a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$

Properties of Minors and Cofactors

(i) The sum of the products of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of any other row (or column) is zero,

i.e. if $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$

and so on.

(ii) The sum of the product of elements of any row (or column) of a determinant with the cofactors of the corresponding elements of the same row (or column) is Δ,

i.e. if $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then $|A| = \Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

(iii) In general, if $|A| = \Delta$, then $|A| = \sum_{i=1}^{n} a_{ij} C_{ij}$

and (adj A) $= \Delta^{n-1}$, where A is a matrix of order $n \times n$.

Applications of Determinants in Geometry

Let the three points in a plane be $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$, then

(i) Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

(ii) If the given points are collinear, then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$

(iii) Let two points are $A(x_1, y_1)$, $B(x_2, y_2)$ and $P(x, y)$ be a point on the line joining points A and B, then the equation of line is given by

$$\frac{1}{2}\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Adjoint of a Matrix

Adjoint of a matrix is the transpose of the matrix of cofactors of the given matrix,

i.e. $$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Properties of Adjoint of a Matrix

If A and B are two non-singular matrices of same order n, then

(i) $A(\text{adj } A) = (\text{adj } A)\, A = |A|\, I$

(ii) $\text{adj}(A') = (\text{adj } A)'$

(iii) $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$

(iv) $\text{adj}(kA) = k^{n-1}(\text{adj } A), k \in R$

(v) $\text{adj}(A^m) = (\text{adj } A)^m$

(vi) $\text{adj}(\text{adj } A) = |A|^{n-2} A$, where A is a non-singular matrix.

(vii) $|\text{adj } A| = |A|^{n-1}$, where A is a non-singular matrix.

(viii) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$, where A is a non-singular matrix.

(ix) $\text{adj}(I_n) = I_n$, $\text{adj}(O) = O$

Note

(i) Adjoint of a diagonal matrix is a diagonal matrix.

(ii) Adjoint of a triangular matrix is a triangular matrix.

(iii) Adjoint of a symmetric matrix is a symmetric matrix.

Inverse of a Matrix

Let A be a non-zero square matrix of order n, then a square matrix B, such that $AB = BA = I$, is called inverse of A, denoted by A^{-1}.

i.e. $$A^{-1} = \frac{1}{|A|}(\text{adj } A) \text{ given in properties}$$

Properties of Inverse of a Matrix

Let A and B be two square matrices of same order n. Then,

(i) $(A^{-1})^{-1} = A$

(ii) $(AB)^{-1} = B^{-1}A^{-1}$

In general, $(A_1A_2A_3 \ldots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1} \ldots A_3^{-1}A_2^{-1}A_1^{-1}$

(iii) $(A')^{-1} = (A^{-1})'$

(iv) $|A^{-1}| = |A|^{-1}$

(v) $AA^{-1} = A^{-1}A = I$

(vi) $(A^k)^{-1} = (A^{-1})^k, k \in N$

(vii) If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ and $abc \neq 0$, then $A^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$.

(viii) If A, B and C are square matrices of the same order and A is a non-singular matrix, then

(a) $AB = AC \Rightarrow B = C$ [left cancellation law]

(b) $BA = CA \Rightarrow B = C$ [right cancellation law]

Note

- Square matrix A is invertible iff it is non-singular.
- If a non-singular square matrix A is symmetric, then A^{-1} is also symmetric.
- A square matrix is invertible iff it is non-singular and every invertible matrix possesses a unique inverse.

Differentiation of Determinant

If $\Delta(x) = \begin{vmatrix} a(x) & b(x) & c(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, then

$$\frac{d\Delta}{dx} = \begin{vmatrix} a'(x) & b'(x) & c'(x) \\ p(x) & q(x) & r(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} a(x) & b(x) & c(x) \\ p'(x) & q'(x) & r'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} a(x) & b(x) & c(x) \\ p(x) & q(x) & r(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Integration of Determinant

If $\Delta(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

$$\int \Delta(x)\,dx = \begin{vmatrix} \int a_{11}(x)\,dx & \int a_{12}(x)\,dx & \int a_{13}(x)\,dx \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

If the elements of more than one column or rows are functions of x, then the integration can be done only after evaluation/expansion of the determinant.

Homogeneous and Non-homogeneous System of Linear Equations

A system of equations $AX = B$, is called a **homogeneous system**, if $B = O$ and if $B \neq O$, then it is called a **non-homogeneous system** of equations.

Solution of System of Linear Equations

The values of the variables satisfying all the linear equations in the system, is called solution of system of linear equations.

1. Solution of System of Equations by Matrix Method

(i) **Non-homogeneous System of Equations** Let $AX = B$ be a system of n linear equations in n variables.

(a) If $|A| \neq 0$, then the system of equations is consistent and has a unique solution given by $X = A^{-1}B$.

(b) If $|A| = 0$ and $(\text{adj } A)B = O$, then the system of equations is consistent and has infinitely many solutions.

(c) If $|A| = 0$ and $(\text{adj } A)\, B \neq O$, then the system of equations is inconsistent i.e. having no solution.

(ii) **Homogeneous System of Equations** Let $AX = O$ is a system of n linear equations in n variables.

(a) If $|A| \neq 0$, then it has only one solution $X = O$, is called the **trivial solution.**

(b) If $|A| = 0$, then the system has infinitely many solutions and it is called **non-trivial solution.**

2. Solution of System of Equations by Rank Method

(i) **Non-homogeneous System of Equations** Let $AX = B$ be a system of n linear equations in n variables, then

Step I Write the augmented matrix $[A : B]$.

Step II Reduce the augmented matrix to Echelon form using elementary row-transformation.

Step III Determine the rank of coefficient matrix A and augmented matrix $[A : B]$ by counting the number of non-zero rows in A and $[A : B]$.

Step IV

(i) If $\rho(A) \neq \rho(A\,B) \rightarrow \rho(A : B)$ then the system of equations is inconsistent.

(ii) If $\rho(A) = \rho(A\,B) \rightarrow \rho(A : B) =$ the number of unknowns, then the system of equations is consistent and has a unique solution.

(iii) If $\rho(A) = \rho(A\,B) \rightarrow \rho(A : B) <$ the number of unknowns, then the system of equations is consistent and has infinitely many solutions.

(ii) **Homogeneous System of Equations**

(a) If $AX = 0$, be a homogeneous system of linear equations and $\rho(A) =$ number of unknown, then $AX = 0$, have a non-trivial solution i.e. $X = 0$.

(b) If $\rho(A) <$ number of unknowns, then $AX = 0$, have a non-trivial solution, with infinitely many solutions.

Solution of Linear Equations by Determinant/Cramer's Rule

***Case* I** The solution of the system of simultaneous linear equations

$$a_1x + b_1y = c_1 \quad \text{...(i)}$$

$$a_2x + b_2y = c_2 \quad \text{...(ii)}$$

is given by $x = \dfrac{D_1}{D}, y = \dfrac{D_2}{D}$

where, $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ provided that $D \neq 0$.

(i) If $D \neq 0$, then the given system of equations is **consistent** and **has a unique solution** given by $x = \dfrac{D_1}{D}, y = \dfrac{D_2}{D}$.

(ii) If $D = 0$ and $D_1 = D_2 = 0$, then the system is **consistent** and has **infinitely many solutions.**

(iii) If $D = 0$ and one of D_1 and D_2 is non-zero, then the system is **inconsistent.**

***Case* II** Let the system of equations be $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$. Then, the solution of the system of equation is $x = \frac{D_1}{D}, y = \frac{D_2}{D}, z = \frac{D_3}{D}$, it is called **Cramer's rule**.

where, $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$

and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$.

(i) If $D \neq 0$, then the system of equations is consistent with unique solution.

(ii) If $D = 0$ and atleast one of the determinant D_1, D_2, D_3 is non-zero, then the given system is inconsistent, i.e. having no solution.

(iii) If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the system is consistent, with infinitely many solutions.

(iv) If $D \neq 0$ and $D_1 = D_2 = D_3 = 0$, then system has only trivial solution, $(x = y = z = 0)$.

Explanation/Value of Some Particular Types of Determinants

(i) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)$

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)[(a^2+b^2+c^2)+(ab+bc+ca)]$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(v) $\begin{vmatrix} x^2 & (x+a)^2 & (x-a)^2 \\ y^2 & (y+a)^2 & (y-a)^2 \\ z^2 & (z+a)^2 & (z-a)^2 \end{vmatrix} = -4a^3(x-y)(y-z)(z-x)$

(vi) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b & a & c \end{vmatrix} = a^2+b^2+c^2-ab-bc-ca$

$= \frac{1}{2}[(b-c)^2+(c-a)^2+(a-b)^2]$

(vii) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$= -(a^3+b^3+c^3-3abc)$

(viii) $\begin{vmatrix} x+a & b & c & d \\ a & x+b & c & d \\ a & b & x+c & d \\ a & b & c & x+d \end{vmatrix} = x^3(x+a+b+c+d)$

Maximum and Minimum Values of Determinants

If $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$, where a_i' s $\in \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

Then, $|A|_{\max}$ when diagonal elements are $\{\min(\alpha_1, \alpha_2, \ldots, \alpha_n)\}$ and non-diagonal elements are $\{\max(\alpha_1, \alpha_2, \ldots, \alpha_n)\}$.

Also, $|A|_{\min} = -|A|_{\max}$