Matrices

Matrix

A matrix is a rectangular arrangement of numbers (real or complex) which may be represented as

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}.$$

Matrix is enclosed by [ ] or ( ).

Compact form the above matrix is represented by $[a_{ij}]_{m\times n}$ or $A=[a_{ij}]$.

Element of a Matrix

The numbers $a_{11}, a_{12}, \dots$ etc., in the above matrix are known as the element of the matrix, generally represented as a_{ij}, which denotes element in ith row and jth column.

Order of a Matrix

In above matrix has m rows and n columns, then A is of order $m \times n$.

Types of Matrices

(i) **Row Matrix** A matrix having only one row and any number of columns is called a row matrix.

(ii) **Column Matrix** A matrix having only one column and any number of rows is called column matrix.

(iii) **Null/Zero Matrix** A matrix of any order, having all its elements are zero, is called a null/zero matrix, i.e. $a_{ij}=0, \forall\ i, j$.

(iv) **Square Matrix** A matrix of order $m \times n$, such that $m = n$, is called square matrix.

(v) **Diagonal Matrix** A square matrix $A=[a_{ij}]_{m\times n}$ is called a diagonal matrix, if all the elements except those in the leading diagonals are zero, i.e. $a_{ij}=0$ for $i \neq j$. It can be represented as $A=\text{diag}\ [a_{11}\ a_{22} \dots a_{nn}]$.

(vi) **Scalar Matrix** A square matrix in which every non-diagonal element is zero and all diagonal elements are equal, is called scalar matrix, i.e. in scalar matrix, $a_{ij} = 0$, for $i \neq j$ and $a_{ij} = k$, for $i = j$.

(vii) **Unit/Identity Matrix** A square matrix, in which every non-diagonal element is zero and every diagonal element is 1, is called unit matrix or an identity matrix,

i.e. $$a_{ij} = \begin{cases} 0, \text{ when } i \neq j \\ 1, \text{ when } i = j \end{cases}$$

(viii) **Rectangular Matrix** A matrix of order $m \times n$, such that $m \neq n$, is called rectangular matrix.

(ix) **Horizontal Matrix** A matrix in which the number of rows is less than the number of columns, is called horizontal matrix.

(x) **Vertical Matrix** A matrix in which the number of rows is greater than the number of columns, is called vertical matrix.

(xi) **Upper Triangular Matrix** A square matrix $A = [a_{ij}]_{n \times n}$ is called a upper triangular matrix, if $a_{ij} = 0, \forall\ i > j$.

(xii) **Lower Triangular Matrix** A square matrix $A = [a_{ij}]_{n \times n}$ is called a lower triangular matrix, if $a_{ij} = 0, \forall\ i < j$.

(xiii) **Submatrix** A matrix which is obtained from a given matrix by deleting any number of rows or columns or both is called a submatrix of the given matrix.

(xiv) **Equal Matrices** Two matrices A and B are said to be equal, if both having same order and corresponding elements of the matrices are equal.

(xv) **Principal Diagonal of a Matrix** In a square matrix, the diagonal from the first element of the first row to the last element of the last row is called the principal diagonal of a matrix.

e.g. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 6 & 5 \\ 1 & 1 & 2 \end{bmatrix}$, then principal diagonal of A is 1, 6, 2.

(xvi) **Singular Matrix** A square matrix A is said to be singular matrix, if determinant of A denoted by det (A) or $|A|$ is zero, i.e. $|A| = 0$, otherwise it is a non-singular matrix.

Algebra of Matrices

1. Addition of Matrices

Let A and B be two matrices each of order $m \times n$. Then, the sum of matrices $A + B$ is defined only if matrices A and B are of same order.

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$. Then, $A + B = [a_{ij} + b_{ij}]_{m \times n}$.

Properties of Addition of Matrices

If A, B and C are three matrices of order $m \times n$, then

(i) **Commutative Law** $A + B = B + A$

(ii) **Associative Law** $(A + B) + C = A + (B + C)$

(iii) **Existence of Additive Identity** A zero matrix (0) of order $m \times n$ (same as of A), is additive identity, if

$$A + 0 = A = 0 + A$$

(iv) **Existence of Additive Inverse** If A is a square matrix, then the matrix $(-A)$ is called additive inverse, if

$$A + (-A) = 0 = (-A) + A$$

(v) **Cancellation Law** $A + B = A + C \Rightarrow B = C$ [left cancellation law]

$B + A = C + A \Rightarrow B = C$ [right cancellation law]

2. Subtraction of Matrices

Let A and B be two matrices of the same order, then subtraction of matrices, $A - B$, is defined as

$$A - B = [a_{ij} - b_{ij}]_{m \times n},$$

where $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$

3. Multiplication of a Matrix by a Scalar

Let $A = [a_{ij}]_{m \times n}$ be a matrix and k be any scalar. Then, the matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k and is denoted by kA, given as

$$kA = [ka_{ij}]_{m \times n}$$

Properties of Scalar Multiplication

If A and B are two matrices of order $m \times n$, then

(i) $k(A + B) = kA + kB$

(ii) $(k_1 + k_2)A = k_1A + k_2A$

(iii) $k_1k_2A = k_1(k_2A) = k_2(k_1A)$

(iv) $(-k)A = -(kA) = k(-A)$

4. **Multiplication of Matrices**

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ are two matrices such that the number of columns of A is equal to the number of rows of B, then multiplication of A and B is denoted by AB, is given by $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$,

where c_{ij} is the element of matrix C and $C = AB$.

e.g. If $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, then

$$AB = \begin{bmatrix} a_1 b_1 + a_2 b_3 & a_1 b_2 + a_2 b_4 \\ a_3 b_1 + a_4 b_3 & a_3 b_2 + a_4 b_4 \end{bmatrix}.$$

Properties of Multiplication of Matrices

(i) **Associative Law** $(AB)C = A(BC)$

(ii) **Existence of Multiplicative Identity** $A \cdot I = A = I \cdot A$, where, I is called multiplicative Identity.

(iii) **Distributive Law** $A(B + C) = AB + AC$

(iv) **Cancellation Law** If A is non-singular matrix, then

$AB = AC \Rightarrow B = C$ [left cancellation law]

$BA = CA \Rightarrow B = C$ [right cancellation law]

(v) **Zero Matrix as the Product of Two Non-zero Matrices** $AB = O$, does not necessarily imply that $A = O$ or $B = O$ or both A and $B = O$.

Note Multiplication of diagonal matrices of same order will be commutative.

Important Points to be Remembered

(i) If A and B are square matrices of the same order, say n, then both the product AB and BA are defined and each is a square matrix of order n.

(ii) In the matrix product AB, the matrix A is called premultiplier (prefactor) and B is called postmultiplier (postfactor).

(iii) The rule of multiplication of matrices is row columnwise (or $\rightarrow \downarrow$ wise) the first row of AB is obtained by multiplying the first row of A with first, second, third,... columns of B respectively; similarly second row of A with first, second, third, ... columns of B, respectively and so on.

Positive Integral Powers of a Square Matrix

Let A be a square matrix. Then, we can define

(i) $A^{n+1} = A^n \cdot A$, where $n \in N$.

(ii) $A^m \cdot A^n = A^{m+n}$.

(iii) $(A^m)^n = A^{mn}, \forall m, n \in N$

Matrix Polynomial

Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n$. Then,
$f(A) = a_0 A^n + a_1 A^{n-2} + \ldots + a_n I_n$ is called the matrix polynomial.

Transpose of a Matrix

Let $A = [a_{ij}]_{m \times n}$, be a matrix of order $m \times n$. Then, the $n \times m$ matrix obtained by interchanging the rows and columns of A is called the transpose of A and is denoted by A' or A^T.

$$A' = A^T = [a_{ji}]_{n \times m}$$

Properties of Transpose

For any two matrices A and B of suitable orders,

(i) $(A')' = A$

(ii) $(A \pm B)' = A' \pm B'$

(iii) $(kA)' = kA'$

(iv) $(AB)' = B' A'$

(v) $(A^n)' = (A')^n$

(vi) $(ABC)' = C' B' A'$

Symmetric and Skew-Symmetric Matrices

(i) A square matrix $A = [a_{ij}]_{n \times n}$ is said to be **symmetric**, if $A' = A$. i.e. $a_{ij} = a_{ji}$, $\forall i$ and j.

(ii) A square matrix A is said to be **skew-symmetric**, if $A' = -A$, i.e. $a_{ij} = -a_{ji}, \forall i$ and j.

Properties of Symmetric and Skew-symmetric Matrices

(i) Elements of principal diagonals of a skew-symmetric matrix are all zero. i.e. $a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0$ or $a_{ii} = 0$, for all values of i.

(ii) If A is a square matrix, then

(a) $A + A'$ is symmetric. (b) $A - A'$ is skew-symmetric matrix.

(iii) If A and B are two symmetric (or skew-symmetric) matrices of same order, then $A + B$ is also symmetric (or skew-symmetric).

(iv) If A is symmetric (or skew-symmetric), then kA (k is a scalar) is also symmetric (or skew-symmetric) matrix.

(v) If A and B are symmetric matrices of the same order, then the product AB is symmetric, iff $BA = AB$.

(vi) Every square matrix can be expressed uniquely as the sum of a symmetric and a skew-symmetric matrix.

i.e. Matrix A can be written as $\frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

(vii) The matrix $B' AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric matrix.

(viii) All positive integral powers of a symmetric matrix are symmetric.

(ix) All positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric are symmetric matrix.

(x) If A and B are symmetric matrices of the same order, then

(a) $AB - BA$ is a skew-symmetric and

(b) $AB + BA$ is symmetric.

(xi) For a square matrix A, AA' and $A' A$ are symmetric matrix.

Elementary Operations (Transformations of a Matrix)

Any one of the following operations on a matrix is called an elementary transformation.

(i) Interchanging any two rows (or columns), denoted by

$$R_i \longleftrightarrow R_j \text{ or } C_i \longleftrightarrow C_j.$$

(ii) Multiplication of the element of any row (or column) by a non-zero scalar quantity and denoted by

$$R_i \to kR_i \text{ or } C_i \to kC_j.$$

(iii) Addition of constant multiple of the elements of any row to the corresponding element of any other row, denoted by

$$R_i \to R_i + kR_j \text{ or } C_i \to C_i + kC_j.$$

Elementary Matrix

A matrix obtained from an identity matrix by a single elementary operation is called an elementary matrix.

Equivalent Matrix

Two matrices A and B are said to be equivalent, if one can be obtained from the other by a sequence of elementary transformation.

The symbol $\approx$ is used for equivalence.

Trace of a Matrix

The sum of the diagonal elements of a square matrix A is called the trace of A, denoted by trace (A) or tr (A).

Properties of Trace of a Matrix

(i) $\text{Trace}(A \pm B) = \text{Trace}(A) \pm \text{Trace}(B)$

(ii) $\text{Trace}(kA) = k\,\text{Trace}(A)$

(iii) $\text{Trace}(A') = \text{Trace}(A)$

(iv) $\text{Trace}(I_n) = n$

(v) $\text{Trace}(O) = 0$

(vi) $\text{Trace}(AB) \neq \text{Trace}(A) \times \text{Trace}(B)$

(vii) $\text{Trace}(AA') \geq 0$

Conjugate of a Matrix

The matrix obtained from a matrix A containing complex number as its elements, on replacing its elements by the corresponding conjugate complex number is called conjugate of A and is denoted by $\overline{A}$.

Properties of Conjugate of a Matrix

Let A and B are two matrices of order $m \times n$ and k be a scalar, then

(i) $\overline{(\overline{A})} = A$

(ii) $\overline{(A + B)} = \overline{A} + \overline{B}$

(iii) $\overline{(AB)} = \overline{A}\,\overline{B}$

(iv) $\overline{(kA)} = k\overline{A}$

(v) $\overline{(A^n)} = (\overline{A})^n$

Transpose Conjugate of a Matrix

The transpose of the conjugate of a matrix A is called transpose conjugate of A and is denoted by A^θ or A^*,

i.e. $\overline{(A')} = (\overline{A})' = A^\theta$ or A^*

Properties of Transpose Conjugate of a Matrix

(i) $(A^*)^* = A$

(ii) $(A + B)^* = A^* + B^*$

(iii) $(kA)^* = \overline{k}A^*$

(iv) $(AB)^* = B^* A^*$

(v) $(A^n)^* = (A^*)^n$

Some Special Types of Matrices

1. **Orthogonal Matrix**

A square matrix of order n is said to be orthogonal, if $AA' = I_n = A'A$

Properties of Orthogonal Matrix

(i) If A is orthogonal matrix, then A' is also orthogonal matrix.

(ii) For any two orthogonal matrices A and B, AB and BA is also an orthogonal matrix.

(iii) If A is an orthogonal matrix, then A^{-1} is also orthogonal matrix.

2. **Idempotent Matrix**

A square matrix A is said to be idempotent, if $A^2 = A$.

Properties of Idempotent Matrix

(i) If A and B are two idempotent matrices, then

(a) AB is idempotent, iff $AB = BA$.

(b) $A + B$ is an idempotent matrix, iff $AB = BA = O$

(c) $AB = A$ and $BA = B$, then $A^2 = A, B^2 = B$

(ii) (a) If A is an idempotent matrix and $A + B = I$, then B is an idempotent and $AB = BA = O$.

(b) Diagonal (1, 1, 1, ...,1) is an idempotent matrix.

3. **Involutory Matrix**

A square matrix A is said to be involutory, if $A^2 = I$

4. **Nilpotent Matrix**

A square matrix A is said to be nilpotent matrix, if there exists a positive integer m such that $A^m = 0$. If m is the least positive integer such that $A^m = 0$, then m is called the index of the nilpotent matrix A.

5. **Unitary Matrix**

A square matrix A is said to be unitary, if $\overline{A'}A = I$

6. **Periodic Matrix**

If $A^{k+1} = A$, where k is a positive integer, then A is known as periodic matrix and k is known as period of matrix A.

Rank of a Matrix

A positive integer r is said to be the rank of a non-zero matrix A, if

(i) there exists at least one minor in A of order r which is not zero.

(ii) every minor in A of order greater than r is zero, rank of a matrix A is denoted by $\rho(A) = r$.

Properties of Rank of a Matrix

(i) The rank of a null matrix is zero i.e. $\rho(O) = 0$

(ii) If I_n is an identity matrix of order n, then $\rho(I_n) = n$.

(iii) (a) If a matrix A does't possess any minor of order r, then $\rho(A) \geq r$.

(b) If atleast one minor of order r of the matrix is not equal to zero, then $\rho(A) \leq r$.

(iv) If every $(r+1)$th order minor of A is zero, then any higher order minor will also be zero.

(v) If A is of order n, then for a non-singular matrix A, $\rho(A) = n$

(vi) $\rho(A') = \rho(A)$

(vii) $\rho(A^*) = \rho(A)$

(viii) $\rho(A + B) \leq \rho(A) + \rho(B)$

(ix) If A and B are two matrices such that the product AB is defined, then rank (AB) cannot exceed the rank of the either matrix.

(x) If A and B are square matrix of same order and $\rho(A) = \rho(B) = n$, then $\rho(AB) = n$

(xi) Every skew-symmetric matrix of odd order has rank less than its order.

(xii) Elementary operations do not change the rank of a matrix.