Probability

Experiment

An operation which produce some well-defined results or outcomes is called an experiment.

Types of Experiments

1. **Deterministic Experiment**

Those experiments, which when repeated under identical conditions produce the same result or outcome are known as deterministic experiment.

2. **Probabilistic/Random Experiment**

Those experiments, which when repeated under identical conditions, do not produce the same outcome every time but the outcome produced is one of the several possible outcomes, are called random experiment.

Some Basic Definitions

(i) **Trial** Performing an experiment is called a trial. The number of times an experiment is repeated is called the number of trials.

(ii) **Sample Space** The set of all possible outcomes of a random experiment is called the sample space of the experiment and it is denoted by S.

(iii) **Sample Point** The outcome of an experiment is called the sample point, i.e. the elements of set S are called the sample points.

(iv) **Event** A subset of the sample space associated with a random experiment is called event or case.

(v) **Elementary** (or Simple) **Event** An event containing only one sample point is called elementary event (or indecomposable event).

(vi) **Compound Event** An event containing more than one sample points is called compound event (or decomposable event).

(vii) **Occurrence of an Event** An event associated to a random experiment is said to occur, if any one of the elementary events associated to it is an outcome.

(viii) **Certain Event** An event which must occur, whatever be the outcomes, is called a certain event (or sure event).

(ix) **Impossible Event** An event which cannot occur in a random experiment, is called an impossible event.

(x) **Favourable Outcomes** Let S be the sample space associated with a random experiment and $E \subset S$. Then, the elementary events belonging to E are known as the favourable outcomes to E.

(xi) **Equally likely Outcomes** The outcomes of a random experiment are said to be equally likely, when each outcome is as likely to occur as the other.

Algebra of Events

Let A and B are two events associated with a random experiment, whose sample space is S. Then,

(i) the event 'not A' is the set A' or $S - A$

(ii) the events A or B is the set $A \cup B$

(iii) the events A and B is the set $A \cap B$

(iv) the events A but not B is the set $A - B$ or $A \cap B'$

Note For more details, see operations on sets.

Probability—

Theoretical (Classical) Approach

If there are n equally likely outcomes associated with a random experiment and m of them are favourable to an event A, then the probability of happening or occurrence of A, denoted by $P(A)$, is given by

$$P(A) = \frac{m}{n} = \frac{\text{Number of favourable outcomes to } A}{\text{Total number of possible outcomes}}$$

Axiomatic Approach

Let $S = \{w_1, w_2, w_3, \dots w_n\}$ be a sample space, then according to axiomatic approach we have the following

(i) $0 \le P(w_i) \le 1$ for each $w_i \in S$

(ii) $P(w_1) + P(w_2) + \dots + P(w_n) = 1$

(iii) For any event A, $P(A) = \Sigma P(w_i)$, $w_i \in A$.

Note

- Theoretical approach is valid only when the outcomes are equally likely and number of total outcomes is known.
- P(sure event) $= P(S) = 1$ and P(impossible event) $= P(\phi) = 0$

Different Types of Events and Their Probabilities

(i) **Equally Likely Events** The given events are said to be equally likely, if none of them is expected to occur in preference to the other.

Thus, if the events E and F are equally likely, then $P(E) = P(F)$

(ii) **Mutually Exclusive Events** A set of events is said to be mutually exclusive, if the happening of one event excludes the happening of the other.

If A and B are mutually exclusive events, then $(A \cap B) = \phi$.

$\therefore$ The probability of mutually exclusive events is $P(A \cap B) = 0$.

(iii) **Probability of Exhaustive Events** A set of events is said to be exhaustive, if atleast one of them necessarily occurs whenever the experiment is performed.

If $E_1, E_2, \dots, E_n$ are exhaustive events, then

$$E_1 \cup E_2 \cup \dots \cup E_n = S.$$

and so $$P(E_1 \cup E_2 \cup E_3 \cup \dots \cup E_n) = 1.$$

Note If $E_i \cap E_j = \phi$ for $i \neq j$ and $\bigcup_{i=1}^{n} E_i = S$, then events $E_1, E_2, \dots, E_n$ are called **mutually exclusive and exhaustive events**.

(iv) **Independent Events** Two events A and B, associated to a random experiment, are independent if the probability of occurrence or non-occurrence of A is not affected by the occurrence or non-occurrence of B.

Note If A and B are independent events associated with a random experiment, then

(a) $P(A \cap B) = P(A)\,P(B)$

(b) $\overline{A}$ and B are independent events.

(c) A and $\overline{B}$ are independent events.

(d) $\overline{A}$ and $\overline{B}$ are independent events.

(v) **Complementary Event** Let A be an event of a sample space S, the complementary event to A is the event containing all sample points other than the sample point in A and it is denoted by A' or $\overline{A}$ i.e. A' or $\overline{A} = \{n : n \in S, n \notin A\}$

$\therefore$ The probability of complementary event to A is

$$P(\overline{A}) = 1 - P(A)$$

Note

(i) $P(A) + P(A') = 1$

(ii) $P(A \cup A') = P(S) = 1$

(iii) $P(A \cap A') = P(\phi) = 0$

(iv) $P(A')' = P(A)$

Partition of a Sample Space

The events $A_1, A_2, \ldots, A_n$ represent a partition of the sample space S, if they are pairwise disjoint, exhaustive and have non-zero probabilities. i.e.

(i) $A_i \cap A_j = \phi;\ i \neq j;\ i, j = 1, 2, \ldots, n$

(ii) $A_1 \cup A_2 \cup \ldots \cup A_n = S$

(iii) $P(A_i) > 0, \forall\ i = 1, 2, \ldots, n$

Important Results on Probability

(i) **Addition Theorem of Probability**

(a) For two events A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(b) For three events A, B and C

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

(c) For n events $A_1, A_2, \ldots, A_n$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i} \sum_{< j \leq n} P(A_i \cap A_j) + \sum_{1 \leq i} \sum_{< j < k} \sum_{\leq n} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

(ii) If A and B are two events associated with a random experiment, then

(a) $P(\overline{A} \cap B) = P(B) - P(A \cap B)$

(b) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

(c) $P[(A \cap \overline{B}) \cup (\overline{A} \cap B)] = P(A) + P(B) - 2P(A \cap B)$

(d) $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

(e) $P(\overline{A} \cup \overline{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B)$

(f) $P(A) = P(A \cap B) + P(A \cap \overline{B})$

(g) $P(B) = P(A \cap B) + P(B \cap \overline{A})$

(iii) (a) P (exactly one of A, B occurs)

$$= P(A) + P(B) - 2P(A \cap B) = P(A \cup B) - P(A \cap B)$$

(b) P(neither A nor B occurs) $= P(A' \cap B') = 1 - P(A \cup B)$

(iv) If $B \subseteq A$, then

(a) $P(A \cap \overline{B}) = P(A) - P(B)$

(b) $P(B) \leq P(A)$

(v) If A and B are two events, then

$$P(A \cap B) \leq P(A) (\text{or } P(B)) \leq P(A \cup B) \leq P(A) + P(B)$$

(vi) If A, B and C are three events, then

(a) P(exactly one of A, B, C occurs)

$$= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C) - 2P(A \cap C) + 3P(A \cap B \cap C)$$

(b) P (atleast two of A, B, C occurs)

$$= P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C)$$

(c) P (exactly two of A, B, C occurs)

$$= P(A \cap B) + P(B \cap C) + P(A \cap C) - 3P(A \cap B \cap C)$$

(vii) (a) $P(A \cup B) = P(A) + P(B)$, if A and B are mutually exclusive events.

(b) $P(A \cup B \cup C) = P(A) + P(B) + P(C)$, if A, B and C are mutually exclusive events.

(viii) If the events $A_1, A_2, \ldots, A_n$ are mutually exclusive, i.e. $A_i \cap A_j = \phi$ for $i \neq j$, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = P(A_1) + (A_2) + \ldots + P(A_n)$$

and $P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(\phi) = 0$

(ix) If $A_1, A_2, \ldots, A_n$ are independent events associated with a random experiment, then probability of occurrence of atleast one

$$= P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - P(\overline{A_1 \cup A_2 \cup \ldots \cup A_n})$$

$$= 1 - P(\overline{A}_1) P(\overline{A}_2) \ldots P(\overline{A}_n)$$

(x) If $A_1, A_2, \ldots, A_n$ are n events associated with a random experiment, then

(a) $P(A_1 \cap A_2 \cap \ldots \cap A_n) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$

(Bonferroni's Inequality)

Or

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) \geq 1 - P(\overline{A}_1) - P(\overline{A}_2) \ldots - P(\overline{A}_n)$$

(b) $P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$ (Booley's Inequality)

Odds in Favour and Against of an Event

(i) Odds in favour of an event E is given by $\dfrac{P(E)}{P(\overline{E})}$

(ii) Odds in against of an event E is given by $\dfrac{P(\overline{E})}{P(E)}$

Note If odds in favour of an event E are $a:b$, then $P(E)=\dfrac{a}{a+b}$ and $P(\overline{E})=\dfrac{b}{a+b}$.

Conditional Probability

Let A and B be two events associated with a random experiment. Then, the probability of occurrence of event A under the condition that B has already occurred and $P(B)\neq 0$, is called the conditional probability and it is given by

$$P(A/B)=\frac{P(A\cap B)}{P(B)}$$

If A has already occurred and $P(A)\neq 0$, then $P(B/A)=\dfrac{P(A\cap B)}{P(A)}$

Note If A and B are independent events, then $P(B/A)=P(B)$ and $P(A/B)=P(A)$.

Properties of Conditional Probability

(i) $P\left(\dfrac{A}{B}\right)+P\left(\dfrac{\overline{A}}{B}\right)=1$

(ii) $P((A\cup B)/F)=P\left(\dfrac{A}{F}\right)+P\left(\dfrac{B}{F}\right)-P\left(\dfrac{(A\cap B)}{F}\right)$, where F is an event of sample space S such that $P(F)\neq 0$.

Multiplication Theorem on Probability

(i) If A and B are two events associated with a random experiment, then

$$P(A\cap B)=P(A)P(B/A),\text{ if } P(A)\neq 0$$

or $$P(A\cap B)=P(B)P(A/B),\text{ if } P(B)\neq 0$$

(ii) If $A_1, A_2, \ldots, A_n$ are n events associated with a random experiment, then

$$P(A_1\cap A_2\cap\ldots\cap A_n)=P(A_1)\,P(A_2/A_1)\,P(A_3/(A_1\cap A_2))\ldots P(A_n/(A_1\cap A_2\cap A_3\cap\ldots\cap A_{n-1}))$$

Theorem of Total Probability

Let S be the sample space and let $E_1, E_2, \ldots, E_n$ be a partition of the sample space S. If A is any event which occurs with E_1 or E_2 or … or E_n, then

$$P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + \ldots + P(E_n)P(A/E_n)$$
$$= \sum_{r=1}^{n} P(E_r)P(A/E_r)$$

Baye's Theorem

Let S be the sample space and let $E_1, E_2, \ldots, E_n$ be a partition of the sample space S. If A is any event which occurs with E_1 or E_2 or … or E_n, then probability of occurrence of E_i, when A occurred, is

$$P(E_i / A) = \frac{P(E_i)P(A/E_i)}{\sum_{i=1}^{n} P(E_i)P(A/E_i)}, i = 1, 2, \ldots, n$$

where, $P(E_i), i = 1, 2, \ldots, n$ is known as the **priori probability**

and $P\left(\frac{E_i}{A}\right), i = 1, 2, \ldots, n$ is known as **posteriori probability**

Important Points to be Remembered

Coin

A coin has two sides, head and tail. If an experiment consists of more than one coin, then coins are considered as distinct, if not otherwise stated.

Die

A die has six face marked with 1, 2, 3, 4, 5 and 6. If an experiment consists of more than one die, then all dice are considered as distinct, if not otherwise stated.

Playing Cards

A pack of playing cards has 52 cards, which are divided into 4 suits (spade, heart, diamond and club) each having 13 cards.

The cards in each suit are ace, king, queen, jack, 10, 9, 8, 7, 6, 5, 4, 3 and 2.

King, queen and jack are called face cards, so there are in all 12 face cards. Also, there are 16 honour cards, 4 of each suit namely ace, king, queen and jack.

The suits, clubs and spades are of black colour while the suits hearts and diamonds are of red colour. So, there are 26 red cards and 26 black cards.

Random Variable

Let S be a sample space associated with a given random experiment. A real valued function X defined on S, i.e.

$X: S \to R$, is called a random variable.

There are two types of random variable

(i) **Discrete Random Variable** If the range of the function $X: S \to R$ is a finite set or countably infinite set of real numbers, then it is called a discrete random variable.

e.g. In tossing of two coins $S = \{HH, HT, TH, TT\}$, let X denotes number of heads in tossing of two coins, then

$$X(HH) = 2, X(TH) = 1, X(HT) = 1, X(TT) = 0$$

(ii) **Continuous Random Variable** If the range of X is an interval (a, b) of R, then X is called a continuous random variable.

Probability Distribution of a Random Variable

If a random variable X takes values $x_1, x_2, \dots, x_n$ with respective probabilities $p_1, p_2, \dots, p_n$, then the representation

X	x_1	x_2	x_3 ...	x_n
P(X)	p_1	p_2	p_3 ...	p_n

is known as the probability distribution of X.

or

Probability distribution gives the values of the random variable along with the corresponding probabilities.

Mathematical Expectation/Mean of a Random Variable

If X is a discrete random variable which assume values $x_1, x_2, \dots, x_n$ with respective probabilities $p_1, p_2, \dots, p_n$, then the mean μ of X is defined as

$$E(X) = \mu = p_1x_1 + p_2x_2 + \dots + p_nx_n = \sum_{i=1}^{n} p_ix_i$$

Variance of a Random Variable

Variance of a random variable is denoted by σ^2 and it is defined as

$$V(X) = \sigma^2 = E(X^2) - [E(X)]^2$$

where, $$E(X^2) = \sum_{i=1}^{n} x_i^2 p_i$$

Standard Deviation

$$\sigma = \sqrt{V(X)} = \sqrt{E(X^2) - (E(X))^2}$$

Some Important Results

(i) If $Y = a X + b$, then

(a) $E(Y) = E(aX + b) = aE(X) + b$

(b) $\sigma_y^2 = V(Y) = a^2 V(X) = a^2 \sigma_x^2$

(c) $\sigma_y = \sqrt{V(Y)} = |a| \sigma_x$

(ii) If $Y = aX^2 + bX + c$, then

$$E(Y) = E(aX^2 + bX + c)$$
$$= aE(X^2) + bE(X) + c$$

Bernoulli Trials and Binomial Distribution

Bernoulli Trials

Trials of a random experiment are called Bernoulli trials, if

(i) number of trials is finite

(ii) trials are independent

(iii) each trial has exactly two outcomes success and failure

(iv) probability of success remains same in each trial.

Binomial Distribution

The probability of r successes in n-Bernaulli trials is denoted by $P(X = r)$ and is given by

$$P(X = r) = {}^nC_r \, p^r q^{n-r}, \; r = 0, 1, 2, \ldots n.$$

where, p = probability of success

q = probability of failure and $p + q = 1$

This can be represented by the following :

X	0	1	2	...	n
P(X)	${}^nC_0 p^0 q^n$	${}^nC_1 p^1 q^{n-1}$	${}^nC_2 p^2 q^{n-2}$	...	${}^nC_n p^n$

The above probability distribution is known as binomial distribution with parameter n and p.

Note

- $P(x = x)$ or $P(x)$ is called the probability function of binomial distribution.
- A binomial distribution with parameter n and p is denoted by $B(n, p)$.

Important Results

(i) If $p = q$, then probability of r successes in n trials is ${}^nC_r\, p^n$.

(ii) Mean $= E(X) = \mu = np$

(iii) Variance $= \sigma_x^2 = npq$

(iv) Standard deviation $= \sigma_x = \sqrt{npq}$

(v) Mean is always greater than variance.

(vi) If the total number of trials is n in any attempt and if there are N such attempts, then the total number of r successes is

$$N({}^nC_r\, p^r q^{n-r})$$

Geometrical Probability

If the total number of possible outcomes of a random experiment is infinite, in such cases, the definition of probability is modified and the general expression for the probability P of occurrence of an event is given by

$$P = \frac{\text{Measure of region occupied by the event}}{\text{Measure of the whole region}}$$

where, measure means length or area or volume of the region, if we are dealing with one, two or three dimensional space respectively.

Important Results to be Remembered

(i) When two dice are thrown, the number of ways of getting a total r is

(a) $(r-1)$, if $2 \le r \le 7$ and (b) $(13-r)$, if $8 \le r \le 12$

(ii) Experiment of insertion of n letters in n addressed envelopes.

(a) Probability of inserting all the n letters in right envelopes $= \dfrac{1}{n!}$

(b) Probability that atleast one letter is not in right envelope $= 1 - \dfrac{1}{n!}$

(c) Probability of keeping all the letters in wrong envelopes

$$= \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}$$

(d) Probability that exactly r letters are in right envelopes

$$= \frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^{n-r}\frac{1}{(n-r)!}\right]$$

(iii) (a) **Selection of Shoes from a Cupboard** Out of n pair of shoes, if k shoes are selected at random, the probability that there is no pair is $P = \frac{{}^nC_k 2^k}{{}^{2n}C_k}$

(b) The probability that there is atleast one pair is $(1 - P)$.

(iv) **Selection of Squares from the Chessboard** If r $(1 \le r \le 7)$ squares are selected at random from a chessboard, then probability that they lie on a diagonal is

$$\frac{4[{}^7C_r + {}^6C_r + \ldots + {}^1C_r] + 2({}^8C_r)}{{}^{64}C_r}$$

(v) If A and B are two finite sets and if a mapping is selected at random from the set of all mapping from A to B, then the probability that the mapping is

(a) a one-one function $= \frac{{}^{n(B)}P_{n(A)}}{n(B)^{n(A)}}$, provided $n(B) \ge n(A)$

(b) a many-one function $= 1 - \frac{{}^{n(B)}P_{n(A)}}{n(B)^{n(A)}}$, provided $n(B) \ge n(A)$

(c) a constant function $= \frac{n(B)}{n(B)^{n(A)}}$

(d) a one-one onto function $= \frac{n(A)!}{n(B)^{n(A)}}$, provided $n(A) = n(B)$