DCAM classes
Dynamic Classes for Academic Mastery

Sequences and Series

Sequence

Sequence is a function whose domain is the set of natural numbers or some subset of the type $\{1, 2, 3, \ldots, k\}$. We represents the images of $1, 2, 3, \ldots, n, \ldots$ as $f_1, f_2, f_3, \ldots, f_n \ldots$, where $f_n = f(n)$.

In other words, a sequence is an arrangement of numbers in definite order according to some rule.

- A sequence containing a finite number of terms is called a **finite sequence**.
- A sequence containing an infinite number of terms is called an **infinite sequence**.
- A sequence whose range is a subset of real number R, is called a **real sequence**.

Progression

A sequence whose terms follow a certain pattern is called a progression.

Series

If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a sequence, then the sum expressed as $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is called a series.

- A series having finite number of terms is called **finite series**.
- A series having infinite number of terms is called **infinite series**.

Arithmetic Progression (AP)

A sequence in which terms increase or decrease regularly by a fixed number. This fixed number is called the common difference of AP.

e.g. $a, a + d, a + 2d, \ldots$ is an AP, where a = first term and d = common difference.

*n*th Term (or General Term) of an AP

If a is the first term, d is the common difference and l is the last term of an AP, i.e. the given AP is $a, a+d, a+2d, a+3d, \ldots, l$, then

(a) nth term is given by $a_n = a + (n-1)d$

(b) nth term of an AP from the last term is given by $a'_n = l - (n-1)d$

Note

(i) $a_n + a'_n = a + l$

i.e. nth term from the begining + nth term from the end
= first term + last term

(ii) Common difference of an AP

$$d = a_n - a_{n-1}, \forall n > 1$$

(iii) $a_n = \frac{1}{2}[a_{n-k} + a_{n+k}], k < n$

Properties of Arithmetic Progression

(i) If a constant is added or subtracted from each term of an AP, then the resulting sequence is also an AP with same common difference.

(ii) If each term of an AP is multiplied or divided by a non-zero constant k, then the resulting sequence is also an AP, with common difference kd or $\frac{d}{k}$ respectively, where d = common difference of given AP.

(iii) If a_n, a_{n+1} and a_{n+2} are three consecutive terms of an AP, then $2a_{n+1} = a_n + a_{n+2}$.

(iv) If the terms of an AP are chosen at regular intervals, then they form an AP.

(v) If a sequence is an AP, then its nth term is a linear expression in n, i.e. its nth term is given by $An + B$, where A and B are constants and A = common difference.

Selection of Terms in an AP

(i) Any three terms in AP can be taken as

$$(a-d), a, (a+d)$$

(ii) Any four terms in AP can be taken as

$$(a-3d), (a-d), (a+d), (a+3d)$$

(iii) Any five terms in AP can be taken as

$$(a-2d), (a-d), a, (a+d), (a+2d)$$

Sum of First *n* Terms of an AP

Sum of first n terms of AP, is given by

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + l], \text{ where } l = \text{last term}$$

Note

(i) A sequence is an AP iff the sum of its first n terms is of the form $An^2 + Bn$, where A and B are constants and common difference in such case will be $2A$.

(ii) $a_n = S_n - S_{n-1}$ i.e.
nth term of AP = Sum of first n terms – Sum of first $(n-1)$ terms

Arithmetic Mean (AM)

(i) If a, A and b are in AP, then A is called the arithmetic mean of a and b and it is given by $A = \frac{a+b}{2}$

(ii) If $a_1, a_2, a_3, \ldots, a_n$ are n numbers, then their AM is given by,

$$A = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

(iii) If $a, A_1, A_2, A_3, \ldots, A_n, b$ are in AP, then

(a) $A_1, A_2, A_3, \ldots, A_n$ are called n arithmetic mean between a and b, where

$$A_1 = a + d = \frac{na + b}{n+1}$$

$$A_2 = a + 2d = \frac{(n-1)a + 2b}{n+1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$A_n = a + nd = \frac{a + nb}{n+1} \text{ and } d = \frac{b-a}{n+1}$$

(b) Sum of n AM's between a and b is nA

i.e. $A_1 + A_2 + A_3 + \ldots + A_n = nA$, where $A = \frac{a+b}{2}$

Important Results on AP

(i) If $a_p = q$ and $a_q = p$, then $a_{p+q} = 0$, $a_r = p + q - r$

(ii) If $pa_p = qa_q$, then $a_{p+q} = 0$

(iii) If $a_p = \frac{1}{q}$ and $a_q = \frac{1}{p}$, then $a_{pq} = 1$

(iv) If $S_p = q$ and $S_q = p$, then $S_{p+q} = -(p+q)$

(v) If $S_p = S_q$, then $S_{p+q} = 0$

(vi) If a^2, b^2 and c^2 are in AP, then

$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ and $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ both are also in AP.

(vii) If $a_1, a_2, \dots, a_n$ are the non-zero terms of an AP, then

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \frac{1}{a_3 a_4} + \dots + \frac{1}{a_{n-1} a_n} = \frac{n-1}{a_1 a_n}$$

Geometric Progression GP

A sequence in which the ratio of any term (except first term) to its just preceding term is constant throughout. The constant ratio is called common ratio (r).

i.e. $$\frac{a_{n+1}}{a_n} = r, \forall\ n \geq 1$$

If a is the first term, r is the common ratio and l is the last term of a GP, then the GP can be written as $a, ar, ar^2, \dots, ar^{n-1}, \dots l$.

*n*th Term (or General Term) of a GP

If a is the first term, r is the common ratio and l is the last term, then

(i) nth term of a GP from the beginning is given by $a_n = ar^{n-1}$

(ii) nth term of a GP from the end is given by $a'_n = \frac{l}{r^{n-1}}$.

(iii) The nth term from the end of a finite GP consisting of m terms is ar^{m-n}.

(iv) $a_n a'_n = al$

i.e. nth term from the beginning $\times$ nth term from the end = first term $\times$ last term

Properties of Geometric Progression

(i) If all the terms of GP are multiplied or divided by same non-zero constant, then the resulting sequence is also a GP with the same common ratio.

(ii) The reciprocal of terms of a given GP also form a GP.

(iii) If each term of a GP is raised to same power, then the resulting sequence also forms a GP.

(iv) If the terms of a GP are chosen at regular intervals, then the resulting sequence is also a GP.

(v) If $a_1, a_2, a_3, \ldots, a_n$ are non-zero and non-negative term of a GP, then $\log a_1, \log a_2, \log a_3, \ldots, \log a_n$ are in an AP and *vice-versa*.

(vi) If a, b and c are three consecutive terms of a GP, then $b^2 = ac$.

Selection of Terms in a GP

(i) Any three terms in a GP can be taken as $\frac{a}{r}$, a and ar.

(ii) Any four terms in a GP can be taken as $\frac{a}{r^3}, \frac{a}{r}$, ar and ar^3.

(iii) Any five terms in a GP can be taken as $\frac{a}{r^2}, \frac{a}{r}$, a, ar and ar^2.

Sum of First *n* Terms of a GP

(i) Sum of first n terms of a GP is given by

$$S_n = \begin{cases} \dfrac{a(1-r^n)}{1-r}, & \text{if } r < 1 \\ \dfrac{a(r^n-1)}{r-1}, & \text{if } r > 1 \\ na, & \text{if } r = 1 \end{cases}$$

(ii) $S_n = \dfrac{a - lr}{1 - r}, r < 1$ or $S_n = \dfrac{lr - a}{r - 1}, r > 1$

where, l = last term of the GP

Sum of Infinite Terms of a GP

(i) If $|r| < 1$, then $S_\infty = \dfrac{a}{1-r}$

(ii) If $|r| \geq 1$, then S_∞ does not exist.

Geometric Mean GM

(i) If a, G, b are in GP, then G is called the geometric mean of a and b and is given by $G = \sqrt{ab}$.

(ii) GM of n positive numbers $a_1, a_2, a_3, \ldots, a_n$ are given by $G = (a_1 a_2 \ldots a_n)^{1/n}$

(iii) If $a, G_1, G_2, G_3, \dots, G_n, b$ are in GP, then

(a) $G_1, G_2, G_3, \dots, G_n$, are called n GM's between a and b, where

$$G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}},$$

$$G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}} \text{ and } r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

(b) Product of n GM's,

$$G_1 \times G_2 \times G_3 \times \dots \times G_n = G^n, \text{ where } G = \sqrt{ab}$$

Important Results on GP

(i) If $a_p = x$ and $a_q = y$, then $a_n = \left(\dfrac{x^{n-q}}{y^{n-p}}\right)^{\frac{1}{p-q}}$

(ii) If $a_{m+n} = p$ and $a_{m-n} = q$, then

$$a_m = \sqrt{pq} \text{ and } a_n = p\left(\frac{q}{p}\right)^{\frac{m}{2n}}$$

(iii) If a, b and c are the pth, qth and rth terms of a GP, then

$$a^{q-r} \times b^{r-p} \times c^{p-q} = 1$$

(iv) Sum of n terms of $b + bb + bbb + \dots$ is

$$a_n = \frac{b}{9}\left(\frac{10(10^n - 1)}{9} - n\right); b = 1, 2, \dots, 9$$

(v) Sum of n terms of $0{\cdot}b + 0{\cdot}bb + 0{\cdot}bbb + \dots$ is

$$a_n = \frac{b}{9}\left(n - \frac{(1 - 10^{-n})}{9}\right); b = 1, 2, \dots, 9$$

(vi) If $a_1, a_2, a_3, \dots, a_n$ and $b_1, b_2, b_3, \dots, b_n$ are in GP, then the sequence $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3 \dots$ will not be a GP.

(vii) If pth, qth and rth term of geometric progression are also in geometric progression, then p, q and r are in arithmetic progression.

(viii) If a, b and c are in AP as well as in GP, then $a = b = c$.

(ix) If a, b and c are in AP, then x^a, x^b and x^c are in geometric progression.

Harmonic Progression (HP)

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ of non-zero numbers is called a Harmonic Progression (HP), if the sequence $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}, \ldots$ is in AP.

*n*th Term (or General Term) of Harmonic Progression

(i) nth term of the HP from the beginning

$$a_n = \frac{1}{\frac{1}{a_1} + (n-1)\left(\frac{1}{a_2} - \frac{1}{a_1}\right)}$$

$$= \frac{a_1 a_2}{a_2 + (n-1)(a_1 - a_2)}$$

(ii) nth term of the HP from the end

$$a_n' = \frac{1}{\frac{1}{l} - (n-1)\left(\frac{1}{a_2} - \frac{1}{a_1}\right)} = \frac{a_1 a_2 l}{a_1 a_2 - l(n-1)(a_1 - a_2)},$$

where l is the last term.

(iii) $\frac{1}{a_n} + \frac{1}{a_n'} = \frac{1}{a} + \frac{1}{l} = \frac{1}{\text{First term of HP}} + \frac{1}{\text{Last term of HP}}$

(iv) $a_n = \frac{1}{a + (n-1)d}$, if a, d are the first term and common difference of the corresponding AP.

Note There is no formula for determining the sum of harmonic series.

Harmonic Mean

(i) If a, H and b are in HP, then H is called the harmonic mean of a and b and is given by $H = \frac{2ab}{a+b}$

(ii) Harmonic Mean (HM) of $a_1, a_2, a_3, \ldots, a_n$ is given by

$$\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots\ldots + \frac{1}{a_n}\right)$$

(iii) If $a, H_1, H_2, H_3, \ldots, H_n, b$ are in HP, then

(a) $H_1, H_2, H_3, \ldots, H_n$

are called n harmonic means between a and b, where

$$H_1 = \frac{(n+1)ab}{a+nb},$$

$$H_2 = \frac{(n+1)ab}{2a+(n-1)b},$$

$$H_3 = \frac{(n+1)\,ab}{3a+(n-2)\,b}$$

$$\vdots \qquad \vdots$$

$$H_n = \frac{(n+1)\,ab}{na+(n-(n-1))\,b} = \frac{(n+1)ab}{na+b}$$

(b) $\frac{1}{H_1} + \frac{1}{H_2} + \frac{1}{H_3} + \ldots + \frac{1}{H_n} = \frac{n}{H}$, where $H = \frac{2ab}{a+b}$

Important Results on HP

(i) If in a HP, $a_m = n$ and $a_n = m$, then

$$a_{m+n} = \frac{mn}{m+n}, a_{mn} = 1, a_p = \frac{mn}{p}$$

(ii) If in a HP, $a_p = qr$ and $a_q = pr$,

then $a_r = pq$

(iii) If H is HM between a and b, then

(a) $(H-2a)(H-2b) = H^2$

(b) $\frac{1}{H-a} + \frac{1}{H-b} = \frac{1}{a} + \frac{1}{b}$

(c) $\frac{H+a}{H-a} + \frac{H+b}{H-b} = 2$

Properties of AM, GM and HM between Two Numbers

1. If A, G and H are arithmetic, geometric and harmonic means of two positive numbers a and b, then

   (i) $A = \frac{a+b}{2}, G = \sqrt{ab}, H = \frac{2ab}{a+b}$

   (ii) $A \geq G \geq H$

   (iii) $G^2 = AH$ and so A, G, H are in GP.

   (iv) $$\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \begin{cases} A, & \text{if } n = 0 \\ G, & \text{if } n = -\frac{1}{2} \\ H, & \text{if } n = -1 \end{cases}$$

2. If A, G, H are AM, GM and HM of three positive numbers a, b and c, then the equation having a, b and c as its root is

$$x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$$

where, $A = \dfrac{a+b+c}{3}$, $G = (abc)^{1/3}$

and $$\frac{1}{H} = \left(\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\right)$$

3. If number of terms in AP/GP/HP are odd, then AM/GM/HM of first and last term is middle term of progression.

4. If A_1, A_2 be two AM's, G_1, G_2 be two GM's and H_1, H_2 be two HM's between two numbers a and b, then

$$\frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2}$$

Arithmetic-Geometric Progression

A sequence in which every term is a product of corresponding term of AP and GP is known as arithmetic-geometric progression.

The series may be written as

$$a, (a+d)r, (a+2d)r^2, (a+3d)r^3, \ldots, [a+(n-1)d]r^{n-1}$$

Then, $$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{\{a+(n-1)d\}r^n}{1-r}, \text{ if } r \neq 1$$

$$S_n = \frac{n}{2}[2a+(n-1)d], \text{ if } r = 1$$

Also, $$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}, \text{ if } |r| < 1$$

Method of Difference

Let $a_1 + a_2 + a_3 + \ldots$ be a given series.

***Case* I** If $a_2 - a_1, a_3 - a_2, \ldots$ are in AP or GP, then a_n and S_n can be found by the method of difference.

Clearly, $S_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$

or $S_n = \quad a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$

So, $S_n - S_n = a_1 + (a_2 - a_1) + (a_3 - a_2) + (a_4 - a_3) + (a_n - a_{n-1}) - a_n$

$\Rightarrow \quad a_n = a_1 + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1})$

$\therefore \quad a_n = a_1 + T_1 + T_2 + T_3 + \ldots + T_{n-1}$

where, $T_1, T_2, T_3, \ldots$ are terms of new series and $S_n = \Sigma a_n$

***Case* II** It is not always necessary that the sequence of first order of differences i.e. $a_2 - a_1, a_3 - a_2, \ldots, a_n - a_{n-1}, \ldots$ is always in AP or in GP. In such cases, we proceed as follows.

Let $a_1 = T_1, a_2 - a_1 = T_2, a_3 - a_2 = T_3, \ldots, a_n - a_{n-1} = T_n$

So, $\quad a_n = T_1 + T_2 + \ldots + T_n \quad$ ...(i)

$a_n = \quad T_1 + T_2 + \ldots + T_{n-1} + T_n \quad$ ...(ii)

On subtracting Eq. (i) from Eq. (ii), we get

$$T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + \ldots + (T_n - T_{n-1})$$

Now, the series $(T_2 - T_1) + (T_3 - T_2) + \ldots + (T_n - T_{n-1})$ is series of second order of differences and if it is either in AP or in GP, then $a_n = \Sigma T_r$.

Otherwise, in the similar way, we find series of higher order of differences and the nth term of the series.

Exponential Series

The sum of the series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \infty$ is denoted by the number e.

$$\therefore \quad e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

(i) e lies between 2 and 3.

(ii) e is an irrational number.

(iii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \infty,\ x \in R$

(iv) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots \infty,\ x \in R$

(v) For any $a > 0$, $a^x = e^{x \log_e^a}$

$$= 1 + x (\log_e a) + \frac{x^2}{2!} (\log_e a)^2 + \frac{x^3}{3!} (\log_e a)^3 + \ldots \infty, x \in R$$

Logarithmic Series

(i) $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \infty, (-1 < x \le 1)$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}, (-1 < x \le 1)$$

(ii) $\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \infty, (-1 \le x < 1)$

$\Rightarrow -\log_e(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots \infty, (-1 \le x < 1)$

(iii) $\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \infty\right), (-1 < x < 1)$

(iv) $\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots \infty$

Some Important Series

(i) $\sum_{n=0}^{\infty}\frac{1}{n!} = e = \sum_{n=1}^{\infty}\frac{1}{(n-1)!} = \sum_{n=k}^{\infty}\frac{1}{(n-k)!} = e$

(ii) $\sum_{n=1}^{\infty}\frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \infty = e - 1$

(iii) $\sum_{n=2}^{\infty}\frac{1}{n!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \infty = e - 2$

(iv) $\sum_{n=0}^{\infty}\frac{1}{(n+1)!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \infty = e - 1$

(v) $\sum_{n=1}^{\infty}\frac{1}{(n+1)!} = \sum_{n=0}^{\infty}\frac{1}{(n+2)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \infty = e - 2$

(vi) $\sum_{n=0}^{\infty}\frac{1}{(2n)!} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots = \frac{e + e^{-1}}{2} = \sum_{n=1}^{\infty}\frac{1}{(2n-2)!}$

(vii) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)!} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots = \frac{e - e^{-1}}{2} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}$

(viii) $e^{ax} = 1 + \frac{(ax)}{1!} + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \ldots + \frac{(ax)^n}{n!} + \ldots \infty$

(ix) $\sum_{n=0}^{\infty}\frac{n}{n!} = e = \sum_{n=1}^{\infty}\frac{n}{n!}$

(x) $\sum_{n=0}^{\infty} \frac{n^2}{n!} = 2e = \sum_{n=1}^{\infty} \frac{n^2}{n!}$

(xi) $\sum_{n=0}^{\infty} \frac{n^3}{n!} = 5e = \sum_{n=1}^{\infty} \frac{n^3}{n!}$

(xii) $\sum_{n=0}^{\infty} \frac{n^4}{n!} = 15e = \sum_{n=1}^{\infty} \frac{n^4}{n!}$

(xiii) $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$

(xiv) $\sum_{r=1}^{n} ka_r = k \sum_{r=1}^{n} a_r$

(xv) $\sum_{r=1}^{n} k = k + k + \dots\ n$ times $= n.k$, where k is a constant.

(xvi) $\sum_{r=1}^{n} r = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$

(xvii) Sum of first n even natural numbers.

i.e. $2 + 4 + 6 + \dots + 2n = n(n+1)$

(xviii) Sum of first n odd natural numbers.

i.e. $1 + 3 + 5 + \dots + (2n-1) = n^2$

(xix) $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(xx) $\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

(xxi) $\sum_{r=1}^{n} r^4 = 1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}$

(xxii) Sum of n terms of series

$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + 7^2 - 8^2 + \dots$

***Case* I** When n is odd $= \frac{n(n+1)}{2}$

***Case* II** When n is even $= \frac{-n(n+1)}{2}$

(xxiii) $2 \sum_{i<j=1}^{n} a_i\, a_j = (a_1 + a_2 + \dots + a_n)^2 - (a_1^2 + a_2^2 + \dots + a_n^2)$