Oscillations

Periodic Motion

A motion which repeats itself identically after a fixed interval of time is called periodic motion, *e.g.* orbital motion of the earth around the sun, motion of arms of a clock etc.

Oscillatory Motion

A periodic motion taking place to and fro or back and forth about a fixed point is called oscillatory motion, *e.g.* motion of a simple pendulum, motion of a loaded spring etc.

Note Every oscillatory motion is periodic motion but every periodic motion is not oscillatory motion.

Harmonic Oscillation

The oscillation which can be expressed in terms of single harmonic function, *i.e.* sine or cosine function is called harmonic oscillation.

Simple Harmonic Motion

A harmonic oscillation of constant amplitude and of single frequency under a restoring force whose magnitude is proportional to the displacement and always acts towards mean position is called Simple Harmonic Motion (SHM).

A simple harmonic oscillation can be expressed as

$$y = a \sin \omega t$$

or

$$y = a \cos \omega t$$

where, a = amplitude of oscillation.

Some Terms Related to SHM

(i) **Time Period** Time taken by the body to complete one oscillation is known as time period. It is denoted by T.

(ii) **Frequency** The number of oscillations completed by the body in one second is called frequency. It is denoted by ν.

$$\text{Frequency} = \frac{1}{\text{Time period}}$$

Its SI unit is hertz or second^{-1}.

(iii) **Angular Frequency** The product of frequency with factor 2π, is called angular frequency. It is denoted by ω.

$$\text{Angular frequency } (\omega) = 2\pi\nu$$

Its SI unit is radian per second.

(iv) **Displacement** A physical quantity which represents change in position with respect to mean position or equilibrium position is called displacement. It is denoted by y.

(v) **Amplitude** The maximum displacement in any direction from mean position is called amplitude. It is denoted by a.

(vi) **Phase** A physical quantity which express the position and direction of motion of an oscillating particle is called phase. It is denoted by ϕ.

Some Important Formulae of SHM

(i) Displacement in SHM at any instant is given by

$$y = a \sin \omega t$$

or

$$y = a \cos \omega t$$

where, a = amplitude

and ω = angular frequency.

(ii) Velocity of a particle executing SHM at any instant is given by

$$v = \omega\sqrt{(a^2 - y^2)}$$

At mean position, $y = 0$ and v is maximum

$$v_{\text{max}} = a\omega$$

At extreme position, $y = a$ and v is zero.

(iii) Acceleration of a particle executing SHM at any instant is given by A or $\alpha = -\omega^2 y$

Negative sign indicates that the direction of acceleration is opposite to the direction in which displacement increases, *i.e.* towards mean position.

At mean position, $y = 0$ and acceleration is also zero.

At extreme position, $y = a$ and acceleration is maximum

$$A_{max} = -a\omega^2$$

(iv) Time period in SHM is given by

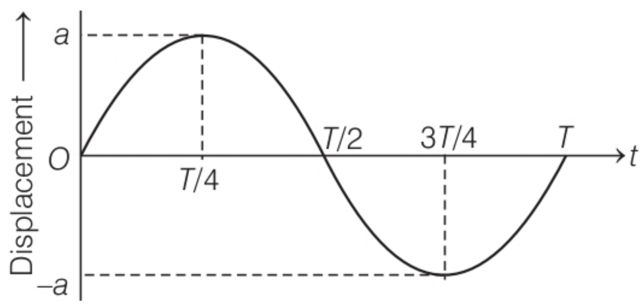
$$T = 2\pi\sqrt{\frac{\text{Inertia factor}}{\text{Spring factor}}}$$

In general, inertia factor $= m$, (mass of the particle)
spring factor $= k$ (force constant)

Graphical Representation of SHM

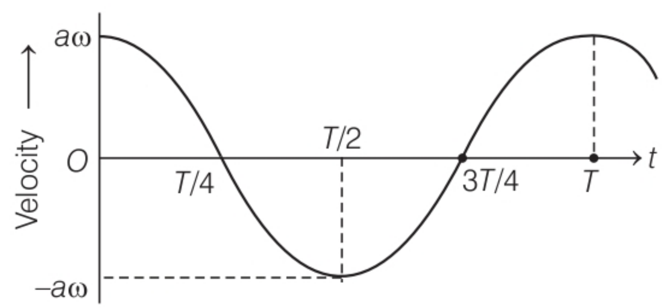
(i) **Displacement-Time Graph**

When $y(t) = a \sin \omega t$



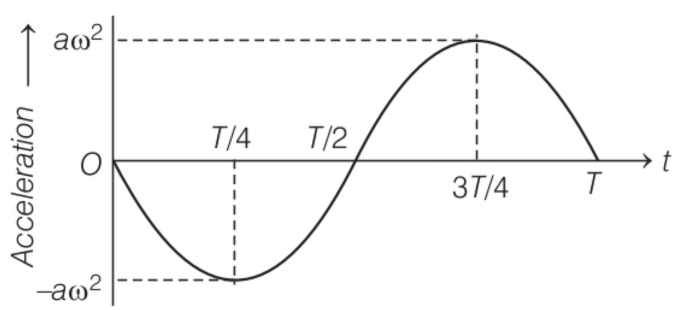
(ii) **Velocity-Time Graph**

When $v(t) = +\omega a \cos \omega t$



(iii) **Acceleration-Time Graph**

When $a(t) = -\omega^2 a \sin \omega t$



Note The acceleration is maximum at a place where the velocity is minimum and *vice-versa*.

From the above mentioned graphs, it can be concluded that for a particle executing SHM, the phase difference between

(i) Instantaneous displacement and instantaneous velocity

$$= \left(\frac{\pi}{2}\right) \text{rad}$$

(ii) Instantaneous velocity and instantaneous acceleration

$$= \left(\frac{\pi}{2}\right) \text{rad}$$

(iii) Instantaneous acceleration and instantaneous displacement

$$= \pi \text{ rad}$$

Note The graph between velocity and displacement for a particle executing SHM is elliptical.

Force in SHM

We know that, the acceleration of body in SHM can be given as $a = -\omega^2 x$.

Applying the equation of motion $\mathbf{F} = m\mathbf{a}$,

We have, $$F = -m\omega^2 x = -kx$$

where, $\omega = \sqrt{\frac{k}{m}}$ and $k = m\omega^2$ is a constant and sometimes it is called the elastic constant.

Thus, in SHM, the force is directly proportional and opposite to the displacement and is always directed towards the mean position.

Energy in SHM

The kinetic energy of the particle is $K = \frac{1}{2} m\omega^2 (a^2 - x^2)$

From this expression, we can see that, the kinetic energy is maximum at the centre $(x = 0)$ and zero at the extremes of oscillation $(x \pm a)$.

The potential energy of the particle is $U = \frac{1}{2} m\omega^2 x^2$.

From this expression, we can see that, the potential energy has a minimum value at the centre $(x = 0)$ and increases as the particle approaches either extreme of the oscillation $(x \pm a)$.

Total energy can be obtained by adding potential and kinetic energies. Therefore,

$$E = K + U$$

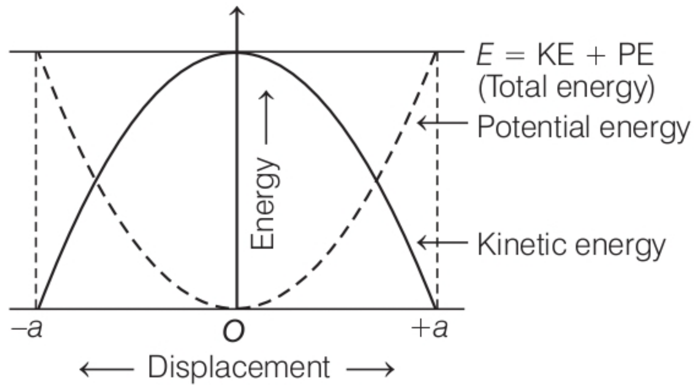
$$= \frac{1}{2}\, m(a^2 - x^2)\, \omega^2 + \frac{1}{2}\, m\omega^2 x^2 = \frac{1}{2}\, m\omega^2 a^2$$

where, a = amplitude,

m = mass of particle executing SHM

and ω = angular frequency.

Changes in kinetic and potential energies during oscillations is represented in the graph given below.



Some points related to energy of the particle executing SHM

(i) The frequency of kinetic energy or potential energy of a particle executing SHM is double than that of the frequency in SHM.

(ii) The frequency of total energy of particles executing SHM is zero as total energy in SHM remains constant at all positions.

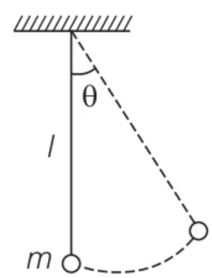
How the different physical quantities (*e.g.* displacement, velocity, acceleration, kinetic energy etc) vary with time or displacement are listed ahead in tabular form.

S.No.	Name of the equation	Expression of the equation	Remarks
1.	Displacement-time	$x = A\cos(\omega t + \phi)$	x varies between $+A$ and $-A$
2.	Velocity-time $\left(v = \frac{dx}{dt}\right)$	$v = -A\omega\sin(\omega t + \phi)$	v varies between $+A\omega$ and $-A\omega$
3.	Acceleration-time $\left(a = \frac{dv}{dt}\right)$	$a = -A\omega^2\cos(\omega t + \phi)$	a varies between $+A\omega^2$ and $-A\omega^2$
4.	Kinetic energy-time $\left(K = \frac{1}{2}mv^2\right)$	$K = \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi)$	K varies between $-\frac{1}{2}mA^2\omega^2$ and $\frac{1}{2}mA^2\omega^2$

S.No.	Name of the equation	Expression of the equation	Remarks
5.	Potential energy-time $\left(U = \frac{1}{2} m\omega^2 x^2\right)$	$U = -\frac{1}{2} m\omega^2 A^2 \cos^2(\omega t + \phi)$	U varies between $\frac{1}{2} mA^2\omega^2$ and 0
6.	Total energy-time $(E = K + U)$	$E = \frac{1}{2} m\omega^2 A^2$	E is constant
7.	Velocity-displacement	$v = \omega\sqrt{A^2 - x^2}$	$v = 0$ at $x = \pm A$ and at $x = 0$, $v = \pm A\omega$
8.	Acceleration-displacement	$a = -\omega^2 x$	$a = 0$ at $x = o$ and $a = \pm \omega^2 A$ at $x = \pm Ax$
9.	Kinetic energy-displacement	$K = \frac{1}{2} m\omega^2 (A^2 - x^2)$	$K = 0$ at $x = \pm A$ and $K = \frac{1}{2} m\omega^2 A^2$ at $x = 0$
10.	Potential energy-displacement	$U = \frac{1}{2} m\omega^2 x^2$	$U = 0$ at $x = 0$ and $U = \frac{1}{2} m\omega^2 A^2$ at $x = \pm A$
11.	Total energy-displacement	$E = \frac{1}{2} m\omega^2 A^2$	E is constant

Simple Pendulum

A simple pendulum consists of a heavy point mass suspended from a rigid support by means of an elastic inextensible string.



The time period of the simple pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where, l = effective length of the pendulum and g = acceleration due to gravity.

If the effective length l of simple pendulum is very large and comparable with the radius of earth (R), then its time period is given by

$$T = 2\pi\sqrt{\frac{Rl}{(l + R)g}}$$

For a simple pendulum of length equal to radius of earth,

$$T = 2\pi\sqrt{\frac{R}{2g}} = 60 \text{ min}$$

For a simple pendulum of infinite length ($l >> R$),

$$T = 2\pi\sqrt{\frac{R}{g}} = 84.6 \text{ min}$$

If the bob of the simple pendulum is suspended by a metallic wire of length l, having coefficient of linear expansion α, then due to increase in temperature by $d\theta$, then

Effective length, $l' = l(1 + \alpha\, d\theta)$

and $$T' = 2\pi\sqrt{\frac{l(1 + \alpha d\theta)}{g}}$$

When a bob of simple pendulum of density ρ oscillates in a fluid of density ρ_0 ($\rho_0 < \rho$), then time period get increased.

Increased time period, $$T' = 2\pi\sqrt{\frac{\rho l}{(\rho - \rho_0)g}}$$

When simple pendulum is in a horizontally accelerated vehicle, then its time period is given by

$$T = 2\pi\sqrt{\frac{l}{\sqrt{(a^2 + g^2)}}}$$

where, $a =$ horizontal acceleration of the vehicle.

When simple pendulum is in a vehicle sliding down an inclined plane, then its time period is given by

$$T = 2\pi\sqrt{\frac{l}{g\cos\theta}}$$

where, $\theta =$ inclination of plane.

When a bob of simple pendulum has positive q and made to oscillate in uniform electric field acting in upward direction, then

$$T = 2\pi\sqrt{\frac{l}{g - \frac{qE}{m}}}$$

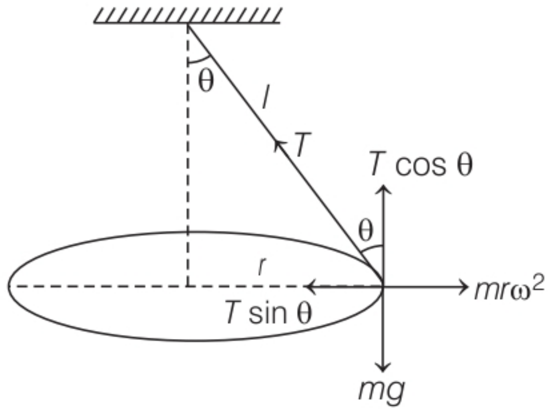
Second's Pendulum

A simple pendulum having time period of 2 seconds is called second's pendulum.

The effective length of a second's pendulum is 99.992 cm of approximately 1 m on earth.

Conical Pendulum

If a simple pendulum is fixed at one end and the bob is rotating in a horizontal circle, then it is called a conical pendulum.



In equilibrium, $T \sin\theta = mr\omega^2$

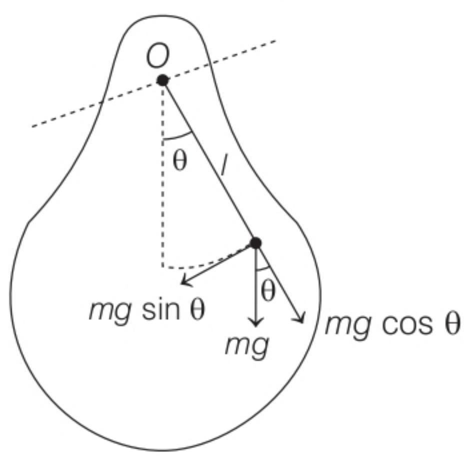
Its time period, $T' = 2\pi\sqrt{\dfrac{mr}{T\sin\theta}}$

Compound Pendulum

Any rigid body mounted, so that it is capable of swinging in a vertical plane about some axis passing through it is called a compound pendulum.

Its time period is given by

$$T = 2\pi\sqrt{\frac{I}{mg\,d}}$$



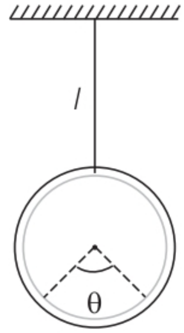
where, I = moment of inertia of the body about an axis passing through the centre of suspension,

m = mass of the body

and d = distance of centre of gravity from the centre of suspension.

Torsional Pendulum

It consists of a disc (or some other object) suspended from a wire suspended to a rigid support, which is then twisted and released, resulting in oscillatory motion.



Time period of torsional pendulum is given by

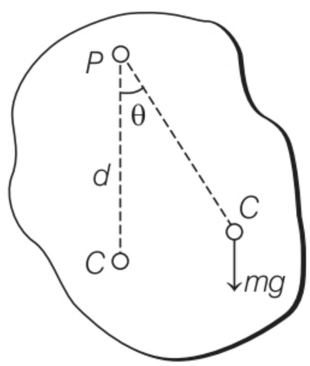
$$T = 2\pi\sqrt{\frac{I}{C}}$$

where, I = moment of inertia of the body about the axis of rotation

and C = restoring couple per unit twist.

Physical Pendulum

When a rigid body of any shape is capable of oscillating about an axis (may or may not be passing through it), it constitutes a physical pendulum.



$$T = 2\pi\sqrt{\frac{I}{mgd}}$$

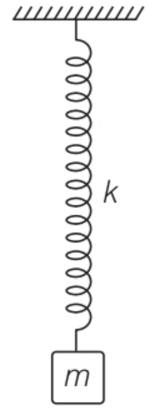
(i) The simple pendulum whose time period is same as that of a physical pendulum is termed as an equivalent simple pendulum.

$$T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{l}{g}}$$

(ii) The length of an equivalent simple pendulum is given by $l = \frac{I}{md}$

Spring Pendulum

A point mass suspended from a massless (or light) spring constitutes a spring pendulum. If the mass is once pulled downwards so as to stretch the spring and then released, the system oscillated up and down about its mean position simple harmonically. Time period and frequency of oscillations are given by

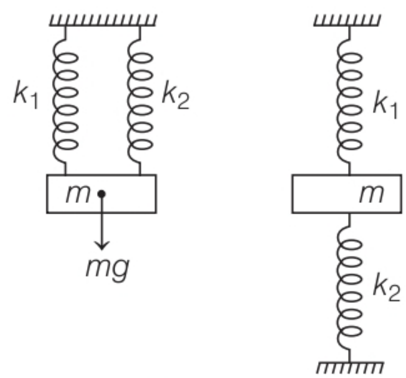


$$T = 2\pi\sqrt{\frac{m}{k}} \text{ or } \nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

If the spring is not light but has a definite mass m_s, then it can be easily shown that period of oscillation will be

$$T = 2\pi\sqrt{\frac{m + \frac{m_s}{3}}{k}}$$

When two springs of force constants k_1 and k_2 are connected in parallel to mass m as shown in figure, then

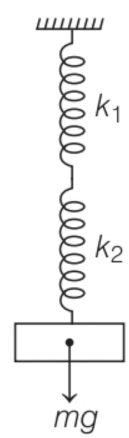


(i) Effective force constant of the spring combination

$$k = k_1 + k_2$$

(ii) Time period, $T = 2\pi\sqrt{\frac{m}{(k_1 + k_2)}}$

When two springs of force constant k_1 and k_2 are connected in series to mass m as shown in figure, then



(i) Effective force constant of the spring combination,

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

(ii) Time period, $T = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$

Oscillations of Liquid in a U-tube

If a liquid is filled up to height h in both limbs of a U-tube and now liquid is depressed upto a small distance y in one limb and then released, then liquid column in U-tube start executing SHM.

The time period of oscillation is given by $T = 2\pi\sqrt{\frac{h}{g}}$

Oscillations of Ball in Bowl

If a small steel ball of mass m is placed at a small distance from O inside a smooth concave surface of radius R and released, it will oscillate about O.

$$T = 2\pi\sqrt{\frac{R}{g}}$$

Oscillations of a Ball in a Tunnel through the Earth

If a ball moves through a tunnel along a diameter of earth, then due to gravitational force between ball and earth a restoring force is set up, due to which the ball performs SHM, whose time period is given by

$$T = 2\pi\sqrt{R/g}$$

where, $R =$ radius of earth.

Free Oscillations

When a body which can oscillate about its mean position is displaced from mean position and then released, it oscillates about its mean position. These oscillations are called free oscillations and the frequency of oscillations is called **natural frequency.**

Damped Oscillations

The oscillations in which amplitude decreases with time are called damped oscillations.

The displacement of the damped oscillator at an instant t is given by

$$x = x_0 e^{-bt/2m} \cos(\omega' t + \phi)$$

where, $x_0 e^{-bt/2m}$ is the amplitude of oscillator which decreases continuously with time t and ω'.

The mechanical energy E of the damped oscillator at an instant t is given by

$$E = \frac{1}{2} k x_0^2 \, e^{-bt/m}$$

Forced Oscillations

Oscillations of any object with a frequency different from its natural frequency under a periodic external force are called forced oscillations.

Resonant Oscillations

When a body oscillates with its own natural frequency with the help of an external periodic force whose frequency is equal to the natural frequency of the body, then these oscillations are called resonant oscillations.

Lissajous' Figures

If two SHMs are acting in mutually perpendicular directions, then due to their superpositions the resultant motion, in general, is a curve/loop. The shape of the curve depends on the frequency ratio of two SHMs and initial phase difference between them. Such figures are called Lissajous' figures.

Let two SHMs be of same frequency (*e.g.* $x = a_1 \sin \omega t$ and $y = a_2 \sin(\omega t + \phi)$, then the general equation of resultant motion is found to be

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2} \cos\phi = \sin^2\phi$$

The equation represents an ellipse. However, if $\phi = 0°$ or π or $n\pi$, then the resultant curve is a straight inclined line.